Homogenization of Kirchhoff plate equation

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[VII Partial differential equations, optimal design and numerics]

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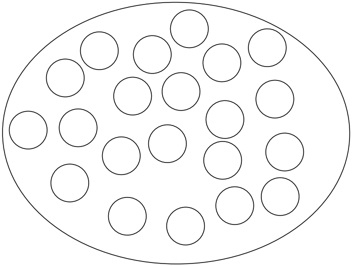
The physical idea of homogenization is to average a heterogeneous media in order to derive effective properties.

$$\begin{cases} Au = f \quad \text{in} \quad \Omega \\ \text{initial/boundary condition} \end{cases}$$



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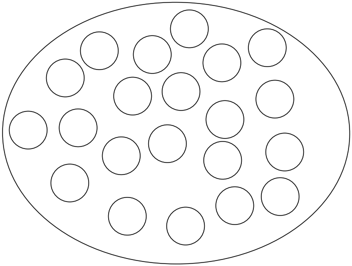

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Kirchhoff plate equation

Homogeneous Dirichlet boundary value problem:

$$\begin{cases} \mathrm{divdiv}(\mathrm{M}\nabla\nabla \mathrm{u}) = \mathrm{f} \quad \mathrm{in} \quad \Omega \\ u \in H_0^2(\Omega). \end{cases}$$

- $\Omega \subseteq \mathbb{R}^2$ bounded domain
- $f \in H^{-2}(\Omega)$ external load
- $M \in \mathfrak{M}_2(\alpha, \beta; \Omega) := \{M \in L^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})) : (\forall \mathrm{S} \in \mathrm{Sym})\, \mathrm{M}(\mathrm{x})\mathrm{S} : \mathrm{S} \geq \alpha \mathrm{S} : \mathrm{S} \text{ and } \mathrm{M}^{-1}\mathrm{S} : \mathrm{S} \geq \frac{1}{\beta}\mathrm{S} : \mathrm{S} \text{ a.e.x}\}$ describes properties of material of the given plate
- $u \in H_0^2(\Omega)$ vertical displacement of the plate

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Antonić, Balenović, 1999.

Definition

A sequence of tensor functions (M^n) in $\mathfrak{M}_2(\alpha,\beta;\Omega)$ H-converges to $M \in \mathfrak{M}_2(\alpha,\beta;\Omega)$ if for any $f \in H^{-2}(\Omega)$ the sequence of solutions (u_n) of problems

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coverges weakly to a limit u in $H^2_0(\Omega)$, while the sequence $(M^n\nabla\nabla u_n)$ converges to $M\nabla\nabla u$ weakly in the space $L^2(\Omega;\mathrm{Sym})$.

Theorem

Let (M^n) be a sequence in $\mathfrak{M}_2(\alpha,\beta;\Omega)$. Then there is a subsequence (M^{n_k}) and a tensor function $M \in \mathfrak{M}_2(\alpha,\beta;\Omega)$ such that (M^{n_k}) H-converges to M.

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Theorem (Locality of the H-convergence)

Let (M^n) and (O^n) be two sequences of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$, which H-converge to M and O, respectively. Let ω be an open subset compactly embedded in Ω. If $M^n(x) = O^n(x)$ in ω, then $M(x) = O(x)$ in ω.

Theorem (Irrelevance of boundary conditions)

Let (M^n) be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to M. For any sequence (z_n) such that

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M^n satisfies $M^n\nabla\nabla z_n \rightharpoonup M\nabla\nabla z$ in $L^2_{\mathrm{loc}}(\Omega; Sym)$.

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Theorem (Energy convergence)

Let (M^n) be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to M. For any $f \in H^{-2}(\Omega)$, the sequence (u_n) of solutions of

$$\begin{cases} \mathrm{divdiv}(\mathrm{M^n \nabla\nabla u_n}) = \mathrm{f} \quad \mathrm{in} \quad \Omega \\ u_n \in H_0^2(\Omega)\,. \end{cases}$$

satisfies $M^n \nabla\nabla u_n : \nabla\nabla u_n \rightharpoonup M\nabla\nabla u : \nabla\nabla u$ weakly-$*$ in the space of Radon measures and
$\int_\Omega M^n \nabla\nabla u_n : \nabla\nabla u_n \, dx \to \int_\Omega M\nabla\nabla u : \nabla\nabla u \, dx$*, where u is the solution of the homogenized equation*

$$\begin{cases} \mathrm{divdiv}(\mathrm{M \nabla\nabla u}) = \mathrm{f} \quad \mathrm{in} \quad \Omega \\ u \in H_0^2(\Omega)\,. \end{cases}$$

