Evolution of networks

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joint work with Harald Garcke and Julia Menzel

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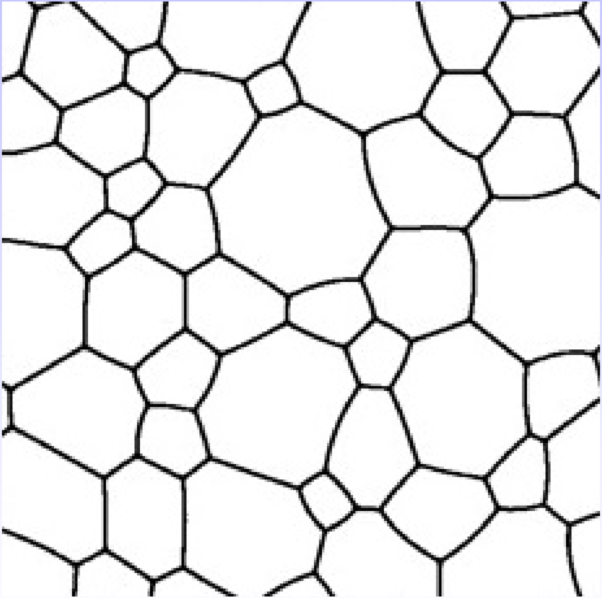
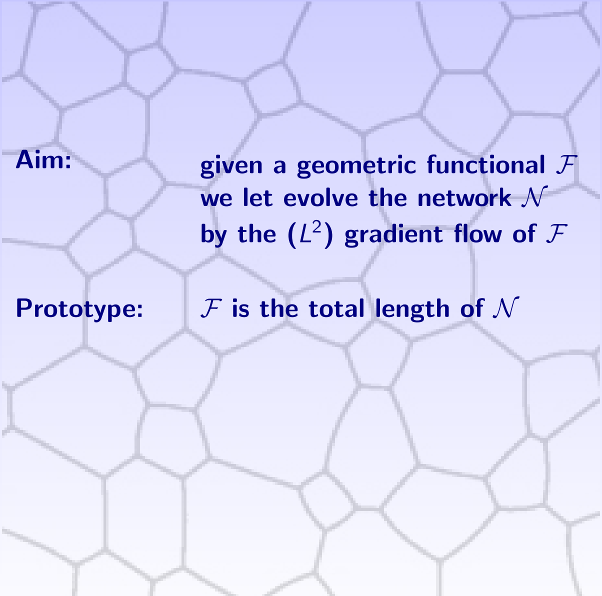
VII PDEs, optimal design and numerics

Curvature, Cycles, and Cohomology

Universität Regensburg



Geometric evolution laws for networks



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Prototype: $\mathcal{F}$ is the total length of $\mathcal{N}$

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Main difficulty: presence of junctions

Geometric L^2 gradient flow of the elastic energy

Given a network $\mathcal{N} = \cup_{i=1}^{N} \gamma^i$, we consider the elastic energy functional defined as

$$E\left(\mathcal{N}\right) = \int_{\mathcal{N}} k^2 \, ds = \sum_{i=1}^{3} \int_{\gamma_i} k^2 \, ds \, .$$

We are interested in the L^2 gradient flow for $E(\mathcal{N})$.

Geometric L^2 gradient flow of the elastic energy

Given a network $\mathcal{N} = \cup_{i=1}^{N} \gamma^i$, we consider the elastic energy type functional defined as

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Formally we derive the motion equation computing the first variation of $F(\mathcal{N})$

$$\frac{d}{dt} F(\widetilde{\mathcal{N}})_{|t=0} = \sum_{i=1}^{3} \int_{\gamma^i} \left\langle \psi^i, \left(2k^i_{ss} + \left(k^i\right)^3 - k^i \right) \nu^i \right\rangle ds$$
$$+ \, boundary \; terms \, ,$$

we obtain

$$(\gamma^i_t)^\perp = -\, 2k^i_{ss} - \left(k^i\right)^3 + k^i$$
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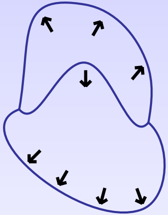

