Modeling and Control of Renewable Resources

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Benasque, 25.08.2017

Introduction: Structured Population Models

u population density

Introduction: Structured Population Models

$u = u(t)$	population density
t	time

Introduction: Structured Population Models

$u = u(t, x)$	population density
t	time
x	age

Introduction: Structured Population Models

$u = u(t, x)$	population density
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$$\partial_t u + \partial_x u = 0$$

Introduction: Structured Population Models

$u = u(t, x)$	population density
t	time
x	biological age/size/(trait)

$$\partial_t u + \partial_x \left([\text{growth/aging}]\, u\right) = 0$$

Introduction: Structured Population Models

$u = u(t,x)$	population density
t	time
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$$\partial_t u + \partial_x \left(g(t, x, \textstyle\int u)\, u \right) = 0$$

Introduction: Structured Population Models

$u = u(t,x)$	population density
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$$\partial_t u + \partial_x \left(g(t, x, \textstyle\int u)\, u \right) = [\text{death}]$$

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$u = u(t,x)$	population density
t	time
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$$\partial_t u + \partial_x \left(g(t,x,\int u)\, u\right) = -d(t,x,u,\int u)$$

Introduction: Structured Population Models

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x	biological age/size/(trait)

$$\partial_t u + \partial_x \left(g(t,x,\textstyle\int u)\, u \right) = -d(t,x,u,\textstyle\int u)$$
$$u(t,0) = [\text{birth}]$$

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$$u(t,0) = b(t,\textstyle\int u)$$

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$$\partial_t u + \partial_x \left(g(t,x,\textstyle\int u)\, u\right) = -d(t,x,u,\textstyle\int u)$$
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$$u(0,x) = [\text{initial datum}]$$

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$$u(t, 0) = b(t, \textstyle\int u)$$

$$u(0, x) = \bar{u}(x)$$

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Balance Law with Boundary
NonLocal flow
NonLocal source
NonLocal boundary data

Introduction: Conservation Laws – Keywords

$$\boxed{\boxed{\partial_t u + \operatorname{div}_x f(t,x,u) = g(t,x,u)}}$$

$$x \in \mathbb{R}^N \qquad u \in \mathbb{R}^n$$

$$\boxed{\boxed{N = 1,\ n \geq 1}} \qquad \boxed{\boxed{N \geq 1,\ n = 1}}$$

Norm:	$\|\cdot\|_{\mathbf{L}^1}$	
x regularity:	$x \to u(t,x)$	$\mathbf{L^1}$, $\mathbf{L^\infty}$, **BV**
t regularity:	$t \to u(t)$	$\mathbf{L^1}$ – Lipschitz
u_o dependence:	$u_o \to u(t)$	$\mathbf{L^1}$ – Lipschitz

Introduction: NonLocal Conservation Laws

- NonLocal Conservation Laws:
  - Elastodynamics
    - (Chen & Christoforou: PAMS, 2007)
    - (Christoforou: JHDE, 2007)

Introduction: NonLocal Conservation Laws

- NonLocal Conservation Laws:
  - Elastodynamics
  - Granular Materials
    - (Amadori & Shen: Comm.PDE, 2009)
    - (Guerra & Shen: JDE, 2014)

Introduction: NonLocal Conservation Laws

- NonLocal Conservation Laws:
  - Elastodynamics
  - Granular Materials
  - Vehicular Traffic
    - (Colombo, Corli & Rosini: ZAMM, 2007)
    - (Colombo, Herty & Mercier: COCV, 2011)
    - (Dong & Tong: NHM, 2011)
    - (Blandin & Goatin: Numer.Math., 2015)

Introduction: NonLocal Conservation Laws

- NonLocal Conservation Laws:
  - Elastodynamics
  - Granular Materials
  - Vehicular Traffic
  - Crowd Dynamics
    - (Colombo, Garavello & Mercier: M3AS, 2011)
    - (Colombo & Mercier: Acta Math.Sc., 2011)
    - (Piccoli & Tosin: ARMA, 2011)
    - (Hoogendoorn et al.: Physica A, 2014)
    - (Goatin & Rossi: CMS, 2017)

Introduction: NonLocal Conservation Laws

- NonLocal Conservation Laws:
  - Elastodynamics
  - Granular Materials
  - Vehicular Traffic
  - Crowd Dynamics
  - Numerical Methods
    - (Betancourt, Burger, Karlsen & Tory: Nonlin., 2011)
    - (Amorim, Colombo & Teixeira: M2AN, 2015)
    - (Aggarwal, Colombo & Goatin: SINUM, 2015)

Introduction: NonLocal Conservation Laws

- NonLocal Conservation Laws:
  - Elastodynamics
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  - Crowd Dynamics
  - Numerical Methods
  - Degasperis–Procesi, Camassa–Holm & Ostrovsky–Hunter
    - (Gesztesy & Holden: Cambridge Studies, 2003)
    - (Holden & Xavier: CPDE, 2007)
    - (Coclite, Ridder & Risebro: Preprint, 2016)

