Controllability of the beam equation

Cristina Urbani

Joint work with P. Cannarsa

GSSI - Gran Sasso Science Institute

VII Partial differential equations, optimal design and numerics

20/08-01/09 Centro de Ciencias de Benasque Pedro Pascual

Outline

- Bilinear control problems

Outline

- Bilinear control problems
- Model

Outline

- Bilinear control problems
- Model
- Previous results

Outline

- Bilinear control problems
- Model
- Previous results
- Controllability of the beam equation

Bilinear control problems, introduction

Dynamical system:

Bilinear control problems, introduction

Dynamical system:

$$\dot{y} = f(y, u)$$

Bilinear control problems, introduction

Dynamical system:

$$\dot{y} = f(y, u)$$

- $y \in Y$ state of the system
- $u \in \mathcal{U}$ control

$$\dot{y} = Ay + uBy$$

Dynamical system:

$$\dot{y} = f(y, u)$$

- $y \in Y$ state of the system
- $u \in \mathcal{U}$ control

$$\dot{y} = Ay + uBy$$

- Problem: local controllability along a trajectory
  Let $(\bar{y}, \bar{u})$ be a trajectory of the control system $\dot{y} = f(y, u)$.
  The control system is *locally controllable along the trajectory* $(\bar{y}, \bar{u})$ if, for every $\varepsilon > 0$, there exists $\nu > 0$ such that, for every $(a, b) \in Y \times Y$ with $|a - \bar{y}(0)| < \nu$ and $|b - \bar{y}(T)| < \nu$, there exists a trajectory (y, u) such that

$$y(0) = a, \quad y(T) = b,$$

$$|u(t) - \bar{u}(t)| \leq \varepsilon, \quad t \in [0, T]$$

Methods

Non linear equations

Methods

Non linear equations

- trajectory $\rightarrow$ linearized system

Methods

Non linear equations

- trajectory $\rightarrow$ linearized system
  - controllability of linearized system $\rightarrow$ inversion theorem

K. Beauchard, 2008 $\Rightarrow$ Nash - Moser Theorem

Non linear equations

- trajectory $\rightarrow$ linearized system
  - controllability of linearized system $\rightarrow$ inversion theorem

K. Beauchard, 2008 $\Rightarrow$ Nash - Moser Theorem

- linearized system $\rightarrow$ moment theory (Ingham inequality)

Beam bending, model

Euler-Bernoulli model, 1750.

Beam bending, model

Euler-Bernoulli model, 1750.

Assumptions:

Beam bending, model

Euler-Bernoulli model, 1750.

Assumptions:

- linear elasticity of the material, Hooke's law

Euler-Bernoulli model, 1750.

Assumptions:

- linear elasticity of the material, Hooke's law
- plane sections remain plane and perpendicular to the neutral axis (transverse vibrations).

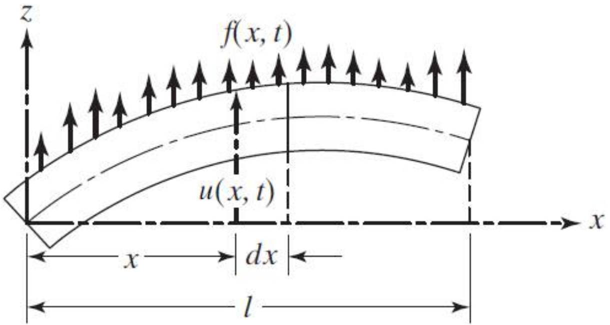

Figure: A beam in bending

If $u(t,x)$ is the beam deflection, the equation for the bending is

$$\rho A(x)\frac{\partial^2 u(t,x)}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 u(t,x)}{\partial x^2}\right]=f(t,x)$$

where ρ density per unit length, $A(x)$ cross-sectional area, E Young elastic modulus, $I(x)$ cross-sectional area moment of inertia (about "z axis"), $f(t,x)$ total external force.

