Control of Partial differential equations involving the fractional Laplacian

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Benasque, August 25th, 2017

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Outline of the talk

We study the controllability problem for the following fractional evolution equations

- Fractional Schrödinger equation: $iu_t + (-\Delta)^s u = 0$
- Fractional wave equation: $u_{tt} + (-\Delta)^{2s} u = 0$
- Fractional heat equation: $u_t + (-\Delta)^s u = 0$.

Main results

SCHRÖDINGER and WAVE:

- $s > 1/2$: null controllability in any time $T > 0$.
- $s = 1/2$: null controllability in time $T > T_0$.
- $s < 1/2$: the problems are not null-controllable.

HEAT:

- $s > 1/2$: null controllability in any time $T > 0$ (in the one-dimensional case).
- $s \in (0, 1)$: approximate controllability in any time $T > 0$.

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Fractional laplacian

For any function u sufficiently regular and for any $s \in (0,1)$, the s-th power of the Laplace operator is given by

$$(-\Delta)^s u(x) = c_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy.$$

Functional setting: fractional Sobolev spaces

- $H^s(\Omega) := \left\{ u \in L^2(\Omega) \,:\, \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{2}+s}} \in L^2(\Omega \times \Omega) \right\}.$
- $\|u\|_{H^s(\Omega)} := \left(\int_\Omega |u|^2 dx + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy \right)^{\frac{1}{2}}.$
- $H^s_0(\Omega) := \left\{ u \in H^s(\mathbb{R}^N) \,:\, u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}.$

Formulation of the problem

We analyse the control problem for the fractional Schrödinger equation

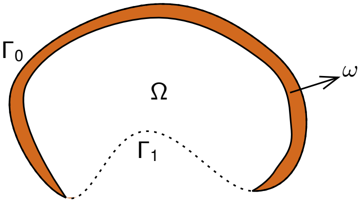
$$iu_t + (-\Delta)^s u = 0$$

on a bounded $C^{1,1}$ domain $\Omega \subset \mathbb{R}^N$. We show null controllability from a neighbourhood of the boundary $\omega \subset \Omega$. As a consequence, we obtain the controllability for the fractional wave equation

$$u_{tt} + (-\Delta)^{2s} u = 0.$$

CONTROL REGION

$$\Gamma_0 := \left\{ x \in \partial\Omega \,\middle|\, (x \cdot \nu) > 0 \right\},$$
$$\Gamma_1 := \left\{ x \in \partial\Omega \,\middle|\, (x \cdot \nu) < 0 \right\},$$
$$\mathcal{O}_\varepsilon := \bigcup_{x \in \Gamma_0} B(x, \varepsilon), \quad \omega := \mathcal{O}_\varepsilon \cap \Omega.$$



Controllability result

Theorem (U.B., PhD Thesis, 2016)

Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{1,1}$ domain with boundary Γ and $s \in [1/2, 1)$. For $u_0 \in L^2(\Omega)$ and $h \in L^2(\omega \times [0,T])$, let $u = u(x,t)$ be the solution of

$$\begin{cases} iu_t + (-\Delta)^s u = h\chi_{(\omega\times[0,T])}, & (x,t) \in Q \\ u \equiv 0, & (x,t) \in \Omega^c \times [0,T] \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1}$$

(i) *If $s \in (1/2, 1)$, for any $T > 0$ the control function h is such that the solution of (1) satisfies $u(x,T) = 0$.*

(ii) *If $s = 1/2$, there exists a minimal time $T_0 > 0$ such that the same result as in (i) holds for any $T \geq T_0$.*

Observability inequality

Proposition

Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{1,1}$ domain with boundary Γ and $s \in [1/2, 1)$. For $v_0 \in L^2(\Omega)$, let $v = v(x,t)$ be the solution of the adjoint system

$$\begin{cases} iv_t + (-\Delta)^s v = 0, & (x,t) \in Q \\ v \equiv 0, & (x,t) \in \Omega^c \times [0,T] \\ v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{2}$$