Introduction: Structured Population

- NonLocal Conservation Laws:
  - Elastodynamics
  - Granular Materials
  - Vehicular Traffic
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  - Numerical Methods
  - Degasperis–Procesi, Camassa–Holm & Ostrovsky–Hunter
  - Measure Valued Conservation Laws
    - (Carrillo, Colombo, Gwiazda & Ulikowska: JDE, 2012)
    - (Gwiazda, Jamróz & Marciniak-Czochra: SIMA, 2012)
    - (Canizo, Carrillo & Cuadrado: Acta Appl.Math., 2013)
    - (Piccoli & Rossi: ARMA, 2014)
    - (Colombo, Gwiazda & Rosinska: COCV, To appear)

Introduction:

- NonLocal Conservation Laws:
  - Elastodynamics
  - Granular Materials
  - Vehicular Traffic
  - Crowd Dynamics
  - Numerical Methods
  - Degasperis–Procesi, Camassa–Holm & Ostrovsky–Hunter
  - Measure Valued Conservation Laws

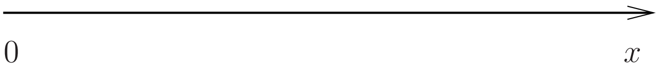
- Structured Population
  - $\approx$ 2000 items in MathSciNet (190 books)
    - (Webb: Theory of Nonlinear Age-Dependent Population Dynamics, 1985)
    - (Diekmann & Heesterbeek: Mathematical Epidemiology of Infectious Diseases, 2000)
    - (Perthame: Transport Equations in Biology, 2008)
    - (Broom & Richtář: Game-Theor. Models in Biology, 2013)

Structured Population on Graphs

(in collaboration with M.Garavello)

Role of the Graph

The standard setting:



Role of the Graph

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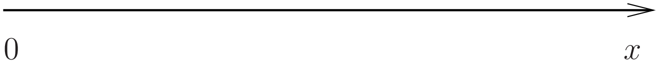
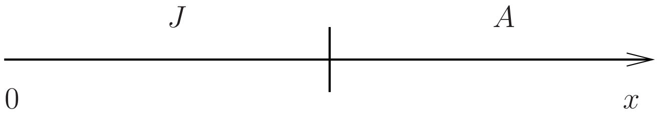


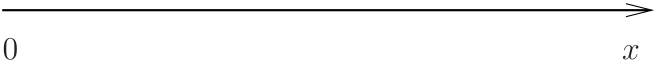
Juvenile – Adult:



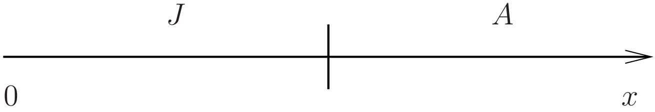
(Carrillo, Cuadrado & Perthame: Math.Biosci., 2007)
(Ackleh & Deng: SIAM Appl.Math., 2009)
(Ackleh & Ma: Numer.Funct.Anal.Opt., 2013)

Role of the Graph

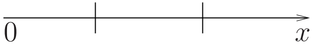
The standard setting:

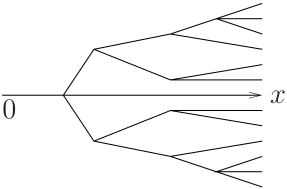


Juvenile – Adult:



Etc. . . .





A General Theorem

$$\begin{cases} \partial_t u_i + \partial_x \big(g_i(t,x)\, u_i\big) = d_i(t,x)\, u_i \\ g_i(t,0)\, u_i(t,0+) = \mathcal{B}_i\big(t, u_1(t), \ldots, u_n(t)\big) \\ u_i(0,x) = \bar{u}_i(x) \end{cases}$$

$t \in \mathbb{R}^+$	time	g_i	growth
$x \in \mathbb{R}^+$	age	$-d_i$	death
$u_i \in \mathbb{R}^+$	density	$\mathcal{B}_i$	birth/change

A General Theorem

$$\begin{cases} \partial_t u_i + \partial_x \left(g_i(t,x)\, u_i\right) = d_i(t,x)\, u_i \\ g_i(t,0)\, u_i(t,0+) = \mathcal{B}_i\left(t, u_1(t), \ldots, u_n(t)\right) \\ u_i(0,x) = \bar{u}_i(x) \end{cases}$$

$$\begin{aligned} \mathcal{B}_i(t, u_1, \ldots, u_n) = {} & \alpha_i\left(t, u_1(\bar{x}_1-), \ldots, u_n(\bar{x}_n-)\right) \\ & + \beta_i\left(\int_{I_1} w_1(x)\, u_1(x)\, \mathrm{d}x\,, \ldots, \int_{I_n} w_n(x)\, u_n(x)\, \mathrm{d}x\right) \end{aligned}$$

u_i lives on $[0, \bar{x}_i]$
u_i is fertile for $x \in I_i$
w_i fertility of u_i

α_i transmission
β_i natality

A General Theorem

$$\begin{cases} \partial_t u_i + \partial_x \left(g_i(t,x)\, u_i\right) = d_i(t,x)\, u_i \\ g_i(t,0)\, u_i(t,0+) = \mathcal{B}_i\big(t, u_1(t), \ldots, u_n(t)\big) \\ u_i(0,x) = \bar u_i(x) \end{cases}$$

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Theorem (Colombo & Garavello: MBE, 2015)

If $g_i \in \mathbf{C}^1$, $\inf g_i > 0$, $\sup_t \left[\mathrm{TV}\left(g_i(t)\right) + \mathrm{TV}\left(\partial_x g_i(t)\right)\right] < +\infty$