Beam equation, previous results

- J. M. Ball, J. E. Marsdent, M, Slemrod, "*Controllability for distributed bilinear systems*", 1982 $\rightarrow$ the beam equation is not controllable in $H_0^2((0,1),\mathbb{R}) \times L^2((0,1),\mathbb{R})$, with control p in $L^2_{loc}([0,+\infty),\mathbb{R})$

Beam equation, previous results

- J. M. Ball, J. E. Marsdent, M, Slemrod, "*Controllability for distributed bilinear systems*", 1982 → the beam equation is not controllable in $H_0^2((0,1),\mathbb{R}) \times L^2((0,1),\mathbb{R})$, with control p in $L^2_{loc}([0,+\infty),\mathbb{R})$
- K. Beauchard, "*Local controllability of one-dimentional beam equation*", 2008 → local controllability in $H^{5+\varepsilon}_{(0)}((0,1),\mathbb{R}) \times H^{3+\varepsilon}_{(0)}((0,1),\mathbb{R})$, with $\varepsilon > 0$ and control p in $H^1_{loc}(\mathbb{R}_+,\mathbb{R})$.

Beam equation with pinned ends, controllability

Control system

$$\begin{cases} u_{tt} + u_{xxxx} + p(t)\mu(x)u_{xx} = 0, & (t,x) \in \mathbb{R}_+ \times (0,1), \\ u(t,0) = u(t,1) = u_{xx}(t,0) = u_{xx}(t,1) = 0. \end{cases} \quad (1)$$

$\mu(x)$ is the effect of an axial force.

Beam equation with pinned ends, controllability

Control system

$$\begin{cases} u_{tt} + u_{xxxx} + p(t)\mu(x)u_{xx} = 0, & (t,x) \in \mathbb{R}_+ \times (0,1), \\ u(t,0) = u(t,1) = u_{xx}(t,0) = u_{xx}(t,1) = 0. \end{cases} \tag{1}$$

$\mu(x)$ is the effect of an axial force.
We introduce the operator A defined by

$$D(A) := H^4 \cap H_0^2((0,1),\mathbb{R}), \qquad Av := \frac{d^4 v}{dx^4}.$$

Beam equation with pinned ends, controllability

Control system

$$\begin{cases} u_{tt} + u_{xxxx} + p(t)\mu(x)u_{xx} = 0, & (t,x) \in \mathbb{R}_+ \times (0,1), \\ u(t,0) = u(t,1) = u_{xx}(t,0) = u_{xx}(t,1) = 0. \end{cases} \tag{1}$$

$\mu(x)$ is the effect of an axial force.
We introduce the operator A defined by

$$D(A) := H^4 \cap H_0^2((0,1),\mathbb{R}), \qquad Av := \frac{d^4 v}{dx^4}.$$

whose eigenvalues and eigenvectors are

$$\lambda_k^2 = \mu_k = (k\pi)^4, \qquad \varphi_k = \sqrt{2}\sin(k\pi x), \qquad \forall k \in \mathbb{N}^*.$$

Beam equation with pinned ends, controllability

Control system

$$\begin{cases} u_{tt} + u_{xxxx} + p(t)\mu(x)u_{xx} = 0, & (t,x) \in \mathbb{R}_+ \times (0,1), \\ u(t,0) = u(t,1) = u_{xx}(t,0) = u_{xx}(t,1) = 0. \end{cases} \quad (1)$$

$\mu(x)$ is the effect of an axial force.
We introduce the operator A defined by

$$D(A) := H^4 \cap H_0^2((0,1),\mathbb{R}), \qquad Av := \frac{d^4 v}{dx^4}.$$

whose eigenvalues and eigenvectors are

$$\lambda_k^2 = \mu_k = (k\pi)^4, \qquad \varphi_k = \sqrt{2}\sin(k\pi x), \qquad \forall k \in \mathbb{N}^*.$$

$\psi_k(t,x) = \varphi_k(x)e^{-i\lambda_k t}$ are solutions of (1) with control $p \equiv 0$.

