NECESSARY CONDITIONS FOR INFINITE HORIZON OPTIMAL CONTROL PROBLEMS WITH STATE CONSTRAINTS

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Abstract. Partial and full sensitivity relations are obtained for nonautonomous optimal control problems with infinite horizon subject to state constraints, assuming that the associated value function be locally Lipschitz in the state. Sufficient structural conditions are given to ensure such a Lipschitz regularity in presence of a positive discount factor, as it is typical of macroeconomics models.

1. Introduction

Consider the infinite horizon optimal control problem $\mathscr{B}_\infty$

$$\text{minimize } \int_{t_0}^{\infty} L(t, x(t), u(t))\, dt \tag{1}$$

over all the trajectory-control pairs subject to the state constrained control system

$$\begin{cases} x'(t) = f(t, x(t), u(t)) & \text{a.e. } t \in [t_0, \infty) \\ x(t_0) = x_0 & \\ u(t) \in U(t) & \text{a.e. } t \in [t_0, \infty) \\ x(t) \in A & t \in [t_0, \infty) \end{cases} \tag{2}$$

where A is a nonempty closed subset of $\mathbb{R}^n$, $U : [0, \infty) \rightrightarrows \mathbb{R}^m$ is a Lebesgue measurable set valued map with closed nonempty images and $(t_0, x_0) \in [0, \infty) \times A$ is the initial datum. Every trajectory-control pair $(x(\cdot), u(\cdot))$ that satisfies the state constrained control system (2) is called feasible. We refer to such $x(\cdot)$ as a feasible trajectory. The infimum of the cost functional in (1) over all feasible trajectory-control pairs, with the initial datum (t_0, x_0), is denoted by $V(t_0, x_0)$ (if no feasible trajectory-control pair exists at (t_0, x_0), we set $V(t_0, x_0) = +\infty$). The function $V : [0, \infty) \times A \to \mathbb{R} \cup \{\pm\infty\}$ is called the value function of the problem $\mathscr{B}_\infty$.

The literature about unconstrained infinite horizon optimal control problems is quite rich. For instance, necessary optimality conditions were derived in absence of state constraints (cfr. e.g. [3] and the reference therein). Usually assumptions on f and L are so that they imply the Lipschitz regularity of $V(\cdot, \cdot)$. Recovering necessary conditions in presence of state constraints appears quite challenging, despite the fact that necessary conditions for Bolza problems in presence of state constraints (cfr. [11]) are well known.

The technique used in this paper is based on reformulating the infinite horizon problem as a Bolza problem with finite horizon and then using known results. For this purpose we observe that, taking $T > 0$, for all $(s, y) \in [0, T] \times A$ (cfr. Lemma 4.2)

$$V(s,y) = \inf \left\{ V(T, x(T)) + \int_s^T L(t, x(t), u(t))\, dt \right\}$$

where the infimum is taken over all the feasible trajectory-control pairs (x, u) satisfying (2) with initial datum (s, y). Hence the problem $\mathscr{B}_\infty$ becomes a Bolza problem on $[0, T]$ with the additional final cost term $\phi^T(\cdot) = V(T, \cdot)$. Then, as it is customary, rewriting the problem in $(n+1)$-dimension, we may consider it as a Mayer problem and apply known results.

Infinite horizon problems have a very natural application in mathematical economics (see, for instance, the Ramsey model in [7]). In this case the planner seeks to find a solution to $\mathscr{B}_\infty$ (dealing with a maximization problem instead of a minimization one) with

$$L(t, x, u) = e^{-\lambda t} l(ug(x)) \quad \& \quad f(t, x, u) = \tilde{f}(x) - ug(x)$$

where $l(\cdot)$ is called the "utility"function, $\tilde{f}(\cdot)$ the "production"function, and $g(\cdot)$ the "consumption"function, while the variable x stands for the "capital"(in many applications one takes as constraint set $A = [0, \infty)$ with $U(\cdot) \equiv [-1, 1]$). The approach used by many authors to address this problem is to find necessary conditions of the first or second order (cfr. [1], [4], [6], [10]). Usually, in mathematical economics, in order to find candidates for optimality of infinite horizon problems under state constraints, the Pontryagin maximum principle is applied to unconstrained infinite horizon problems and then one considers as candidates for optimal solutions only the trajectories that satisfy the state constraints. For instance consider the following problem: maximize $J(u) := \int_0^\infty e^{-\lambda t}(x_u(t) + u(t))\, dt$ over all trajectory-control pair satisfying $x'(t) = -au(t)$ and $x(0) = 1$, with $u(t) \in [-1, 1]$ a.e., subject to the state constraints $x(t) \leqslant 1$, where $a > \lambda > 0$. It is straightforward to show that, applying the maximum principle stated for unconstrained problems, any optimal trajectory-control pair satisfies either one of the following three relations: (i) $x^-(t) = 1 + at$ associated to $u^-(t) \equiv -1$; (ii) $x^+(t) = 1 - at$ associated to $u^+(t) \equiv +1$; (iii) $x^\pm(t) = (1 - at)\chi_{[0,\bar{t}]}(t) + (1 - a\bar{t} + a(t - \bar{t}))\chi_{(\bar{t},\infty)}(t)$ associated to $u^\pm(t) = \chi_{[0,\bar{t}]}(t) - \chi_{(\bar{t},\infty)}(t)$, for some $\bar{t} > 0$. Excluding now trajectories x^- and $x^\pm$, since are not feasible, then this analysis leads to deduce that x^+ is the only candidate for optimality. But this conclusion is not correct. Indeed, one can easily see that the feasible trajectory $\bar{x}(t) \equiv 1$ associated to the control $\bar{u}(t) \equiv 0$ verifies $J(\bar{u}) > J(u^+)$ (cfr. Example 5.6).