$d_i \in (\mathbf{C}^1 \cap \mathbf{L}^\infty)$, $\sup_t \mathrm{TV}\left(d_i(t,\cdot)\right) < +\infty$

$\alpha_i, \beta_i, w_i \in \mathbf{C}^{0,1}$, $\alpha_i(t,0) = 0$, $\beta_i(0) = 0$, $\inf_x w_i > 0$

A General Theorem

$$\begin{cases} \partial_t u_i + \partial_x \left(g_i(t,x)\, u_i\right) = d_i(t,x)\, u_i \\ g_i(t,0)\, u_i(t,0+) = \mathcal{B}_i\left(t, u_1(t), \ldots, u_n(t)\right) \\ u_i(0,x) = \bar{u}_i(x) \end{cases}$$

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$\alpha_i, \beta_i, w_i \in \mathbf{C^{0,1}}$, $\alpha_i(t,0) = 0$, $\beta_i(0) = 0$, $\inf_x w_i > 0$

Then: *Existence of a solution*

Uniqueness of the solution

Continuous dependence from the initial data

Stability with respect to α_i, β_i, w_i

A General Theorem

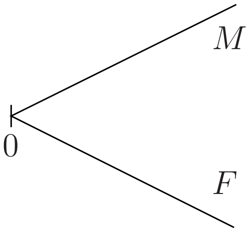
$$\begin{cases} \partial_t u_i + \partial_x \left(g_i(t,x)\, u_i\right) = d_i(t,x)\, u_i \\ g_i(t,0)\, u_i(t,0+) = \mathcal{B}_i\left(t, u_1(t), \ldots, u_n(t)\right) \\ u_i(0,x) = \bar{u}_i(x) \end{cases}$$

$$\mathcal{B}_i(t, u_1, \ldots, u_n) = \alpha_i\left(t, u_1(\bar{x}_1-), \ldots, u_n(\bar{x}_n-)\right) + \beta_i\left(\int_{I_1} w_1(x)\, u_1(x)\, \mathrm{d}x, \ldots, \int_{I_n} w_n(x)\, u_n(x)\, \mathrm{d}x\right)$$

Theorem (Colombo & Garavello: MBE, 2015)

$$\left\| u_i'(t) - u_i''(t) \right\|_{\mathbf{L}^1} \le \mathcal{K}(t) \sum_{j=1}^{n} \left[\left\| \bar{u}_j' - \bar{u}_j'' \right\|_{\mathbf{L}^1} + t \left\| \bar{u}_j' - \bar{u}_j'' \right\|_{\mathbf{L}^\infty} \right] + \mathcal{H}(t) \sum_{j=1}^{n} \left[\left\| \alpha_j' - \alpha_j'' \right\|_{\mathbf{C}^0} + \left\| \beta_j' - \beta_j'' \right\|_{\mathbf{C}^0} + \left\| w_j' - w_j'' \right\|_{\mathbf{C}^0} \right]$$

Example: Age & Sex Structured Population



M males $\quad$ F females

(N. Keyfitz: VI Berkeley Symp. Math. Stat. Prob., 1972)

Example: Age & Sex Structured Population

$$
\left\{
\begin{array}{l}
\partial_t M + \partial_a M = -\boldsymbol{\kappa}\, \boldsymbol{\mu}\, M \\
\partial_t F + \partial_a F = -(1-\boldsymbol{\kappa})\, \boldsymbol{\mu}\, F \\
\\
M(t,0) + F(t,0) = \boldsymbol{\nu} \min\left\{ \boldsymbol{\vartheta} \int_{m_1}^{m_2} M(t,a)\,\mathrm{d}a\,, (1-\boldsymbol{\vartheta}) \int_{f_1}^{f_2} F(t,a)\,\mathrm{d}a \right\} \\
\boldsymbol{\eta}\, M(t,0) = (1-\boldsymbol{\eta})\, F(t,0)
\end{array}
\right.
$$

$\boldsymbol{\eta}$ relative natality in $[0,1]$
$\boldsymbol{\kappa}$ relative mortality in $[0,1]$
$\boldsymbol{\vartheta}$ coupling habits in $[0,1]$

$\boldsymbol{\mu}$ mortality in $\mathbb{R}^+$
$\boldsymbol{\nu}$ natality in $\mathbb{R}^+$

Example: Age & Sex Structured Population

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$\kappa = 0$ no Male dies
$\kappa = 1$ no Female dies

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$\eta = 0$ no Female is born
$\eta = 1$ no Male is born

Example: Age & Sex Structured Population

$$\begin{cases} \partial_t M + \partial_a M = -\boldsymbol{\kappa}\, \boldsymbol{\mu}\, M \\ \partial_t F + \partial_a F = -(1-\boldsymbol{\kappa})\, \boldsymbol{\mu}\, F \\ M(t,0) + F(t,0) = \boldsymbol{\nu} \min\left\{ \boldsymbol{\vartheta} \int_{m_1}^{m_2} M(t,a)\, \mathrm{d}a\,, (1-\boldsymbol{\vartheta}) \int_{f_1}^{f_2} F(t,a)\, \mathrm{d}a \right\} \\ \boldsymbol{\eta}\, M(t,0) = (1-\boldsymbol{\eta})\, F(t,0) \end{cases}$$