Beam equation with pinned ends, controllability

The system

$$\begin{cases} u_{tt} + u_{xxxx} + p(t)\mu(x)u_{xx} + f(t) = 0, & (t,x) \in \mathbb{R}_+ \times (0,1), \\ u(t,0) = u(t,1) = u_{xx}(t,0) = u_{xx}(t,1) = 0, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x) \end{cases} \tag{2}$$

can be transformed into the Cauchy problem

$$\begin{cases} \frac{d\psi}{dt} = -\mathcal{A}\psi - p(t)\mu(x)\mathcal{B}\psi + F(t), \\ \psi(0) = \psi_0 \end{cases} \tag{3}$$

The system

$$\begin{cases} u_{tt} + u_{xxxx} + p(t)\mu(x)u_{xx} + f(t) = 0, & (t,x) \in \mathbb{R}_+ \times (0,1), \\ u(t,0) = u(t,1) = u_{xx}(t,0) = u_{xx}(t,1) = 0, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x) \end{cases} \tag{2}$$

can be transformed into the Cauchy problem

$$\begin{cases} \frac{d\psi}{dt} = -\mathcal{A}\psi - p(t)\mu(x)\mathcal{B}\psi + F(t), \\ \psi(0) = \psi_0 \end{cases} \tag{3}$$

where the linear operators $\mathcal{A}$ e $\mathcal{B}$ are defined as follows

$$D(\mathcal{A}) := H^4_{(0)} \times H^2_0((0,1),\mathbb{R}), \qquad D(\mathcal{B}) := H^2_0 \times L^2((0,1),\mathbb{R})$$

$$\mathcal{A}\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} := \begin{pmatrix} -\psi^2 \\ \psi^1_{xxxx} \end{pmatrix}, \qquad \mathcal{B}\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} := \begin{pmatrix} 0 \\ \psi^1_{xx} \end{pmatrix}$$

and $F : (0,T) \to H^2_0 \times L^2((0,1),\mathbb{R})$.

By choosing $\psi = (u, u_t)$, $\psi_0 = (u_0, u_1)$, and $F = (0, -f)$ the problems (2) e (3) are equivalent.

By choosing $\psi = (u, u_t)$, $\psi_0 = (u_0, u_1)$, and $F = (0, -f)$ the problems (2) e (3) are equivalent.

Let us consider as solution of the homogeneus problem

$$\hat{\psi}_1(t,x) = (\psi_1(t,x), (\psi_1(t,x))_t)$$

and initial datum

$$\psi(0,x) = \hat{\varphi}_1(x) = (\varphi_1(x), \lambda_1 \varphi_1(x))$$

.

Beam equation with pinned ends, controllability

Theorem

Let $T > 0$ *and* $\mu \in H^3((0,1),\mathbb{R})$ *such that*

$$\exists c > 0 \quad \text{such that} \quad |\langle \mu(\varphi_1)_{xx}, \varphi_k \rangle| \geq \frac{c}{k^3}, \quad \forall k \in \mathbb{N}^*. \qquad (4)$$

There exists $\delta > 0$ *and a* C^1 *map*

$$\Gamma : \mathcal{V}_T \to L^2((0,T),\mathbb{R}),$$

$$\mathcal{V}_T := \{\Psi_f \in \mathcal{S}^2 \cap H^5_{(0)} \times H^3_{(0)}((0,1),\mathbb{C}); ||\Psi_f - \hat{\psi}_1(T)||_{H^5_{(0)} \times H^3_{(0)}} < \delta\}$$

such that, $\Gamma(\hat{\psi}_1(T)) = 0$ *and for all* $\Psi_f \in \mathcal{V}_T$ *the solution of (3), with* $\psi = (u, u_t)$, $\psi_0 = (u_0, u_1)$, *initial condition*

$$\psi_0 = \hat{\varphi}_1 \qquad (5)$$

and control $p = \Gamma(\Psi_f)$, *satisfies* $\psi(T) = \Psi_f$.

