Homogenization results for forced mean curvature flow

Annalisa Cesaroni

Università di Padova

VII Partial differential equations, optimal design and numerics, Benasque, 2017

The model problem

Forced mean curvature flow of hypersurfaces $\Gamma(t) \subseteq \mathbb{R}^{n+1}$

$$v = H + g$$

v is the (inward) normal velocity
H the mean curvature (positive on convex sets)

g forcing term, $\mathbf{Z}^{n+1}$ periodic, models
the spatially heterogeneous environment.

Motivation: motion of phase boundaries through heterogeneous material.

Main goal

Describe the the effective front and its effective velocity on a large space-time scale.
The rescaled evolution law is

$$v_\varepsilon(x) = \varepsilon H(x) + g\left(\frac{x}{\varepsilon}\right) \qquad x \in \Gamma_\varepsilon(t),$$

Aim: study the convergence, as $\varepsilon \to 0$,

$$\Gamma_\varepsilon(t) \longrightarrow \overline{\Gamma}(t)$$

and the limit propagation law $v_\varepsilon(x) \longrightarrow v(x) = c_g(\nu(x))$ where ν is the normal vector.
Related problem: long time behaviour of $\Gamma(t)$.

Setting of the problem

Let $x = (y, z) \in \mathbb{R}^{n+1}$, $y \in \mathbb{R}^n$, $z \in \mathbb{R}$.

We consider the motion of **entire graphs**:

- $g(x) = g(y)$, i.e. it does not depend on z.
- $\Gamma_\varepsilon(0)$ is the graph (in the vertical direction z) of a function $u_0 : \mathbb{R}^n \to \mathbb{R}$.

So $\Gamma_\varepsilon(t)$ are the graphs (in the vertical directions) of the solutions to

$$(PDE)_\varepsilon \begin{cases} u_t(t,y) = \sqrt{1+|Du|^2}\left(\varepsilon \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) + g\left(\frac{y}{\varepsilon}\right)\right) \\ u(0,y) = u_0(y). \end{cases}$$

The evolution takes place in $(0,1)^n \times \mathbb{R}$ with periodic bc.

PDE approach

Aim : describe appropriate assumptions on the forcing term g assuring that $u_\varepsilon \to u$ and characterize u as solution of an appropriate PDE.

Two main issues:

- **1) Identification of the limit problem**:
  cell problem: $\forall\ p \in \mathbb{R}^n$ find $c(p) \in \mathbb{R}$ such that

$$(C) \qquad -\operatorname{div}\left(\frac{D\chi + p}{\sqrt{1+|D\chi+p|^2}}\right) = g(y) + \frac{c(p)}{\sqrt{1+|D\chi+p|^2}}$$

  has a bounded (periodic) solution χ_p.

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- **2) Convergence** :
  as $\varepsilon \to 0$, u_ε solutions to $(PDE)_\varepsilon$ converge to u, solution to

$$(PDE) \begin{cases} u_t(t,y) = c(Du) \\ u(0,y) = u_0(y). \end{cases}$$

Sketch of proof: formal ansatz

Formal asymptotic expansion of the solution u_ε to $(PDE)_\varepsilon$:

$$u_\varepsilon(y,t) = u(y,t) + \varepsilon\chi\left(\frac{y}{\varepsilon}, y, t\right) + o(\varepsilon)$$

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We plug it into the perturbed equation $(\chi(\xi, y, t))$,

$$u_t = \sqrt{1+|Du+D_\xi\chi|^2}\left(\mathrm{div}_\xi\left(\frac{Du+D_\xi\chi}{\sqrt{1+|Du+D_\xi\chi|^2}}\right) + g\left(\frac{y}{\varepsilon}\right)\right) + O(\varepsilon)$$

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let χ_p be a solution of the cell problem

$$-\mathrm{div}\left(\frac{D\chi+p}{\sqrt{1+|D\chi+p|^2}}\right) = g(y) + \frac{c(p)}{\sqrt{1+|D\chi+p|^2}}$$

with $p = Du(y,t)$, then u solves

$$u_t = c(Du).$$

Sketch of the proof: Perturbed test function method

Perturbed test function method [Evans, '89, '92]:

Step 1 u_ε are loc. equibounded w.r.t. ε (by comparison with appropriate barriers)

Step 2 We compute the relaxed semilimits
$\underline{u}(y,t) = \liminf_{\varepsilon\to 0,\ (w,s)\to(y,t)} u_\varepsilon(w,s)$ and
$\overline{u}(y,t) = \limsup \ldots$

Step 3 $\overline{u}$ and $\underline{u}$ are sub and supersolution to the limit problem: if ϕ is a test function at (x,t) for $\overline{u}$ (or $\underline{u}$), then

$$\phi_\varepsilon(y,s) = \phi(y,s) + \varepsilon\chi_{D\phi(y,t)}\left(\frac{y}{\varepsilon}\right)$$

is a test function for u_ε at $(y_\varepsilon, t_\varepsilon) \to (y,t)$.

Step 4 $\underline{u} = \overline{u} = u$ by comparison principle for the limit problem. So $u_\varepsilon \to u$ locally uniformly.

Perturbed test function method

This method works well in the graph case

but

in the general case (motion of general sets–the equation is *singular* at $Du = 0$)
or in the graph case with different scaling,
something more is required, e.g. some uniform estimates on the curvature of the level set of χ and regularity of the correctors χ with respect to p.

Cell problem and traveling waves

If $(c(p), \chi_p)$ solve the cell problem

$$(C) \qquad -\operatorname{div}\left(\frac{D\chi + p}{\sqrt{1+|D\chi+p|^2}}\right) = g(y) + \frac{c(p)}{\sqrt{1+|D\chi+p|^2}}$$

then

$$\chi_p(y) + p \cdot y + c(p)t$$

is a **travelling wave solution** to the forced mean curvature flow $(\varepsilon = 1)$.

$\chi_p(y) + p \cdot y$ gives the almost planar profile
$c(p)$ is the speed of propagation

Homogenization result is related to stability properties of traveling waves.

Literature, motivations

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- **[Lions, Souganidis '05]**: solution to the cell problem by viscosity solution methods.
  Assume $g^2(z) > n^2|Dg(z)|$ then the solutions to

$$\lambda v_\lambda - \sqrt{1+|Dv_\lambda + p|^2}\left(\operatorname{div}\left(\frac{Dv_\lambda + p}{\sqrt{1+|Dv_\lambda + p|^2}}\right) - g(x)\right) = 0$$

are equilipschitz (unif. w.r.t λ),
$\lambda v_\lambda \to -c(p)$, $v_\lambda - v_\lambda(0) \to \chi_p$.

Literature, planar case

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- $g(y,z) > 0$, **[Caffarelli, Monneau, '12]**, general case. Homogenization even for g non positive, under the assumption there exists a subsolution with compact support expanding in all directions.

Case of trapping, $c(p) \equiv 0$

We assume $\int_{(0,1)^n} g(y) = 0$.
Cell problem is

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Proposition

If g satisfy $\int_{(0,1)^n} g(y) = 0$ *and*

$$(G1) \quad \forall E \subseteq [0,1]^n \quad \int_E g(x) < \mathrm{Per}(E, \mathbb{T}^n) \qquad \mathbb{T}^n = \mathbb{R}^n \setminus \mathbb{Z}^n.$$

then there exists for every p a periodic (smooth) solution χ *to the cell problem (unique up to addition of constants).*
Moreover $p \mapsto \chi_p(y)$ *is smooth.*

Case of trapping, literature

- $n = 1$, $g(x,y) = g(x)$, **[Cardaliaguet, Lions, Souganidis '09]**: homogenization (pinning) if
  $(g) \quad \max \int_0^x g - \min \int_0^x g < 2$ (this is exactly condition $(G1)$).
  No homogenization if $\max \int_0^x g - \min \int_0^x g > 2$.

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- **[Caffarelli, DeLaLlave '01]**: existence of plane-like minimizers to

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Remark

Regularity $p \mapsto \chi_p$ is an important issue. Much more delicate in the general case (non graph case). Problem related to weak KAM theory, differentiability of the stable norm.