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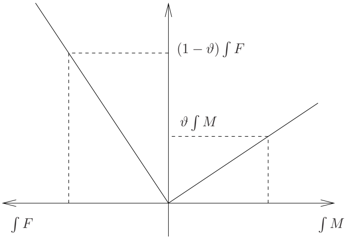

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$\boldsymbol{\eta}$	relative natality	in $[0,1]$
$\boldsymbol{\kappa}$	relative mortality	in $[0,1]$
$\boldsymbol{\vartheta}$	coupling habits	in $[0,1]$

$\boldsymbol{\mu}$	mortality	in $\mathbb{R}^+$
$\boldsymbol{\nu}$	natality	in $\mathbb{R}^+$

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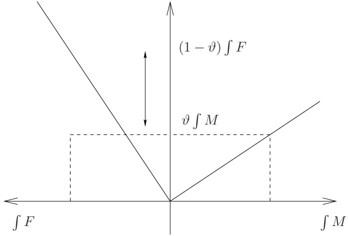
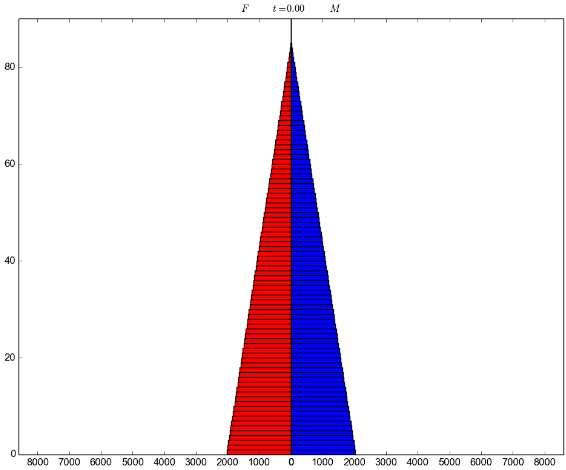


Example: Age & Sex Structured Population

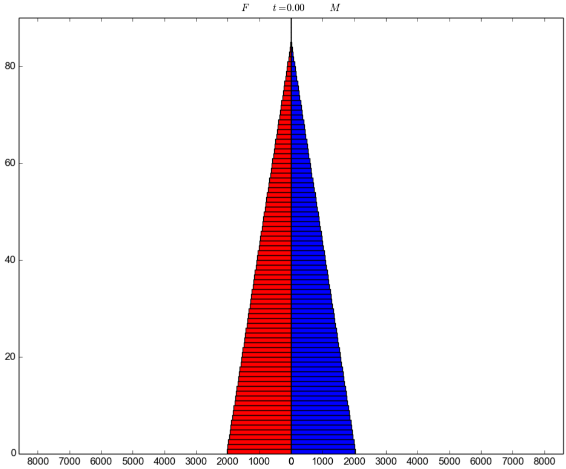
$\kappa = 0.600 \qquad \eta = 0.485 \qquad \vartheta = 0.7067$



Age pyramid - *"slowly"* growing

Example: Age & Sex Structured Population

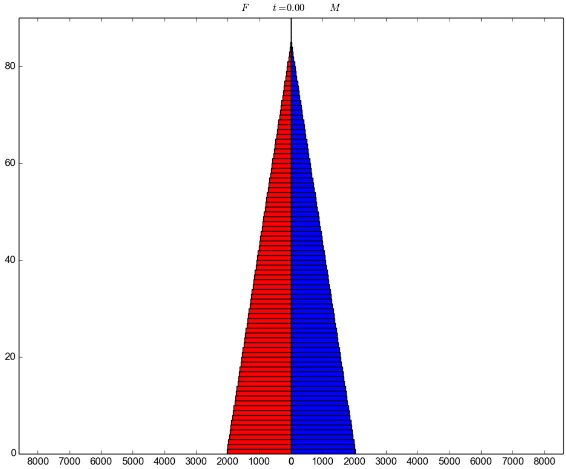
$\kappa = 0.600 \qquad \eta = 0.300 \qquad \vartheta = 0.7067$