Sketch of the proof:

- proof of existence, uniqueness and $C^0([0,T], H_0^2 \times L^2((0,1),\mathbb{R}))$ regularity of the weak solution of

$$\begin{cases} \frac{d\psi}{dt} = -\mathcal{A}\psi - p(t)\mu(x)\mathcal{B}\psi + F(t), \\ \psi(0) = \psi_0, \end{cases} \tag{6}$$

Sketch of the proof:

- proof of existence, uniqueness and $C^0([0,T], H_0^2 \times L^2((0,1),\mathbb{R}))$ regularity of the weak solution of

$$\begin{cases} \frac{d\psi}{dt} = -\mathcal{A}\psi - p(t)\mu(x)\mathcal{B}\psi + F(t), \\ \psi(0) = \psi_0, \end{cases} \tag{6}$$

- definition of

$$V_T := \{\xi = (\xi^1, \xi^2) \in L^2 \times L^2(0,1); \Im\langle i\lambda_1\xi^1 + \xi^2, \psi_1(T)\rangle = 0\}$$

and the othogonal projection onto V_T

$$P_T : L^2 \times L^2((0,T),\mathbb{R}) \to V_T.$$

Beam equation with pinned ends, controllability

- definition of the end point map

$$\begin{aligned}\Theta_T : L^2((0,T),\mathbb{R}) &\to V_T \cap H_0^2 \times L^2(0,1)\\ u &\mapsto P_T[\psi(T)],\end{aligned}$$

Beam equation with pinned ends, controllability

- definition of the end point map

$$\begin{aligned}\Theta_T : L^2((0,T),\mathbb{R}) &\to V_T \cap H_0^2 \times L^2(0,1)\\ u &\mapsto P_T[\psi(T)],\end{aligned}$$

- proof of C^1 regularity of Θ_T,
  $d\Theta_T(p)\cdot q = P_T(\Psi(T))$ where Ψ is solution of the linearized system,

$$\begin{cases} \Psi_t = -\mathcal{A}\Psi - p(t)\mu(x)\mathcal{B}\Psi - q(t)\mu(x)\mathcal{B}\psi,\\ \Psi(t,0) = \Psi(t,1) = 0, \Psi(0,x) = 0 \end{cases}$$

Beam equation with pinned ends, controllability

- definition of the end point map

$$\Theta_T : L^2((0,T),\mathbb{R}) \to V_T \cap H_0^2 \times L^2(0,1)$$
$$u \mapsto P_T[\psi(T)],$$

- proof of C^1 regularity of Θ_T,
  $d\Theta_T(p)\cdot q = P_T(\Psi(T))$ where Ψ is solution of the linearized system,

$$\begin{cases} \Psi_t = -\mathcal{A}\Psi - p(t)\mu(x)\mathcal{B}\Psi - q(t)\mu(x)\mathcal{B}\psi, \\ \Psi(t,0) = \Psi(t,1) = 0, \Psi(0,x) = 0 \end{cases}$$

- proof of the existence of the map

$$d\Theta_T(0)^{-1} : V_T \cap H_0^2 \times L^2(0,1) \to L^2((0,T),\mathbb{R})$$

and its C^0 regularity (controllability of the linearized system)

Beam equation with pinned ends, controllability

- definition of the end point map

$$\begin{aligned}\Theta_T : L^2((0,T),\mathbb{R}) &\to V_T \cap H_0^2 \times L^2(0,1)\\ u &\mapsto P_T[\psi(T)],\end{aligned}$$

- proof of C^1 regularity of Θ_T,
  $d\Theta_T(p)\cdot q = P_T(\Psi(T))$ where Ψ is solution of the linearized system,

$$\begin{cases} \Psi_t = -\mathcal{A}\Psi - p(t)\mu(x)\mathcal{B}\Psi - q(t)\mu(x)\mathcal{B}\psi, \\ \Psi(t,0) = \Psi(t,1) = 0, \Psi(0,x) = 0 \end{cases}$$

- proof of the existence of the map

$$d\Theta_T(0)^{-1} : V_T \cap H_0^2 \times L^2(0,1) \to L^2((0,T),\mathbb{R})$$

  and its C^0 regularity (controllability of the linearized system)
  - moment problem $\to$ (Corollary of) Ingham inequality