Theorem (Ordering property)

Let (M^n) and (O^n) be two sequences of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converge to the homogenized tensors M and O, respectively. Assume that, for any n,

$$M^n\xi : \xi \leq O^n\xi : \xi, \quad \forall \xi \in \mathrm{Sym}.$$

Then the homogenized limits are also ordered:

$$M\xi : \xi \leq O\xi : \xi, \quad \forall \xi \in \mathrm{Sym}.$$

Theorem

Let (M^n) be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that either converges strongly to a limit tensor M in $L^1(\Omega; L(\mathrm{Sym}, \mathrm{Sym}))$, or converges to M almost everywhere in Ω. Then, M^n also H-converges to M.

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Theorem (Compensated compactness result)

Let the following convergences be valid: $w_n \rightharpoonup w$ *in* $H^2_{\mathrm{loc}}(\Omega)$ *and* $D^n \rightharpoonup D$ *in* $L^2_{\mathrm{loc}}(\Omega; M_{2\times 2})$ *with an additional assumption that the sequence* $(\mathrm{divdivD^n})$ *is contained in a precompact (for the strong topology) set of the space* $H^{-2}_{\mathrm{loc}}(\Omega)$*. Then we have that* $E^n : D^n \rightharpoonup E : D$ *weakly-*$*$ *in the space of Radon measures, where we denote* $E^n := \nabla\nabla w^n$*, for* $n \in \mathbb{N} \cup \{\infty\}$.

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Definition

Let (M^n) be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to a limit M. Let $(w_n^{ij})_{1 \leq i,j \leq N}$ be a family of test functions satisfying

$$w_n^{ij} \rightharpoonup \frac{1}{2} x_i x_j \quad \text{in} \quad \mathrm{H}^2(\Omega)$$

$$\mathrm{divdiv}(\mathrm{M^n \nabla\nabla w_n^{ij}}) \to \cdot \quad \mathrm{in}\, \mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)$$

$$M^n \nabla\nabla w_n^{ij} \rightharpoonup \cdot \quad \text{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathrm{Sym}).$$

The tensor W^n defined as $[a_{ijkm}]_{ij} = [\nabla\nabla w_n^{km}]_{ij}$ is called a corrector tensor.

Theorem

Let (M^n) be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to a tensor M. A sequence of correctors (W^n) is unique in the sense that, if there exist two sequences of correctors (W^n) and $(\tilde{W}^n)$, their difference $(W^n - \tilde{W}^n)$ converges strongly to zero in $L^2_{\mathrm{loc}}(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$.

Theorem (Corrector result)

Let (M^n) be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ which H-converges to M. For $f \in H^{-2}(\Omega)$, let (u_n) be the solution of

$$\begin{cases} \mathrm{divdiv}(\mathrm{M^n}\nabla\nabla \mathrm{u_n}) = \mathrm{f} \quad \mathrm{in} \quad \Omega \\ u_n \in H_0^2(\Omega)\,. \end{cases}$$

Let u be the weak limit of (u_n) in $H_0^2(\Omega)$, i.e., the solution of the homogenized equation

$$\begin{cases} \mathrm{divdiv}(\mathrm{M}\nabla\nabla \mathrm{u}) = \mathrm{f} \quad \mathrm{in} \quad \Omega \\ u \in H_0^2(\Omega)\,. \end{cases}$$

Then, $r_n := \nabla\nabla u_n - W^n \nabla\nabla u \to 0$ strongly in $L^1_{\mathrm{loc}}(\Omega; Sym)$.