Lower $\eta \Rightarrow$ too many M are born $\Rightarrow$ extinction

Example: Age & Sex Structured Population

$\kappa = 0.400$ $\eta = 0.485$ $\vartheta = 0.7067$



Lower $\kappa \Rightarrow$ too many F die $\Rightarrow$ extinction

Example: Age & Sex Structured Population

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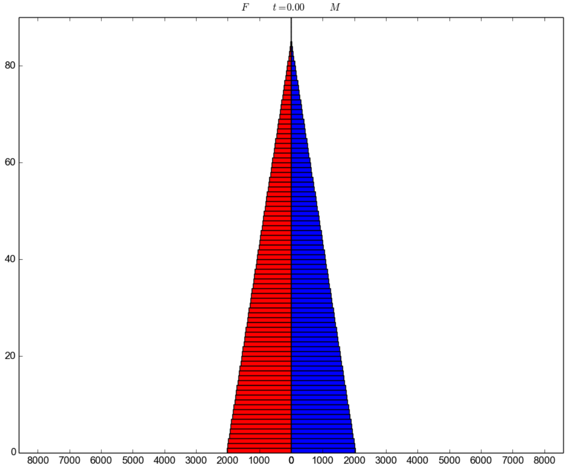
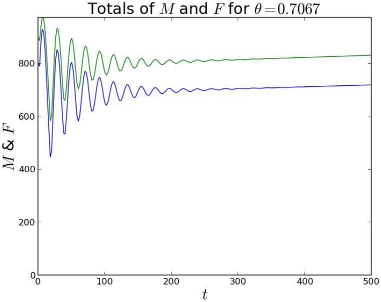
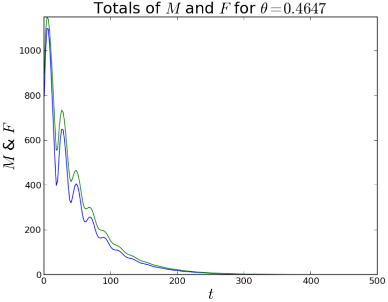
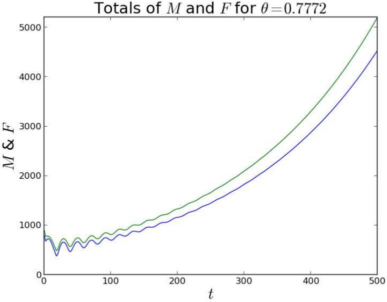


Lower $\vartheta \Rightarrow$ extinction

Example: Age & Sex Structured Population







Example: Age & Sex Structured Population

Optimal Mating Ratio: $$\vartheta(t) = \frac{\int_{f_1}^{f_2} F(t,a)\,\mathrm{d}a}{\int_{m_1}^{m_2} M(t,a)\,\mathrm{d}a + \int_{f_1}^{f_2} F(t,a)\,\mathrm{d}a}$$

Optimal Fertility Rate: $$\frac{\nu}{\frac{1}{\int_{m_1}^{m_2} M(t,a)\,\mathrm{d}a} + \frac{1}{\int_{f_1}^{f_2} F(t,a)\,\mathrm{d}a}}$$

(Colombo & Garavello: MBE, 2015)

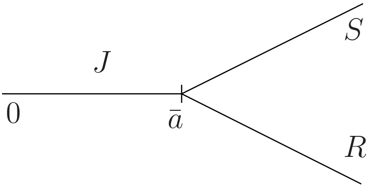
Control

$$\Longleftrightarrow$$

Management of a Biological Resource

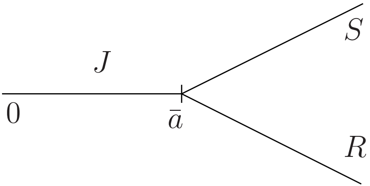
(in collaboration with M. Garavello)

Management of a Biological Resource



J	Juveniles	$a \in [0, \bar{a}]$
S	Sold	$a \in [\bar{a}, +\infty[$
R	Reproduction	$a \in [\bar{a}, a_{\max}]$

Management of a Biological Resource



$$\begin{cases} \partial_t J + \partial_a \big(g_J(t,a)\, J\big) = d_J(t,a)\, J \\ \partial_t S + \partial_a \big(g_S(t,a)\, S\big) = d_S(t,a)\, S \\ \partial_t R + \partial_a \big(g_R(t,a)\, R\big) = d_R(t,a)\, R \\ g_J(t,0)\, J(t,0) = \int w(\alpha)\, R(t,\alpha)\, \mathrm{d}\alpha \\ g_S(t,\bar a)\, S(t,\bar a) = \boldsymbol{\eta}\, g_J(t,\bar a)\, J(t,\bar a) \\ g_R(t,\bar a)\, R(t,\bar a) = (1-\boldsymbol{\eta})\; g_J(t,\bar a)\, J(t,\bar a) \end{cases}$$

J	Juveniles	$a \in [0, \bar a]$
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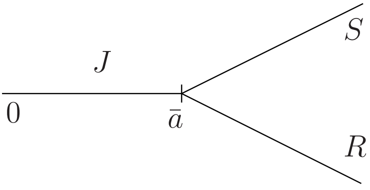
Management of a Biological Resource



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\partial_t R + \partial_a \big(g_R(t,a)\, R \big) = d_R(t,a)\, R \\[1ex]
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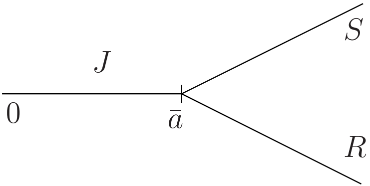
\end{cases}
$$

$$
\text{Profit} = [\text{income from } S] - [\text{costs of } J, S, R]
$$

Management of a Biological Resource



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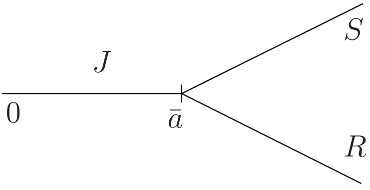
\end{cases}
$$

Profit = [income from S] − [costs of J, S, R]