Moment problem to solve

$$-\int_0^T q(s)e^{-i(\lambda_k-\lambda_1)s}ds = d_{k-1}(\Psi_f) :=$$
$$= \left(i\frac{\langle \Psi_f^1(x), \varphi_k(x)\rangle}{\langle \mu(x)(\varphi_1(x))_{xx}, \varphi_k(x)\rangle}\lambda_k + \frac{\langle \Psi_f^2(x), \varphi_k(x)\rangle}{\langle \mu(x)(\varphi_1(x))_{xx}, \varphi_k(x)\rangle}\right) e^{-i\lambda_k T}.$$

Moment problem to solve

$$-\int_0^T q(s)e^{-i(\lambda_k-\lambda_1)s}ds = d_{k-1}(\Psi_f) :=$$

$$= \left(i\frac{\langle \Psi_f^1(x), \varphi_k(x)\rangle}{\langle \mu(x)(\varphi_1(x))_{xx}, \varphi_k(x)\rangle}\lambda_k + \frac{\langle \Psi_f^2(x), \varphi_k(x)\rangle}{\langle \mu(x)(\varphi_1(x))_{xx}, \varphi_k(x)\rangle}\right) e^{-i\lambda_k T}.$$

We need to show that given

$$\Psi_f = (\Psi_f^1, \Psi_f^2) \in V_T \cap H_0^2 \times L^2(0,1),$$

there exists

$$q \in L^2((0,T),\mathbb{R})$$

that satisfies the moment problem.

Beam equation with pinned ends, controllability

Corollary (of Ingham Theorem)

Let $T > 0$ *and* $(\omega_k)_{k\in\mathbb{N}}$ *an increasing sequence in* $[0,+\infty)$ *such that* $\omega_0 = 0$, *and*

$$\omega_{k+1} - \omega_k \to +\infty \qquad \text{when } k \to +\infty.$$

There exist a linear and continuous map

$$L : l^2_r(\mathbb{N},\mathbb{C}) \to L^2((0,T),\mathbb{R}),$$
$$d \mapsto L(d)$$

such that, for all $d = (d_k)_{k\in\mathbb{N}} \in l^2_r(\mathbb{N},\mathbb{C})$, *the function* $v := L(d)$ *solves*

$$\int_0^T v(t)e^{i\omega_k t}dt = d_k, \qquad \forall k \in \mathbb{N}.$$

Beam equation with pinned ends, controllability

We should ensure that

- $d_0 \in \mathbb{R}$:

$$d_0 = \frac{\langle i\lambda_1 \Psi_f^1 + \Psi_f^2, \varphi_1 \rangle}{\langle \mu(x)(\varphi_1)_{xx}, \varphi_1 \rangle} e^{-i\lambda_1 T} \in \mathbb{R}$$

Beam equation with pinned ends, controllability

We should ensure that

- $d_0 \in \mathbb{R}$:

$$d_0 = \frac{\langle i\lambda_1 \Psi_f^1 + \Psi_f^2, \varphi_1 \rangle}{\langle \mu(x)(\varphi_1)_{xx}, \varphi_1 \rangle} e^{-i\lambda_1 T} \in \mathbb{R}$$

- $(d_k)_{k\in\mathbb{N}} \in l^2(\mathbb{N}, \mathbb{C})$:

We should ensure that

- $d_0 \in \mathbb{R}$:

$$d_0 = \frac{\langle i\lambda_1 \Psi_f^1 + \Psi_f^2, \varphi_1 \rangle}{\langle \mu(x)(\varphi_1)_{xx}, \varphi_1 \rangle} e^{-i\lambda_1 T} \in \mathbb{R}$$