(i) *If $s \in (1/2, 1)$, then for every $T > 0$ there exists a positive constant C, depending only on s, T, N and Ω, such that*

$$\|v_0\|^2_{L^2(\Omega)} \leq C \int_0^T \|v(t)\|^2_{L^2(\omega)}\, dt. \tag{3}$$

(ii) *If $s = 1/2$, then (3) holds for any $T \geq T_0$, where T_0 is the minimal time introduced before.*

Pohozaev identity

Identity for the elliptic problem

$$\int_\Omega (-\Delta)^s u\,(x\cdot\nabla u)\;dx = \frac{2s-N}{2}\int_\Omega u(-\Delta)^s u\,dx - \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\left(\frac{u}{\delta^s}\right)^2(x\cdot\nu)\,d\sigma^{\,1},$$

[1] X. Ros-Oton and J. Serra, Arch. Ration. Mech. Anal., 2014

Identity for the Schrödinger equation

$$\Gamma(1+s)^2\int_\Sigma\left(\frac{|u|}{\delta^s}\right)^2(x\cdot\nu)\,d\sigma dt = 2s\int_0^T\left\|(-\Delta)^{s/2}u(t)\right\|^2_{L^2(\mathbb{R}^N)}dt + \Im\int_\Omega \bar{u}(x\cdot\nabla u)\,dx\Big|_0^T + \Re\int_Q f\left(N\bar{u}+2x\cdot\nabla\bar{u}\right)\,dxdt.$$

Boundary observability

Proposition

There exists two positive constants A_1 and A_2, depending only on s, T, N and Ω, such that

(i) *if $s \in (1/2, 1)$, then for any $T > 0$ and for all v solution of (2) it holds*

$$A_1 \|u_0\|^2_{H^s(\Omega)} \leq \int_\Sigma \left(\frac{|u|}{\delta^s}\right)^2 (x \cdot \nu) d\sigma dt \leq A_2 \|u_0\|^2_{H^s(\Omega)}; \tag{4}$$

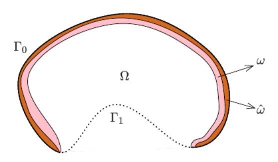
(ii) *if $s = 1/2$, there exists a minimal time $T_0 > 0$ such that (4) holds for any $T > T_0$.*

A technical result

Lemma

Let $1/2 < s < 1$, $\psi \in H_0^s(\Omega)$ *and* $\eta \in C^\infty(\mathbb{R}^N)$ *be a cut-off function such that*

$$\begin{aligned} \eta(x) = 1, \quad & x \in \hat{\omega} \\ 0 \le \eta(x) \le 1, \quad & x \in \omega \setminus \hat{\omega} \\ \eta(x) = 0, \quad & x \in \omega^c. \end{aligned}$$



Then $(-\Delta)^s(\psi\eta) = \psi(-\Delta)^s\eta + R$ *and*

$$\|R\|_{L^2(\mathbb{R}^N)} \le C\left[\|\psi\|_{H^s(\omega)} + \|\psi\|_{L^2(\omega^c)}\right].$$

Fourier analysis for the Schrödinger equation

Theorem

The exponent $s = 1/2$ is sharp for the control.

Let $s \in (0,1)$. For the eigenvalues associated to the problem

$$\begin{cases} (-d_x^2)^s \phi_k(x) = \lambda_k \phi_k(x), & x \in (-1,1) \\ \phi_k(x) \equiv 0, & x \in (-1,1)^c \end{cases}$$

it holds

$$\lambda_k = \left(\frac{k\pi}{2} - \frac{(2-2s)\pi}{8} \right)^{2s} + O\left(\frac{1}{k}\right), \quad \textit{as } k \to +\infty.^{2} \tag{5}$$

