Non null-controllability of the Grushin equation in dimension 2

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Some controllability results for the Grushin equation

Grushin equation

$$\begin{aligned} \Omega &= [-1, 1] \times \mathbb{T} \\ \partial_t f - \partial_x^2 f - x^2 \partial_y^2 f &= 1_\omega u \\ f_{|\partial\Omega} &= 0 \end{aligned}$$

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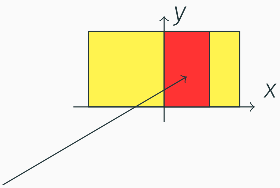

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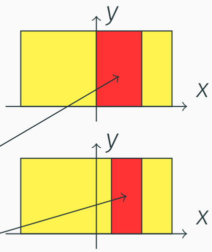

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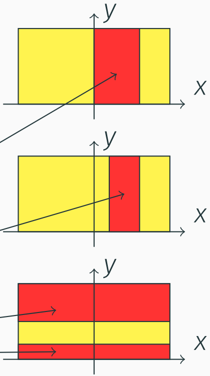

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- Approximation of $-\partial_x^2 + (nx)^2$ on $(-1,1)$ by itself on $\mathbf{R}$: we expect $v_n \sim \left(\frac{n}{4\pi}\right)^{1/4} e^{-nx^2/2}$ et $\lambda_n \sim n$
- Observability inequality with $f(t,x,y) = \sum a_n v_n(x) e^{iny-\lambda_n t}$, heuristics $\lambda_n = n$ and $\int v_n v_m = 1$:

$$\int_{\mathbb{T}} \left|\sum a_n e^{-nT} e^{iny}\right|^2 \mathrm{d}y \leq C \int_{[0,T]\times\omega} \left|\sum a_n e^{-nt} e^{iny}\right|^2 \mathrm{d}t\,\mathrm{d}y$$

The toy model: $D = \sqrt{-\Delta}$

Theorem

Let $H = \{\sum_{n\geq 0} a_n e^{iny}, \sum |a_n|^2 < +\infty\}$ *and* $D \sum a_n e^{iny} = \sum n a_n e^{iny}$.
Let ω *be a strict open set of the unit circle.*

The equation $\partial_t f + Df = 1_\omega u$ *is never null-controllable.*

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Observability inequality $f(t,y) = \sum a_n e^{-nt} e^{iny}$:

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It is the approximate observability inequality of the Grushin equation!

Proof of the non null-controllability of the toy model

Observability inequality with $f(t,y) = \sum a_n e^{-nt} e^{iny}$:

$$\sum |a_n|^2 e^{-2nT} \leq C \int_{[0,T]\times\omega} \left| \sum a_n e^{-nt} e^{iny} \right|^2 \, \mathrm{d}t \, \mathrm{d}y$$

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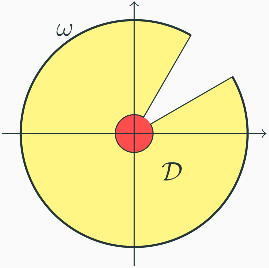
Let $z = e^{-t+iy} = \mu + i\nu$ and $f(z) = \sum_{n\geq 1} a_n z^{n-1}$

$$\begin{aligned}\int_{z\in D(0,e^{-T})} |f(z)|^2 \,\mathrm{d}\mu\,\mathrm{d}\nu &\leq \pi \sum |a_n|^2 e^{-2nT} \\ &\leq \pi C \int_{z\in\mathcal{D}} |f(z)|^2 \,\mathrm{d}\mu\,\mathrm{d}\nu\end{aligned}$$



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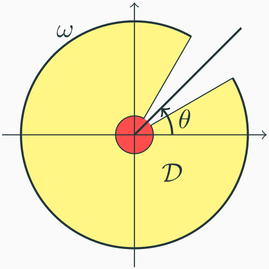

False thanks to Runge's theorem (take $f_k \longrightarrow 1/z$ uniformly on every compact subset of $\mathbb{C} \setminus e^{i\theta}\mathbb{R}_+$) □