- $(d_k)_{k\in\mathbb{N}} \in l^2(\mathbb{N}, \mathbb{C})$:

$$\sum_{k=1}^{\infty} |d_k(\Psi_f)|^2 =$$

$$= \sum_{k=1}^{\infty} \left| \left(i \frac{\langle \Psi_f^1(x), \varphi_k(x) \rangle}{\langle (\mu(x)\varphi_1(x))_{xx}, \varphi_k(x) \rangle} \lambda_k + \frac{\langle \Psi_f^2(x), \varphi_k(x) \rangle}{\langle (\mu(x)\varphi_1(x))_{xx}, \varphi_k(x) \rangle} \right) e^{-i\lambda_k T} \right|^2$$

$$\leq 2 \sum_{k=1}^{\infty} \left| \frac{\langle \Psi_f^1(x), \varphi_k(x) \rangle}{\langle (\mu(x)\varphi_1(x))_{xx}, \varphi_k(x) \rangle} k^2 \pi^2 \right|^2 + \left| \frac{\langle \Psi_f^2(x), \varphi_k(x) \rangle}{\langle (\mu(x)\varphi_1(x))_{xx}, \varphi_k(x) \rangle} \right|^2$$

$$\leq 2 \sum_{k=1}^{\infty} \left| \frac{|\langle \Psi_f^1(x), \varphi_k(x) \rangle|}{c} k^5 \pi^2 \right|^2 + \left| \frac{|\langle \Psi_f^2(x), \varphi_k(x) \rangle|}{c} k^3 \right|^2$$

$$= 2 \sum_{k=1}^{\infty} \frac{|\langle k^5 \Psi_f^1(x), \varphi_k(x) \rangle|^2}{c} \pi^4 + \frac{|\langle k^3 \Psi_f^2(x), \varphi_k(x) \rangle|^2}{c}$$

Recall: the space

$$H^s_{(0)}(I,\mathbb{C})$$

is equipped with the norm

$$||\phi||_{H^s_{(0)}} = \sum_{k=1}^{\infty} |k^s \langle \phi, \varphi_k \rangle|^2.$$

Recall: the space

$$H^s_{(0)}(I,\mathbb{C})$$

is equipped with the norm

$$||\phi||_{H^s_{(0)}} = \sum_{k=1}^{\infty} |k^s \langle \phi, \varphi_k \rangle|^2.$$

We can conclude the proof defining

$$\Gamma(\Psi_f) =: \Theta_T^{-1}[P_T \Psi_f].$$

Recall: the space

$$H^s_{(0)}(I, \mathbb{C})$$

is equipped with the norm

$$||\phi||_{H^s_{(0)}} = \sum_{k=1}^{\infty} |k^s \langle \phi, \varphi_k \rangle|^2.$$

We can conclude the proof defining

$$\Gamma(\Psi_f) =: \Theta_T^{-1}[P_T \Psi_f].$$

Therefore the solution of the control system with $p = \Gamma(\Psi_f)$ satisfies

$$\psi(T) = P_T(\psi(T)) + \sqrt{1 - ||P_T \psi(T)||^2_{L^2 \times L^2}} \tilde{\psi}(T) =$$

$$= P_T(\Psi_f) + \sqrt{1 - ||P_T \Psi_f||^2_{L^2 \times L^2}} \tilde{\psi}(T) = \Psi_f$$

Beam equation with sliding ends, controllability

Let us consider the system

$$\begin{cases} u_{tt} + u_{xxxx} + p(t)\mu(x)u_{xx} = 0, & (t,x) \in \mathbb{R}_+ \times (0,1), \\ u_x(t,0) = u_x(t,1) = u_{xxx}(t,0) = u_{xxx}(t,1) = 0. \end{cases} \quad (7)$$

Beam equation with sliding ends, controllability

Let us consider the system

$$\begin{cases} u_{tt} + u_{xxxx} + p(t)\mu(x)u_{xx} = 0, & (t,x) \in \mathbb{R}_+ \times (0,1), \\ u_x(t,0) = u_x(t,1) = u_{xxx}(t,0) = u_{xxx}(t,1) = 0. \end{cases} \tag{7}$$

We introduce the operator A defined by

$$D(A) := H^4 \cap H_0^2((0,1),\mathbb{R}), \qquad Av := \frac{d^4 v}{dx^4}.$$

Let us consider the system

$$\begin{cases} u_{tt} + u_{xxxx} + p(t)\mu(x)u_{xx} = 0, & (t,x) \in \mathbb{R}_+ \times (0,1), \\ u_x(t,0) = u_x(t,1) = u_{xxx}(t,0) = u_{xxx}(t,1) = 0. \end{cases} \tag{7}$$