Small-amplitude homogenization

$$A^n_\gamma(x) := A_0 + \gamma B^n(x),\ \gamma \in \mathbf{R}$$

$$A_\gamma := A_0 + \gamma B_0 + \gamma^2 C_0 + o(\gamma^2),\ \gamma \in \mathbf{R}$$

Small-amplitude homogenization

$$A_\gamma^n(x) := A_0 + \gamma B^n(x),\ \gamma \in \mathbf{R}$$

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Theorem

Let $M^n : \Omega \times P \to \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})$ *be a sequence of tensors, such that* $M^n(\cdot, p) \in \mathfrak{M}_2(\alpha, \beta; \Omega)$*, for* $p \in P$*, where* $P \subseteq \mathbf{R}$ *is an open set. Assume that (for some* $k \in \mathbf{N}_0$*) a mapping* $p \mapsto M^n(\cdot, p)$ *is of class* C^k *from P to* $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$*, with derivatives which are equicontinuous on every compact set* $K \subseteq P$ *up to order k:*

$$(\forall K \in \mathcal{K}(P))\,(\forall \varepsilon > 0)(\exists \delta > 0)(\forall p, q \in K)(\forall n \in \mathbf{N})$$
$$(\forall i \in \{0, \dots, k\})$$
$$|p - q| < \delta \Rightarrow \|(M^n)^{(i)}(\cdot, p) - (M^n)^{(i)}(\cdot, q)\|_{\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))} < \varepsilon.$$

Then there is a subsequence (M^{n_k}) *such that for every* $p \in P$

$$M^{n_k}(\cdot, p) \xrightarrow{H} M(\cdot, p) \quad \text{in} \quad \mathfrak{M}_2(\alpha, \beta; \Omega)$$

and $p \mapsto M(\cdot, p)$ is a C^k mapping from P to $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$.

Periodic case

- $Y = [0,1]^d$
- $M^n(x) := M(nx), x \in \Omega$
- $H^2_\#(Y) := \{f \in H^2_{\text{loc}}(\mathbf{R}^d)\,\text{such that f is Y} - \text{periodic}\}$ with the norm $\|\cdot\|_{H^2(Y)}$
- $H^2_\#(Y)/\mathbf{R}$ equipped with the norm $\|\nabla\nabla\cdot\|_{L^2(Y)}$
- $E_{ij}, 1 \le i,j \le d$ are $M_{d\times d}$ matrices defined as

$$[E_{ij}]_{kl} = \begin{cases} 1, & \text{if } i=j=k=l \\ \frac{1}{2}, & \text{if } i \neq j, (k,l) \in \{(i,j),(j,i)\} \\ 0, & \text{otherwise.} \end{cases}$$

Theorem

Let (M^n) be a sequence of tensors defined by $M^n(x) := M(nx), x \in \Omega$. Then (M^n) H-converges to a constant tensor $M^* \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ defined as

$$m^*_{klij} = \int_Y M(y)(E_{ij} + \nabla\nabla w_{ij}(y)) : (E_{kl} + \nabla\nabla w_{kl}(y))\, dy,$$

where $(w_{ij})_{1 \le i,j \le d}$ is the family of unique solutions in $H^2_{\#}(Y)/\mathbf{R}$ of boundary value problems

$$\begin{cases} \mathrm{div}\,\mathrm{div}\,(M(y)(E_{ij} + \nabla\nabla w_{ij}(y))) = 0 \text{ in } \mathrm{Y, i, j} = 1, \dots, \mathrm{d} \\ y \to w_{ij}(y). \end{cases}$$

$$A^n_\gamma(x) := A_0 + \gamma B^n(x)$$

- A_0 constant, coercive tensor
- $B^n(x) := B(nx)$ Y-periodic, L^∞ tensor function and $\displaystyle\int_Y B(y)\,dy = 0$

$$A_\gamma := A_0 + \gamma B_0 + \gamma^2 C_0 + o(\gamma^2)$$

$$B_0 E_{ij} : E_{kl} = 0$$

$$\begin{aligned} C_0 E_{ij} : E_{kl} =& (2\pi i)^2 \int_Y \sum_{k \in J} a_{-k} B_k k \cdot k^T : E_{kl} \, dy \\ &+ (2\pi i)^4 \int_Y \sum_{k \in J} A_0 k \cdot k^T a_k : a_{-k} k \cdot k^T \, dy \\ &+ (2\pi i)^2 \int_Y \sum_{k \in J} B_k E_{ij} : a_{-k} k \cdot k^T \, dy \end{aligned}$$

Thank you for your attention!