Find $\boldsymbol{\eta}$ to maximize the profit

Stability estimates ⇒ existence of an optimal $\boldsymbol{\eta}$

Management of a Biological Resource



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g_S(t,\bar a)\, S(t,\bar a) = \eta\, g_J(t,\bar a)\, J(t,\bar a) \\
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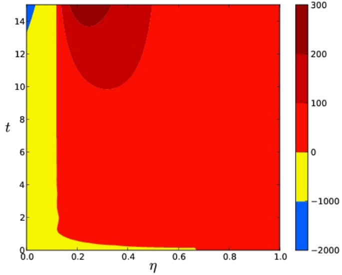
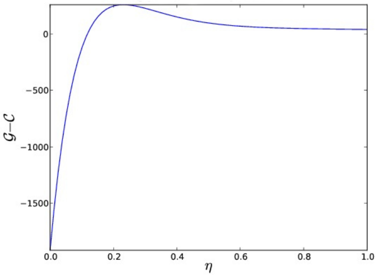
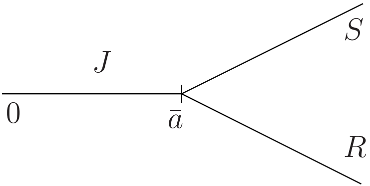
\end{cases}
$$





(Colombo & Garavello: MBE, 2015)

Management of a Biological Resource



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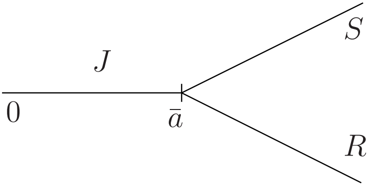
Profit = [income from S] − [costs of J, S, R]

Find $\boldsymbol{\eta}$ to maximize the profit

Stability estimates ⇒ existence of an optimal $\boldsymbol{\eta}$

The Profit is Differentiable w.r.t $\boldsymbol{\eta}$!

Management of a Biological Resource



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\begin{cases}
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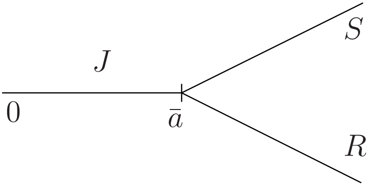
\end{cases}
$$

Management of a Biological Resource



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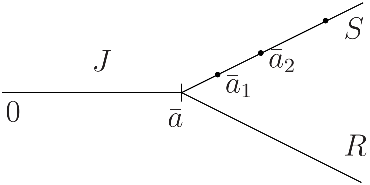
\end{cases}$$

$$\mathcal{C}_J = \iint C_J \left(t, a, J(t,a)\right) \mathrm{d}a\, \mathrm{d}t$$
$$\mathcal{C}_S = \iint C_S \left(t, a, S(t,a)\right) \mathrm{d}a\, \mathrm{d}t$$
$$\mathcal{C}_R = \iint C_R \left(t, a, R(t,a)\right) \mathrm{d}a\, \mathrm{d}t$$

Management of a Biological Resource



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S(t,\bar a_i+) = \big(1 - \boldsymbol{\vartheta_i}(t)\big)\, S(t,\bar a_i-)
\end{cases}
$$

$$\mathcal{C}_J = \iint C_J\,\big(t,a,J(t,a)\big)\,\mathrm{d}a\,\mathrm{d}t$$

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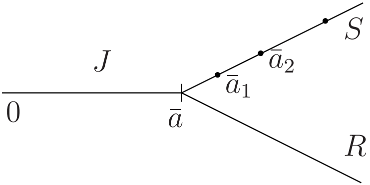
$$\mathcal{C}_R = \iint C_R\,\big(t,a,R(t,a)\big)\,\mathrm{d}a\,\mathrm{d}t$$

$$\mathcal{I} = \int \sum_i \boldsymbol{\vartheta_i(t)}\, P_i(t)\, S(t,\bar a_i-)\,\mathrm{d}t$$

Management of a Biological Resource



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g_R(t,\bar a)\, R(t,\bar a) = \big(1 - \boldsymbol{\eta(t)}\big)\, g_J(t,\bar a)\, J(t,\bar a) \\
S(t,\bar a_i +) = \big(1 - \boldsymbol{\vartheta_i}(t)\big)\, S(t,\bar a_i -)
\end{cases}$$

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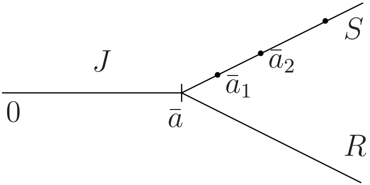
$$\mathcal{I} = \int \sum_i \boldsymbol{\vartheta_i(t)}\, P_i(t)\, S(t,\bar a_i -)\, \mathrm{d}t$$

$$\mathcal{P} = \mathcal{I} - (\mathcal{C}_J + \mathcal{C}_S + \mathcal{C}_R)$$

Management of a Biological Resource



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$$\mathcal{I} = \int \sum_i \boldsymbol{\vartheta}_{\mathbf{i}}(\mathbf{t})\, P_i(t)\, S(t,\bar a_i -)\,\mathrm{d}t$$

$$\mathcal{P} = \mathcal{I} - (\mathcal{C}_J + \mathcal{C}_S + \mathcal{C}_R)$$

$\mathcal{P}$ is Gateaux differentiable w.r.t. $\boldsymbol{\eta}$ and $\boldsymbol{\vartheta}$

(Colombo & Garavello: ESAIM – COCV, 2017)

Management of a Biological Resource

How to find the optimal control?
maximal profit?

Management of a Biological Resource

How to find the optimal control? / maximal profit?