We introduce the operator A defined by

$$D(A) := H^4 \cap H_0^2((0,1),\mathbb{R}), \qquad Av := \frac{d^4 v}{dx^4}.$$

whose eigenvalues and eigenvectors are

$$\mu_k = (k\pi)^4, \qquad \varphi_k = \sqrt{2}\cos(k\pi x), \qquad \forall k \in \mathbb{N}^*.$$

Let us consider the system

$$\begin{cases} u_{tt} + u_{xxxx} + p(t)\mu(x)u_{xx} = 0, & (t,x) \in \mathbb{R}_+ \times (0,1), \\ u_x(t,0) = u_x(t,1) = u_{xxx}(t,0) = u_{xxx}(t,1) = 0. \end{cases} \tag{7}$$

We introduce the operator A defined by

$$D(A) := H^4 \cap H_0^2((0,1),\mathbb{R}), \qquad Av := \frac{d^4 v}{dx^4}.$$

whose eigenvalues and eigenvectors are

$$\mu_k = (k\pi)^4, \qquad \varphi_k = \sqrt{2}\cos(k\pi x), \qquad \forall k \in \mathbb{N}^*.$$

Again the system can be transformed into

$$\begin{cases} \frac{d\psi}{dt} = -\mathcal{A}\psi - p(t)\mu(x)\mathcal{B}\psi + F(t), \\ \psi(0) = \psi_0 \end{cases}$$

Beam equation with sliding ends, controllability

Theorem

Let $T > 0$ *and* $\mu \in H^3((0,1),\mathbb{R})$ *such that*

$$\exists c > 0 \quad \textit{tale che} \quad |\langle \mu(\varphi_1)_{xx}, \varphi_k \rangle| \geq \frac{c}{k^2}, \quad \forall k \in \mathbb{N}^*.$$

There exists $\Gamma > 0$ *and a* C^1 *map*

$$\Gamma : \mathcal{V}_T \to L^2((0,T),\mathbb{R}),$$

$$\mathcal{V}_T := \{\Psi_f \in \mathcal{S}^2 \cap H^4_{(0)} \times H^2_{(0)}((0,1),\mathbb{C}); ||\Psi_f - \hat{\psi}_1(T)||_{H^4_{(0)} \times H^2_{(0)}} < \delta\}$$

such that, $\Gamma(\hat{\psi}_1(T)) = 0$ *and for every* $\Psi_f \in \mathcal{V}_T$ *the solution of (7), with* $\psi = (u, u_t)$, $\psi_0 = (u_0, u_1)$, *initial condition*

$$\psi_0 = \hat{\varphi}_1$$

and control $p = \Gamma(\Psi_f)$, *satisfies* $\psi(T) = \Psi_f$.

If we compute explicitly

$$\langle \mu(x)(\varphi_1)_{xx}, \varphi_k \rangle = \mu'(1) \left[\frac{(-1)^k 2(k^2+1)}{(k^2-1)^2} \right] + \mu'(0) \left[\frac{2(k^2+1)}{(k^2-1)^2} \right] + \\ - \int_0^1 \mu'''(x) \left(\frac{\sin((k+1)\pi x)}{(k+1)^3\pi} + \frac{\sin((k-1)\pi x)}{(k-1)^3\pi} \right) dx$$