Differences between the Grushin equation and the Toy model

"Holomorphic" observability inequality (we know it is false):

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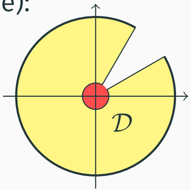

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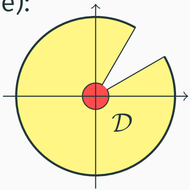

Observability inequality of the Grushin equation

- Let $\lambda_n = n + \rho_n$ and $z = \mu + i\nu$:

$$\sum |a_n|^2 e^{-2nT} e^{-2\rho_n T} \leq C \int_{-1}^{1} \int_{z \in \mathcal{D}} \left| \sum v_n(x) a_n z^n |z|^{\rho_n} \right|^2 \, \mathrm{d}\mu \, \mathrm{d}\nu \, \mathrm{d}x$$

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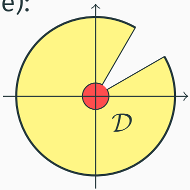

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- Solution:
  - $e^{-2\rho_n T}$ in the lhs: not a problem
  - Treat x as a parameter
  - Write $\gamma_n = v_n(x)|\zeta|^{\rho_n}$ and prove

$$\left|\sum \gamma_n a_n z^n\right|_{L^\infty(\mathcal{D})} \le C \left|\sum a_n z^n\right|_{L^\infty(U)}$$

Conclusion

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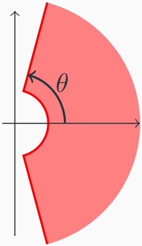
That's all folks!

Tools

Definition

Let $S(r)$ the space of functions γ such that for all $0 < \theta < \pi/2$:

- γ is holomorphic on $\{|z| > r(\theta), |\arg(z)| < \theta\}$
- γ has sub-exponential growth on each of these domains



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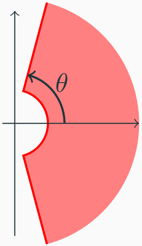

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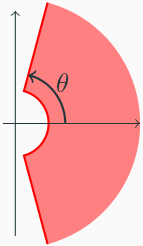

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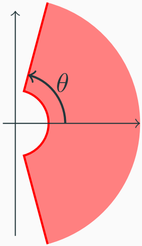

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Let γ in $S(r)$, and $H_\gamma(\sum a_n z^n) = \sum \gamma(n) a_n z^n$. Let U a bounded domain star-shaped with respect to 0, let $\delta > 0$ and $U^\delta = \{z, distance(z, U) < \delta\}$. For all polynomials f:

$$|H_\gamma(f)|_{L^\infty(U)} \leq C|f|_{L^\infty(U^\delta)}$$

Holomorphic and exponential estimate of the eigenvalues

Theorem

There exists γ in some $S(r)$ such that:

$$\lambda_\alpha = \alpha + \gamma(\alpha)e^{-\alpha} \qquad \gamma \sim \frac{4}{\sqrt{\pi}}\alpha^{3/2}$$

Idea of the proof.

- Explicitly solve the ODE (solution as an integral on some complex path)
- Deduce from that solution an implicit equation between α and λ_α
- Solve that equation with Newton's method (necessary estimates provided by the stationary phase theorem) □

Estimates on some operators on entire functions

Theorem

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$$|H_\gamma(f)|_{L^\infty(U)} \leq C|f|_{L^\infty(U^\delta)}$$

Idea of the proof.

- Write $H_\gamma(f)(z) = \oint_{\partial D(0,R)} \frac{1}{\zeta} K_\gamma\left(\frac{z}{\zeta}\right) f(\zeta)\,\mathrm{d}\zeta$ with $K_\gamma(\zeta) = \frac{1}{2i\pi}\sum \gamma(n)\zeta^n$ (Cauchy's integral formula)
- Extend K_γ to $\mathbb{C} \setminus [1, +\infty[$ thanks to Poisson's summation formula (we can do it because the holomorphy of γ allows us to extend the Fourier transform of γ to $\mathbb{C} \setminus i\mathbb{R}_+$)
- Change the path of integration □