Necessary conditions in form of the maximum principle and partial sensitivity relations, for infinite horizon problems under smooth functional constraints, such as $h(t, x(t)) \geqslant 0$ are known. Roughly speaking (cfr. [8]) if $(\bar{x}, \bar{u})$ is optimal at (t_0, x_0) for the problem

$$\begin{cases} \text{maximize } \int_{t_0}^{\infty} L(t, x(t), u(t))\, dt & \\ x'(t) = f(t, x(t), u(t)) & \text{a.e. } t \in [t_0, \infty) \\ x(t_0) = x_0 & \\ u(t) \in U & \text{a.e. } t \in [t_0, \infty) \\ h(t, x(t)) \geqslant 0 & t \in [t_0, \infty), \end{cases}$$

with U closed convex subset of $\mathbb{R}^m$, $h \in C^2$, f and L continuous and with continuous partial derivatives with rspect to x and u, then there exist $q^0 \in \{0, 1\}$, a co-state $q(\cdot)$, and a nondecreasing function $\mu(\cdot)$, constant on any interval where $h(t, \bar{x}(t)) > 0$, such that $(q^0, q(t_0)) \neq (0, 0)$, $\mu(t_0) = 0$ and $q(\cdot)$ satisfies

$$q(t) = q(t_0) - \int_{t_0}^{t} \nabla_x H(s, \bar{x}(s), q(s), \bar{u}(s))\, ds - \int_{[t_0,t]} \nabla_x h(s, \bar{x}(s))\, d\mu(s)$$

and the maximum principle

$$H(t, \bar{x}(t), q(t), \bar{u}(t)) = \max_{u \in U} H(t, \bar{x}(t), q(t), u) \;\; \text{a.e. } t \in [t_0, \infty),$$

where $H(t, x, p, u) := \langle p, f(t, x, u) \rangle + q^0 L(t, x, u)$. Furthermore in [5], using the language of calculus of variations, the authors provide partial results on sensitivity relations showing that, if A is convex and $\operatorname{int} A \neq \emptyset$, then for any optimal trajectory $\bar{x}(\cdot)$ of the problem $\mathscr{B}_\infty$ there exists an absolutely continuous function $q(\cdot)$, solving the adjoint equation, such that $q(t) \in \partial_x V(t, \bar{x}(t))$ for all $t \in [t_0, \infty)$.

In the present work, for the first time we provide partial and full sensitivity relations, together with a transversality condition at the initial time, under mild assumption on the dynamic and constraint set. To describe our results, assume for the sake of simplicity that $L(t, x, u) = e^{-\lambda t} l(x, u)$ is regular, $U(\cdot) \equiv U$ is a closed subset of $\mathbb{R}^m$, $V(t, \cdot)$ is regular, and denote by $N_A(y)$ the limiting normal cone to A at y. If $(\bar{x}, \bar{u})$ is optimal for $\mathscr{B}_\infty$ at (t_0, x_0) then (cfr. Theorem 4.3 below) there exist an absolutely continuous co-state $p(\cdot)$, a nonnegative locally finite Borel measure μ on $[t_0, \infty)$ and a Borel measurable selection $\nu(\cdot) \in \overline{\text{co}}\, N_A(\bar{x}(\cdot)) \cap \mathbb{B}$ such that $p(\cdot)$ satisfies the adjoint equation

$$-p'(t) = \mathrm{d}_x^* f(t, \bar{x}(t), \bar{u}(t))\, (p(t) + \eta(t)) - e^{-\lambda t} \nabla_x l(\bar{x}(t), \bar{u}(t)) \quad \text{a.e. } t \in [t_0, \infty),$$

the maximality condition

$$\begin{aligned}&\langle\, p(t)+\eta(t), f(t,\bar{x}(t),\bar{u}(t))\rangle - e^{-\lambda t} l(\bar{x}(t),\bar{u}(t))\\&= \max_{u\in U}\left\{\langle\, p(t)+\eta(t), f(t,\bar{x}(t),u)\rangle - e^{-\lambda t} l(\bar{x}(t),u)\right\} \quad \text{a.e. } t\in[t_0,\infty),\end{aligned}$$

and the transversality and sensitivity relations

$$-p(t_0)=\nabla_x V(t_0,\bar{x}(t_0)), \quad -(p(t)+\eta(t))=\nabla_x V(t,\bar{x}(t)) \text{ a.e. } t\in(t_0,\infty) \tag{3}$$

where $\eta(t_0)=0$ and $\eta(t)=\int_{[t_0,t]}\nu(s)\,d\mu(s)$ for ll $t\in(t_0,\infty)$. Observe that if $\bar{x}(\cdot)\in \operatorname{int} A$ then $\nu(\cdot)\equiv 0$ and the usual maximum principle holds true. But if $\bar{x}(t)\in\partial A$ for some time t, then a measure multiplier factor, $\int_{[0,t]}\nu\,d\mu$, may arise modifying the adjoint equation. Furthermore, the transversality condition and sensitivity relation in (3) lead to a significant economic interpretation (cfr. [2], [9]): the co-state $p+\eta$ can be regarded as the "shadow price"or "marginal price", i.e. (3), describes the contribution to the value function (the optimal total utility) of a unit increase of capital x.

The outline of the paper is as follows. In Section 2, we provide basic definitions, terminology, and facts from nonsmooth analysis. In Section 3, we give a bound on the total variation of measures associated to Mayer problems. In Section 4, we focus on the main result, investigating the problem $\mathscr{B}_\infty$ and stating sensitivity relations and transversality condition on the co-state. Finally, in the last Section, it is proved the uniform locally Lipschitz continuity of a large class of value functions.

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