Thanks to (5), we have

$$\liminf_{k \to +\infty} (\lambda_{k+1} - \lambda_k) = \gamma_\infty > 0, \quad \textit{for } s \geq 1/2,$$

$$\liminf_{k \to +\infty} (\lambda_{k+1} - \lambda_k) = 0, \quad \textit{for } s < 1/2.$$

[2] M. Kwaśnichi, J. Funct. Anal., 2012

Fractional wave equation

Let us consider the problem

$$\begin{cases} u_{tt} + (-\Delta)^{2s} u = h\chi_{\{\omega\times[0,T]\}}, & (x,t)\in Q \\ u \equiv (-\Delta)^s u \equiv 0, & (x,t)\in \Omega^c \times [0,T] \\ u(x,0) = u_0(x) & \\ u_t(x,0) = u_1(x) \quad , & x\in\Omega. \end{cases}$$

Definition (Higher order fractional Laplacian)

$$(-\Delta)^{2s} u(x) := (-\Delta)^s (-\Delta)^s u(x), \quad s \in [1/2, 1),$$

$$\mathcal{D}\Big((-\Delta)^{2s}\Big) = \Big\{ u \in H_0^s(\Omega) \Big|\, (-\Delta)^s u|_{\Omega^c} \equiv 0,\ (-\Delta)^{2s} u \in L^2(\Omega) \Big\}.$$

Controllability result

Theorem

Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{1,1}$ domain and $s \in [1/2, 1)$. For any couple of initial data $(u_0, u_1) \in H^{2s}(\Omega) \times L^2(\Omega)$ and $h \in L^2(\omega \times [0, T])$, let us consider the following equation

$$\begin{cases} u_{tt} + (-\Delta)^{2s} u = h\chi_{\{\omega \times [0,T]\}}, & (x,t) \in Q \\ u \equiv (-\Delta)^s u \equiv 0, & (x,t) \in \Omega^c \times [0, T] \\ u(x,0) = u_0(x) & \\ u_t(x,0) = u_1(x) \quad , & x \in \Omega. \end{cases} \tag{6}$$

(i) *If $s \in (1/2, 1)$, for any $T > 0$ the control function h is such that the solution of (6) satisfies $u(x, T) = u_t(x, T) = 0$.*

(ii) *If $s = 1/2$, there exists a minimal time $T_0 > 0$ such that the same result as in (i) holds for $T > T_0$.*

Control results

We consider the following parabolic problem

$$\begin{cases} u_t + (-d_x^2)^s u = g\chi_{(\omega\times[0,T])}, & (x,t)\in(-1,1)\times[0,T] \\ u\equiv 0, & (x,t)\in(-1,1)^c\times[0,T] \\ u(x,0)=u_0(x), & x\in(-1,1). \end{cases} \quad (7)$$

Theorem

For all $u_0 \in L^2(-1,1)$ the parabolic problem (7) is null-controllable with a control function $g \in L^2((-1,1)\times(0,T))$ if and only if $s > 1/2$.

Theorem

Let $s \in (0,1)$. For all $u_0 \in L^2(-1,1)$, there exists a control function $g \in L^2(\omega \times (0,T))$ such that the unique solution u to the parabolic problem (7) is approximately controllable.

Proofs (sketch)

- NULL CONTROLLABILITY: the result is equivalent to the condition

$$\sum_{k\geq 1}\frac{1}{\lambda_k} < +\infty$$

which holds for $s > 1/2$ and fails for $s \leq 1/2$.

- APPROXIMATE CONTROLLABILITY: it holds for all $s \in (0,1)$, since the Fractional Laplacian possess the Unique Continuation property.[3]

[3] M.M. Fall and V. Felli, Comm. Partial Differential Equations, 2014..

MORE DETAILS (WITH NUMERICS) NEXT WEEK.

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$$\|R\|_{L^2(\mathbb{R}^N)} \leq C\left[\|\psi\|_{H^s(\omega)} + \|\psi\|_{L^2(\omega^c)}\right].$$

This estimates for the $L^2(\mathbb{R}^N)$-norm of the remainder term R can be applied also for proving local elliptic [4] and parabolic [5] regularity for the fractional Laplacian.