Trapping

Proposition

Under condition (G1),

$$v^\varepsilon = \varepsilon H + g\left(\frac{y}{\varepsilon}\right) \to 0.$$

Moreover, if (G1) is violated, i.e. there exists a measurale set A such that

$$\int_A g > \mathrm{Per}(A, \mathbb{T}^n)$$

no homogenization (i.e. in general v^ε is not converging).

A more detailed result (rescaling time): convergence of

$$v_\varepsilon = H + \frac{1}{\varepsilon} g\left(\frac{y}{\varepsilon}\right) \longrightarrow ? \qquad \text{as } \varepsilon \to 0$$

Literature: homogenization of interfacial energies

The equation

$$H(y) + \frac{1}{\varepsilon} g\left(\frac{y}{\varepsilon}\right) = 0 \qquad y \in \Gamma^\varepsilon$$

is the Euler-Lagrange equation of the functional

$$\mathcal{F}_\varepsilon(E) = \mathrm{Per}(E, \mathbb{T}^n) + \frac{1}{\varepsilon}\int_E g\left(\frac{y}{\varepsilon}\right) dy.$$

[Chambolle, Thouroude '09] proved that, under $(G1)$,

$$\mathcal{F}_\varepsilon(E) \xrightarrow{\Gamma} \mathcal{F}_0(E) := \int_{\partial^\star E} \phi(\nu) d\mathcal{H}^n$$

The problem we are considering is related: study the convergence of the gradient flow of $\mathcal{F}_\varepsilon$.

A first homogenization result

Theorem (Barles, C., Novaga, '11)

Under assumption $(G1)$,

$$v_\varepsilon = \left[H(y) + \frac{1}{\varepsilon} g\left(\frac{y}{\varepsilon}\right)\right] \longrightarrow G_g(\nu, D\nu)$$

For $n = 1$, $G_g(\nu, D\nu) \equiv \alpha_g(\nu)H$, where

- for every $g \not\equiv 0$, there exists $K_g > 0$ s.t.

  $$0 < \alpha_g(\nu) \leq K_g|\nu_2|$$

  then $\lim_{\nu_2 \to 0} \alpha_g(\nu) = 0$;

- $$0 < \alpha_g(\nu) \leq 1 \qquad \forall \nu$$

  and $\alpha_0(\nu) = 1$.

General case. Existence of the generalized correctors

The cell problem

$$(C) \qquad -\operatorname{div}\left(\frac{D\chi + p}{\sqrt{1+|D\chi+p|^2}}\right) = g(y) + \frac{c(p)}{\sqrt{1+|D\chi+p|^2}}$$

Proposition (C., Novaga, '12)

If g satisfes

$(G2) \quad \exists A \subseteq [0,1]^n \quad \int_A g(x) > \mathrm{Per}(A, \mathbb{T}^n) \qquad \mathbb{T}^n = \mathbb{R}^n \setminus \mathbb{Z}^n$

- $\forall\ p\ \exists$ *a positive constant* $\int_{[0,1]^n} g \leq c(p) \leq \max g$
- *a maximal support set* E_p *(open) and a function* $\chi_p \in \mathcal{C}^{2,\alpha}(E_p)$, $\chi_p = -\infty$ *in* $[0,1]^n \setminus E_p$, *solution to* (C) *in* E_p *with b.c.*

$$\forall \phi \in \mathcal{C}^1 \qquad \chi_p - \phi \quad \textit{achieves its maximum in } E_p$$

Generalized correctors and homogenization

Remark

The existence of a generalized corrector is not sufficient to get homogenization.

Counterexample (see **[Caffarelli, Monneau, '12]**):

Assume $(G2)$ holds, then there exists an open set E (say a ball) and a travelling wave $c(0)t + \chi_0(y)$, equal to $-\infty$ outside E.

If outside E g is sufficiently small, we can construct a cylindrical supersolution to the evolution, going to $+\infty$ in E and moving with speed $c'(0) < c(0)$.