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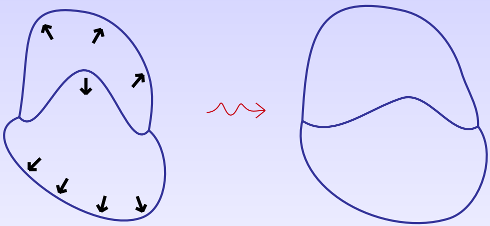

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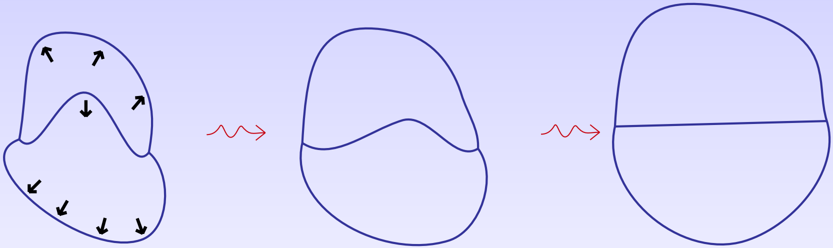

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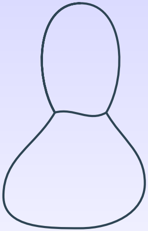

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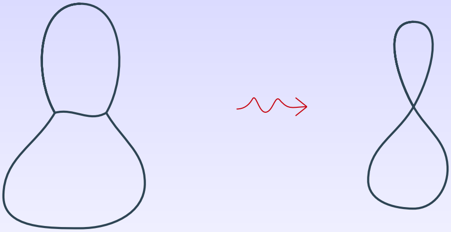

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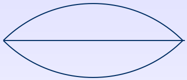

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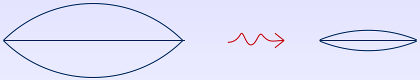

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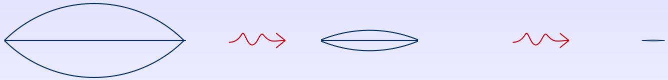

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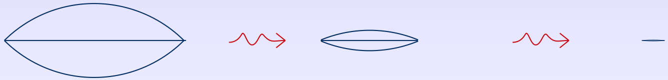
- vanishing of a curve;
- vanishing of the network;
- convergence to a stationary point of F.



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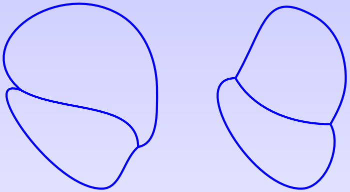
Theta–network



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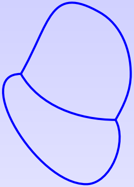

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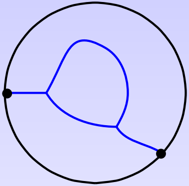

Case 3:
Lens
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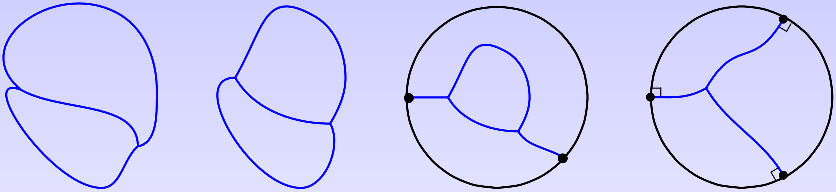

Case 1:
Theta–network

Case 2:
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with fixed
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Case 3:
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Case 4:
Triod
with fixed
points and
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Problem

Consider an admissible initial Theta network $\Theta = \cup_{i=1}^{3} \gamma^i$, we study its evolution by

$$(v^i)^\perp = -(2k^i_{ss} + (k^i)^3 - k^i)\nu^i =: -A^i \nu^i \,,$$

coupled with the following conditions at the triple junctions:

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with φ^i *admissible initial data.*

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$$\begin{cases} \gamma^i_t = -2\frac{\gamma^i_{xxxx}}{\left|\gamma^i_x\right|^4} + 12\frac{\gamma_{xxx}\left\langle\gamma_{xx},\gamma_x\right\rangle}{\left|\gamma_x\right|^6} + 5\frac{\gamma_{xx}\left|\gamma_{xx}\right|^2}{\left|\gamma_x\right|^6} & \\ +8\frac{\gamma^i_{xx}\left\langle\gamma^i_{xxx},\gamma^i_x\right\rangle}{\left|\gamma^i_x\right|^6} - 35\frac{\gamma^i_{xx}\left\langle\gamma^i_{xx},\gamma^i_x\right\rangle^2}{\left|\gamma^i_x\right|^8} + \frac{\gamma^i_{xx}}{\left|\gamma^i_x\right|^2} & & \textit{motion} \\ \gamma^1(t,y) = \gamma^2(t,y) = \gamma^3(t,y) & \text{for } y \in \{0,1\} & \textit{concurrency condition} \\ \gamma^i_{xx}(t,y) = 0 & \text{for } y \in \{0,1\} & \textit{second order condition} \\ \sum_{i=1}^3 \left(2k^i_s\nu^i - \tau^i\right)(t,y) = 0 & \text{for } y \in \{0,1\} & \textit{third order condition} \\ \gamma^i(0,x) = \varphi^i(x) & \text{for } x \in [0,1] & \textit{initial data} \end{cases}$$

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with φ^i admissible initial data.

Theorem (Short time existence)

Let $(\varphi^i)_{i=1,2,3}$ be an admissible inital data. Then there exists a strictly positive time T such that the system (1) has a unique solution in $C^{\frac{4+\alpha}{4},4+\alpha}([0,T]\times[0,1]) =: \mathbb{E}_T$.

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The proof is based on the following steps:

- linearisation of the system;
- resolution of the linearised system;
- fixed point argument.

We introduce the following notation: given a Theta–network $\Theta = \cup_{i=1}^N \gamma^i$, with $\gamma^i : [0,1] \to \mathbb{R}^2$, we denote with γ the triple $(\gamma^1, \gamma^2, \gamma^3)$.

Moreover

$$\begin{cases} 0 = \gamma^i_t + 2\frac{\gamma^i_{xxxx}}{|\gamma^i_x|^4} - \tilde{f}(\gamma^i_{xxx}, \gamma^i_{xx}, \gamma^i_x) & =: \mathcal{M}(\gamma^i) \\ \gamma^1(t,y) = \gamma^2(t,y) = \gamma^3(t,y) & \\ \gamma^i_{xx}(t,y) = 0 & \\ 0 = \sum_{i=1}^3 \frac{1}{|\gamma^i_x|^3} \left\langle \gamma^i_{xxx}, \nu^i \right\rangle \nu^i - \sum_{i=1}^3 \frac{\gamma^i_x}{|\gamma^i_x|} & =: \mathcal{B}(\gamma) \end{cases}$$

We fix an admissible initial data $\Theta_0 = \bigcup_{i=1}^{3} \varphi^i$.
We linearise $\mathcal{M}(\gamma^i)$ and $\mathcal{B}(\gamma)$ around the initial data:

$$\gamma^i_t + \frac{2}{|\varphi^i_x|^4}\gamma^i_{xxxx} = \left(\frac{2}{|\varphi^i_x|^4} - \frac{2}{|\gamma^i_x|^4}\right)\gamma^i_{xxxx} + \tilde{f}(\gamma^i_{xxx}, \gamma^i_{xx}, \gamma^i_x) =: f^i\,,$$