[4] U. B., M. Warma and E. Zuazua, Adv. Nonlinear Stud., 2017.
[5] U. B., M. Warma and E. Zuazua, Preprint, 2017.

Theorem

Let $1 < p < \infty$. Given $f \in L^p(\Omega)$, let u be the unique weak solution to the Dirichlet problem

$$(-\Delta)^s u = f, \quad x \in \Omega, \qquad u = 0, \quad x \in \mathbb{R}^N \setminus \Omega.$$

Then $u \in \left(\mathscr{L}^p_{2s}\right)_{\mathrm{loc}}(\Omega)$. As a consequence we have the following result.

1. *If $1 < p < 2$ and $s \neq 1/2$, then $u \in (B^{2s}_{p,2})_{\mathrm{loc}}(\Omega)$.*
2. *If $1 < p < 2$ and $s = 1/2$, then $u \in W^{2s,p}_{\mathrm{loc}}(\Omega) = W^{1,p}_{\mathrm{loc}}(\Omega)$.*
3. *If $2 \leq p < \infty$, then $u \in W^{2s,p}_{\mathrm{loc}}(\Omega)$.*

POTENTIAL SPACE:

$$\mathscr{L}^p_{2s}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) : \ (-\Delta)^s u \in L^p(\mathbb{R}^N) \right\}, \quad 1 \leq p \leq \infty, \quad s \geq 0,$$

$$(\mathscr{L}^p_{2s})_{\mathbf{loc}}(\Omega) := \left\{ u \in L^p(\Omega) : u\eta \in \mathscr{L}^p_{2s}(\mathbb{R}^N), \ \forall \eta \in \mathcal{D}(\Omega) \right\}.$$

Theorem

Let $1 < p < \infty$. Given $f \in L^p(\Omega \times (0,T))$, let u be the unique weak solution to the parabolic problem

$$\begin{cases} u_t + (-\Delta)^s u = f, & (x,t) \in \Omega \times (0,T), \\ u = 0, & (x,t) \in (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(\cdot, 0) = 0, & x \in \Omega. \end{cases}$$

Then $u \in L^p\Big((0,T); \big(\mathscr{L}^p_{2s}\big)_{\text{loc}}(\Omega)\Big)$. As a consequence we have the following result.

1. *If $1 < p < 2$ and $s \neq 1/2$, then $u \in L^p\Big((0,T); (B^{2s}_{p,2})_{\text{loc}}(\Omega)\Big)$.*
2. *If $1 < p < 2$ and $s = 1/2$, then $u \in L^p\Big((0,T); W^{2s,p}_{\text{loc}}(\Omega)\Big) = L^p\Big((0,T); W^{1,p}_{\text{loc}}(\Omega)\Big)$.*
3. *If $2 \le p < \infty$, then $u \in L^p\Big((0,T); W^{2s,p}_{\text{loc}}(\Omega)\Big)$.*

Proofs (sketch)

- The proof of the elliptic regularity is obtained by means of a cut-off argument, employing known results for the fractional Poisson equation on $\mathbb{R}^N$.[6]
- The parabolic regularity is a consequence of the elliptic one, employing general results from semi-group theory.[7]
- We mention that the elliptic regularity can be obtain also employing the theory of pseudo-differential operators.[8]

[6] E. Stein, 1970.
[7] D. Lamberton, J. Funct. Anal., 1987.
[8] G. Grubb, Adv. Math., 2015.

Open problems

- Develop Geometric Optics expansions exhibiting the propagation of pulses along rays, leading to sharp geometric results on controllability of these models.
- Carleman estimates for the fractional Laplacian on a domain and application to the controllability of fractional heat equations.
- Analyse the global regularity up to $\partial\Omega$ for the solutions of the elliptic and parabolic problem associated to the fractional Laplacian.

THANK YOU FOR YOUR ATTENTION!

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