After rescaling, we use these two generalized travelling waves as moving barriers for the evolution, obtaining a counterexample to homogenization.

The support set E

Remark

Homogenization takes place when $E_p = (0,1)^n$, for all p.

Proposition

Properties of E_p:

- $E_p \times \mathbb{R}$, *where* E_p *is thought as a subset of* $\mathbb{T}^n$, *is a local minimizer of the geometric functional*

$$\mathcal{F}_{c(p)}(\Sigma, \mathbb{T}^n \times \mathbb{R}) = \int_{\partial_\star \Sigma} e^{c(p)z} d\mathcal{H}^n - \int_{\Sigma} e^{c(p)z} g(y).$$

- E_p *satisfies* $H = g$ *on* $\partial E_p \setminus S_p$.
- ∂E_p *is a* $\mathcal{C}^{2,\alpha}$ *hypersurface, up to a singular set* S_p *of Hausdorff dimension less than* $n-8$.

Sketch of the proof

- variational interpretation of cell problem: we look for solutions to (C) which are minimizers of appropriate exponentially weighted area functionals with a volume term

$$F_{c(p)} = \int_{(0,1)^n} e^{c(p)(\chi_p(y)+p\cdot y)} \left(\sqrt{1+|D\chi_p(y)+p|^2} - \frac{g(y)}{c} \right) dy$$

- $\mathcal{F}_{c(p)}$ is a geometric representation of $F_{c(p)}$
- the epigraph of $\chi_p(y) + p \cdot y$ is a local minimizer of $\mathcal{F}_{c(p)}$
- the class of minimizers is invariant by vertical shift
- Regularity from classical theory of surfaces with prescribed curvature

Existence of correctors, general case.

The cell problem

$$(C) \quad -\operatorname{div}\left(\frac{D\chi + p}{\sqrt{1+|D\chi+p|^2}}\right) = g(y) + \frac{c(p)}{\sqrt{1+|D\chi+p|^2}}$$

Proposition (C., Novaga, '12)

Assume $\int_{(0,1)^n} g > 0$ and
(G3) if $E \times \mathbb{R}$ is a minimizer under compact perturbations of $\mathcal{F}_{c(p)}$, then $E \in \{\mathbb{T}^n, \emptyset\}$.
Then there exists a periodic solution χ_p to (C) for every p.

Sufficient conditions for $(G3)$

- $n = 1$, $g > 0$ (since $H = g$ on ∂E implies $g = 0$ on ∂E),
- $n > 1$, $g > 0$, $\max g < C_n 2^{1/n}$,
- $n > 1$, $g > 0$, $\max g \geq C_n 2^{1/n}$, and $\max g - \min g < \max g \left[\left(\frac{\max g}{C_n} \right)^n - 1 \right]^{-1}$,
- $\min g \leq 0$, $\max g - \min g < C_n 2^{1/n}$,

where C_n is the isoperimetric constant of the torus.
Condition $(G3)$ is related to the condition of Caffarelli and Monneau on the existence of an expanding subsolution of the forced mean curvatue flow.

Homogenization result

Theorem (C., Novaga '12)

Assume $\int_{(0,1)^n} g > 0$ *and* $(G3)$. *Let* u_ε *be the unique continuous viscosity solution to* $(PDE)_\varepsilon$. *Then* u_ε *locally uniformly on* $[0,+\infty) \times \mathbb{R}^n$, *as* $\varepsilon \to 0$, *to the unique continuous viscosity solution* u *of*

$$\begin{cases} u_t = c(Du) \\ u(0,y) = u_0(y) \end{cases}$$

where the function $c(\cdot)$ *is continuous and is given by the previous proposition.*

Extension

- A straightforward extension: case of forced mean curvature with a (regular) trasport term

$$v_\varepsilon = \varepsilon H + g\left(\frac{y}{\varepsilon}\right) + Df\left(\frac{y}{\varepsilon}\right) \cdot \nu(y)$$

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- stochastic homogenization

$$v_\varepsilon = \varepsilon H + g\left(y, \omega\right)$$

with g stationary ergodic. Very difficult problem: existence of plane like minimizers?