Gateaux differentiability $\Rightarrow$ gradient methods

Management of a Biological Resource

How to find the optimal control? / maximal profit?

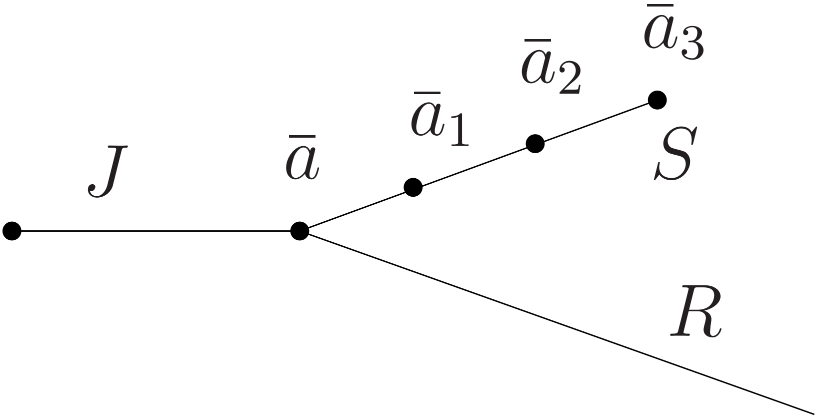
Bang–Bang controls

Theorem (Colombo & Garavello: ESAIM – COCV, 2017)

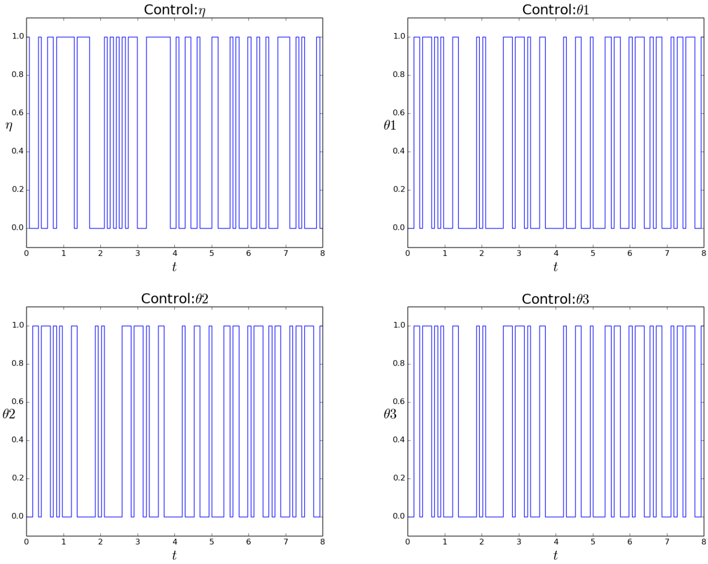
For any $\varepsilon > 0$ *there exists a bang–bang control* $(\eta_\varepsilon, \vartheta_\varepsilon)$ *such that*

$$\mathcal{P}(\eta_\varepsilon, \vartheta_\varepsilon) \geq \sup \mathcal{P}(\eta, \vartheta) - \varepsilon$$

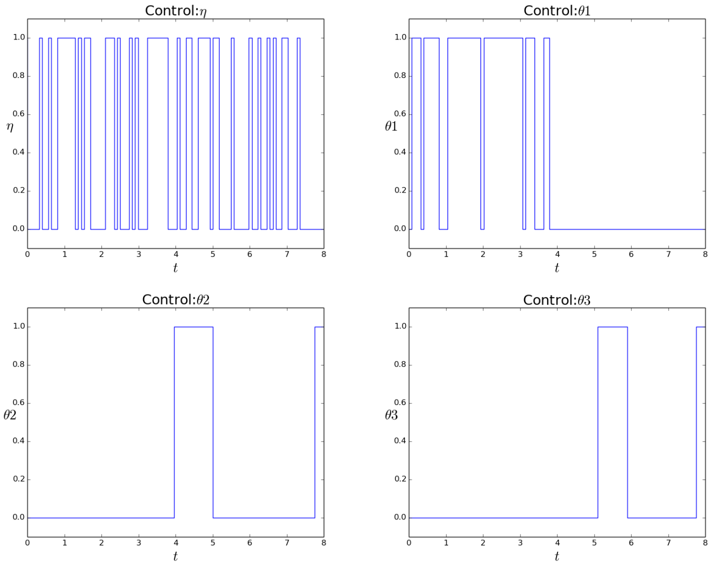
Management of a Biological Resource



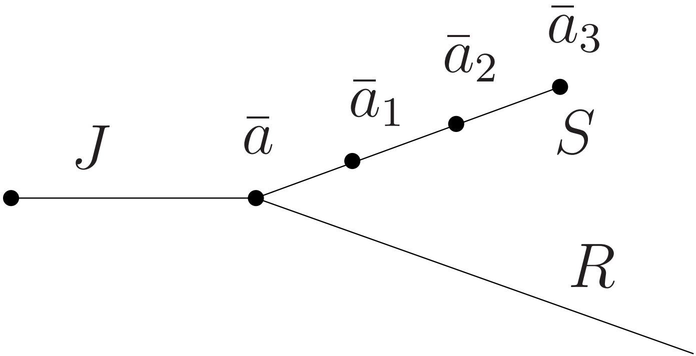
Management of a Biological Resource



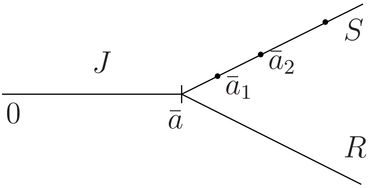
Management of a Biological Resource



Management of a Biological Resource



Management of a Biological Resource



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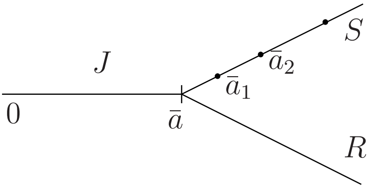
Theorem (Colombo & Garavello: Nonlinear An. RWA, 2017)