$$-\sum_{i=1}^{3} \frac{1}{|\varphi^i_x|^3}\left\langle \gamma^i_{xxx}, \nu^i_0 \right\rangle \nu^i_0 = -\sum_{i=1}^{3} \frac{1}{|\varphi^i_x|^3}\left\langle \gamma^i_{xxx}, \nu^i_0 \right\rangle \nu^i_0 + \sum_{i=1}^{3} \frac{1}{|\gamma^i_x|^3}\left\langle \gamma^i_{xxx}, \nu^i \right\rangle \nu^i + h^i(\gamma^i_x) =: b\,.$$

obtaining the linear operator

$$L_T : \mathbb{E}_T \to C_t^{\frac{\alpha}{4},\alpha}{}_x([0,T)\times[0,1];(\mathbb{R}^2)^3) \times C_t^{\frac{1+\alpha}{4},1+\alpha}{}_x([0,T)\times\{0,1\};\mathbb{R}^2) =: \mathbb{F}_T$$

defined by

$$L_T(\gamma) = \begin{pmatrix} \left(\gamma^i_t + \frac{2}{|\varphi^i_x|^4}\gamma^i_{xxxx}\right)_{i\in\{1,2,3\}} \\ -\mathrm{tr}_{\partial[0,1]} \sum_{i=1}^{3} \frac{1}{|\varphi^i_x|^3}\left\langle \gamma^i_{xxx}, \nu^i_0 \right\rangle \nu^i_0 \end{pmatrix} =: \begin{pmatrix} L_{T,1}(\gamma^i) \\ L_{T,2}(\gamma) \end{pmatrix}$$

The associated linearised system is given by

$$\begin{cases} \gamma^i_t + \frac{2}{|\varphi^i_x|^4}\gamma^i_{xxxx} & = f^i & \text{motion} \\ \gamma^1 - \gamma^2 & = 0 & \text{concurrency} \\ \gamma^1 - \gamma^3 & = 0 & \text{concurrency} \\ \gamma^i_{xx} & = 0 & \text{second order} \\ -\sum_{i=1}^3 \frac{1}{|\varphi^i_x|^3}\left\langle \gamma^i_{xxx}, \nu^i_0 \right\rangle \nu^i_0 & = b & \text{third order} \end{cases} \tag{2}$$

Theorem

Let $\alpha \in (0,1)$. There exists $T > 0$ such that if

- $f^i \in C_t^{\frac{\alpha}{4}, \alpha}{}_x([0,T) \times [0,1]; \mathbb{R}^2)$ *for* $i \in \{1,2,3\}$ *;*
- $b \in C_t^{\frac{1+\alpha}{4}, 1+\alpha}{}_x([0,T) \times \{0,1\}; \mathbb{R}^2)$ *;*
- *linear compatibility conditions are fulfilled;*

then system (2) has a unique solution $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ *in* $C_t^{\frac{4+\alpha}{4}, 4+\alpha}{}_x([0,T) \times [0,1]; (\mathbb{R}^2)^3)$.

We define $N_T : \mathbb{E}_T \to \mathbb{F}_T$ as

$$N_T(\gamma) = \begin{pmatrix} L_{T,1}(\gamma^i) - \mathcal{M}(\gamma^i) \\ L_{T,2}(\gamma) - \mathcal{B}(\gamma) \end{pmatrix} = \begin{pmatrix} f^i \\ b \end{pmatrix}$$

and the map $K_T : L_T^{-1} N_T : \mathbb{E}_T \to \mathbb{E}_T$.

Proposition

For any positive radius M there exists a strictly positive time $T(M)$ such that for all $T \in (0, T(M)]$ the map $K_T : \mathbb{E}_T \cap \overline{B_M} \to \mathbb{E}_T \cap \overline{B_M}$ is a contraction.

As the solutions of (1) in $C^{\frac{4+\alpha}{4}, 4+\alpha}([0, T] \times [0, 1]) \cap \overline{B_M}$ are precisely the fixed points of K_T in $\mathbb{E}_T^{\varphi} \cap \overline{B_M}$, the short time existence Theorem follows.