If we compute explicitly

$$\langle \mu(x)(\varphi_1)_{xx}, \varphi_k \rangle = \mu'(1)\left[\frac{(-1)^k 2(k^2+1)}{(k^2-1)^2}\right] + \mu'(0)\left[\frac{2(k^2+1)}{(k^2-1)^2}\right] + \\ - \int_0^1 \mu'''(x)\left(\frac{\sin((k+1)\pi x)}{(k+1)^3\pi} + \frac{\sin((k-1)\pi x)}{(k-1)^3\pi}\right) dx$$

we have

$$\sum_{k=1}^{\infty} |d_k(\Psi_f)|^2 = \sum_{k=1}^{\infty} \left| \left(i \frac{\langle \Psi_f^1(x), \varphi_k(x)\rangle}{\langle (\mu(x)\varphi_1(x))_{xx}, \varphi_k(x)\rangle} \lambda_k + \frac{\langle \Psi_f^2(x), \varphi_k(x)\rangle}{\langle (\mu(x)\varphi_1(x))_{xx}, \varphi_k(x)\rangle} \right) e^{-i\lambda_k T} \right|^2$$

$$\leq 2 \sum_{k=1}^{\infty} \left| \frac{|\langle \Psi_f^1(x), \varphi_k(x)\rangle|}{|\langle (\mu(x)\varphi_1(x))_{xx}, \varphi_k(x)\rangle|} k^2\pi^2 \right|^2 + \left| \frac{|\langle \Psi_f^2(x), \varphi_k(x)\rangle|}{|\langle (\mu(x)\varphi_1(x))_{xx}, \varphi_k(x)\rangle|} \right|^2$$

$$\leq 2 \sum_{k=1}^{\infty} \left| \frac{\langle \Psi_f^1(x), \varphi_k(x)\rangle}{c} k^4\pi^2 \right|^2 + \left| \frac{|\langle \Psi_f^2(x), \varphi_k(x)\rangle|}{c} k^2 \right|^2$$

$$= 2 \sum_{k=1}^{\infty} \frac{|\langle k^4 \Psi_f^1(x), \varphi_k(x)\rangle|^2}{c} \pi^4 + \frac{|\langle k^2 \Psi_f^2(x), \varphi_k(x)\rangle|^2}{c}$$

Beam equation with sliding ends, controllability

If we compute explicitly

$$\langle \mu(x)(\varphi_1)_{xx}, \varphi_k \rangle = \mu'(1)\left[\frac{(-1)^k 2(k^2+1)}{(k^2-1)^2}\right] + \mu'(0)\left[\frac{2(k^2+1)}{(k^2-1)^2}\right] + \\ - \int_0^1 \mu'''(x)\left(\frac{\sin((k+1)\pi x)}{(k+1)^3\pi} + \frac{\sin((k-1)\pi x)}{(k-1)^3\pi}\right) dx$$

we have

$$\sum_{k=1}^{\infty} |d_k(\Psi_f)|^2 = \sum_{k=1}^{\infty} \left| \left(i \frac{\langle \Psi_f^1(x), \varphi_k(x) \rangle}{\langle (\mu(x)\varphi_1(x))_{xx}, \varphi_k(x) \rangle} \lambda_k + \frac{\langle \Psi_f^2(x), \varphi_k(x) \rangle}{\langle (\mu(x)\varphi_1(x))_{xx}, \varphi_k(x) \rangle} \right) e^{-i\lambda_k T} \right|^2$$

$$\leq 2 \sum_{k=1}^{\infty} \left| \frac{|\langle \Psi_f^1(x), \varphi_k(x) \rangle|}{|\langle (\mu(x)\varphi_1(x))_{xx}, \varphi_k(x) \rangle|} k^2 \pi^2 \right|^2 + \left| \frac{|\langle \Psi_f^2(x), \varphi_k(x) \rangle|}{|\langle (\mu(x)\varphi_1(x))_{xx}, \varphi_k(x) \rangle|} \right|^2$$

$$\leq 2 \sum_{k=1}^{\infty} \left| \frac{\langle \Psi_f^1(x), \varphi_k(x) \rangle}{c} k^4 \pi^2 \right|^2 + \left| \frac{|\langle \Psi_f^2(x), \varphi_k(x) \rangle|}{c} k^2 \right|^2$$

$$= 2 \sum_{k=1}^{\infty} \frac{|\langle k^4 \Psi_f^1(x), \varphi_k(x) \rangle|^2}{c} \pi^4 + \frac{|\langle k^2 \Psi_f^2(x), \varphi_k(x) \rangle|^2}{c}$$

Thanks for your attention!