If $\eta(t) = \sum_i \eta_i\, \chi_{[\tau_{i-1},\tau_i[}$ *and* $\vartheta(t) = \sum_i \vartheta_i\, \chi_{[\tau_{i-1},\tau_i[}$, *then*

Management of a Biological Resource



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$\mathcal{P}$ *is a* **polynomial** *in* $\eta_1, \eta_2, \ldots, \vartheta_1, \vartheta_2, \ldots$

(+ bounds on the degree)

Management of a Biological Resource – Example

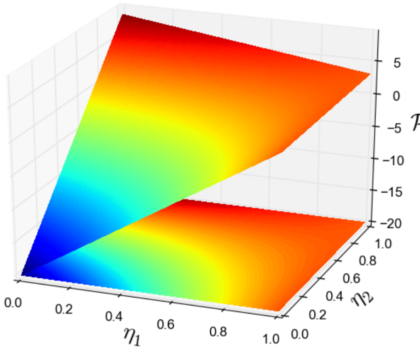
$$
\begin{array}{cccc}
g_J(t,a) = 1.00 & d_J(t,a) = 1.50 & c_J(t,a) = 0.25 & J_o(a) = 1.00 \\
g_S(t,a) = 1.00 & d_S(t,a) = 0.50 & c_S(t,a) = 0.00 & S_o(a) = 0.00 \\
g_r(t,a) = 1.00 & d_R(t,a) = 0.75 & c_R(t,a) = 0.00 & R_o(a) = 0.00 \\
\bar{a} = 1.00 & \bar{a}_1 = 1.50 & N = 1 & \\
p(a) = 0.00 & p_1(t) = 8.00 & w(a) = 120.00\, \chi_{[1.00,\,4.00]}(a) &
\end{array}
$$

Management of a Biological Resource – Example

$$g_J(t,a) = 1.00 \quad d_J(t,a) = 1.50 \quad c_J(t,a) = 0.25 \quad J_o(a) = 1.00$$
$$g_S(t,a) = 1.00 \quad d_S(t,a) = 0.50 \quad c_S(t,a) = 0.00 \quad S_o(a) = 0.00$$
$$g_r(t,a) = 1.00 \quad d_R(t,a) = 0.75 \quad c_R(t,a) = 0.00 \quad R_o(a) = 0.00$$
$$\bar{a} = 1.00 \quad \bar{a}_1 = 1.50 \quad N = 1$$
$$p(a) = 0.00 \quad p_1(t) = 8.00 \quad w(a) = 120.00\, \chi_{[1.00,\, 4.00]}(a)$$

$$\eta = \eta_1\, \chi_{[0,1]} + \eta_2\, \chi_{[1,2]}$$

$$\mathcal{P} = -19.97 + 23.10\, \eta_1 + 28.18\, \eta_2 - 28.18\, \eta_1\, \eta_2$$

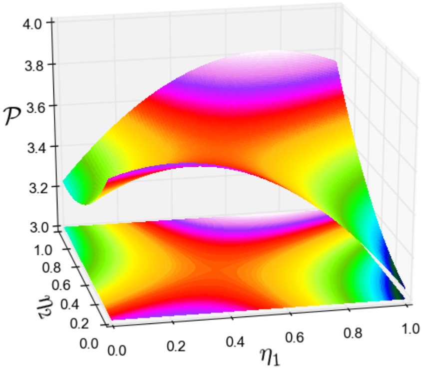


Management of a Biological Resource – Example

$$\bar a = 1.00 \quad d_J(t,a) = 0.50 \quad c_J(t,a) = 0.25 \quad p(a) = 1.00 \quad J_o(a) = 1.00$$
$$N = 1 \quad d_S(t,a) = 1.00 \quad c_S(t,a) = 0.25 \quad p_1(t) = 8.20 \quad S_o(a) = 0.00$$
$$\bar a_1 = 1.50 \quad d_R(t,a) = 1.50 \quad c_R(t,a) = 0.25 \quad w(a) = 10.00 \quad R_o(a) = 0.00$$

$$\eta = \eta_1\,\chi_{[0,0.5]} + \eta_2\,\chi_{[0.5,1]} + \eta_1\,\chi_{[1,1.5]} + \eta_2\,\chi_{[1.5,2]}$$

$$\mathcal{P}(\eta_1,\eta_2) = 3.65 + 0.46\,\eta_1 - 0.88\,\eta_2 + 1.11\,\eta_1\,\eta_2 - 1.06\,{\eta_1}^2 + 0.46\,{\eta_2}^2$$



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