Robust terminal tracking problem for one-dimensional heat equation

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Ort Braude College of Engineering

- Location: Karmiel, Galilee, Israel
- Area: 120000 m^2.
- Students: 5500
- Undergraduate programs (B. Sc.):
    - Software engineering;
    - Information systems;
    - Industrial engineering;
    - Electrical engineering;
    - Mechanical engineering;
    - Biotechnology engineering;
    - Applied mathematics;
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System

- Heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + p(x,t) + q(x,t) \, , \; (x,t) \in \Omega = \{0 < x < 1, 0 < t \leq t_f\}$$

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Control,
$p(\cdot) \in L_2(\Omega)$

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Control
$p(\cdot) \in L_2(\Omega)$

Disturbance
$q(\cdot) \in L_2(\Omega)$

System

- Heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + p(x,t) + q(x,t) \ , \ (x,t) \in \Omega = \{0 < x < 1, 0 < t \leq t_f\}$$

- Initial condition:

$$u(x,0) = g(x), \quad g(\cdot) \in L_2[0,1]$$

- Boundary conditions:

$$u_x(0,t) = 0, \quad u_x(1,t) + Au(1,t) = 0, \quad A > 0$$

Robust terminal tracking problem

For any $\zeta > 0$ and for a given $\nu > 0$, to construct a feedback strategy $p_{\zeta\nu}(x,t,u)$ such that

$$G(u(\cdot)) = \int_0^1 [u(x,t_f) - u^*(x)]^2 \, \mathrm{d}x \leq \zeta,$$

for any $q(x,t) \in L_2(\Omega)$, satisfying

$$\int_0^{t_f} \int_0^1 q^2(x,t) \mathrm{d}x \mathrm{d}t < \nu,$$

$u^*(x)$ is a desired terminal temperature profile.

Auxiliary cost functional

$$J_{\alpha\beta} = J_{\alpha\beta}(p(\cdot), q(\cdot)) =$$

$$\int_0^1 [u(x,t_f) - u^*(x)]^2 \, \mathrm{d}x \; + \alpha \int_0^{t_f} \int_0^1 p^2(x,t) \mathrm{d}x\mathrm{d}t \; - \beta \int_0^{t_f} \int_0^1 q^2(x,t) \mathrm{d}x\mathrm{d}t$$

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- Terminal tracking term

Auxiliary cost functional

$$J_{\alpha\beta} = J_{\alpha\beta}(p(\cdot), q(\cdot)) =$$

$$\int_0^1 [u(x,t_f) - u^*(x)]^2 \, dx \; + \alpha \int_0^{t_f} \int_0^1 p^2(x,t) dx dt \; - \beta \int_0^{t_f} \int_0^1 q^2(x,t) dx dt$$

- Terminal tracking term
- First player's control effort

Auxiliary cost functional

$$J_{\alpha\beta} = J_{\alpha\beta}(p(\cdot), q(\cdot)) =$$

$$\int_0^1 [u(x,t_f) - u^*(x)]^2 \,\mathrm{d}x + \alpha \int_0^{t_f}\int_0^1 p^2(x,t)\mathrm{d}x\mathrm{d}t - \beta \int_0^{t_f}\int_0^1 q^2(x,t)\mathrm{d}x\mathrm{d}t$$

- Terminal tracking term
- First player's control effort
- Second player's control effort

Auxiliary cost functional

$$J_{\alpha\beta} = J_{\alpha\beta}(p(\cdot), q(\cdot)) =$$

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- Terminal tracking term
- First player's control effort
- Second player's control effort
- $\alpha, \beta > 0$ - penalty coefficients

Auxiliary linear-quadratic differential game

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = p(x,t) + q(x,t), \quad (x,t) \in \Omega \\ u(x,0) = g(x), \\ u_x(0,t) = 0, \quad u_x(1,t) + Au(1,t) = 0 \end{cases}$$

$$J_{\alpha\beta} \to \min_{p(\cdot)} \max_{q(\cdot)}$$

LQDG solution

- Optimal feedback strategies:

$$p^0_{\alpha\beta}(x,t,u) = -\frac{1}{2\alpha} l^0_{\alpha\beta}, \quad q^0_{\alpha\beta}(x,t,u) = \frac{1}{2\beta} l^0_{\alpha\beta},$$

$$l^0_{\alpha\beta}(x,t,u) = r_{\alpha\beta} + \int_0^1 \left[R_{\alpha\beta}(x,s,t) + R_{\alpha\beta}(s,x,t)\right]\left[u(s,t) - u^*(s)\right] \mathrm{d}s$$

- Game value:

$$J^0_{\alpha\beta} = \int_0^1 \int_0^1 R_{\alpha\beta}(x,s,t)\left[u(x,t) - u^*(x)\right]\left[u(s,t) - u^*(s)\right] \mathrm{d}s\mathrm{d}x +$$

$$\int_0^1 r_{\alpha\beta}(x,t)\left[u(x,t) - u^*(x)\right] \mathrm{d}x + \rho_{\alpha\beta}(t)$$

LQDG solution: $R_{\alpha\beta}(x, s, t)$

- Riccati-type equation:

$$R_t(x,s,t) + R_{xx}(x,s,t) + R_{xx}(s,x,t) = \frac{\beta - \alpha}{4\alpha\beta} R_1(x,s,t),$$

$$R_1(x,s,t) = \int_0^1 \left[R(y,s,t) + R(s,y,t)\right]\left[R(y,x,t) + R(x,y,t)\right] \mathrm{d}y,$$

- Terminal condition: $R(x,s,t_f) = \delta(s-x)$
- Boundary conditions for $x = 0$: $R_x(0,s,t) = R_x(s,0,t) = 0$
- Boundary conditions for $x = 1$:
  $R_x(1,s,t) + AR(1,s,t) = R_x(s,1,t) + AR(s,1,t) = 0$

LQDG solution: $r_{\alpha\beta}(x,t)$

- Linear equation:

$$r_t(x,t)+r_{xx}(x,t)+R_2(x,t) = \frac{\beta-\alpha}{4\alpha\beta}\int_0^1 \left[R(x,s,t)+R(s,x,t)\right] r(s,t)\mathrm{d}s,$$

$$R_2(x,t) = \int_0^1 \left[R_{ss}(x,s,t)+R_{ss}(s,x,t)\right] u^*(s)\mathrm{d}s,$$

- Terminal condition: $r(x,t_f)=0$
- Boundary condition for $x=0$: $r_x(0,t)=0$
- Boundary condition for $x=1$: $r_x(1,t)+Ar(1,t)=0$

LQDG solution: $\rho_{\alpha\beta}(t)$

- Trivial equation:

$$\rho_t(t) + \int_0^1 r_{xx}(x,t)u^*(x)\mathrm{d}x = \frac{\beta-\alpha}{4\alpha\beta}\int_0^1 r^2(x,t)\mathrm{d}x,$$

- Terminal condition: $\rho(t_f) = 0$

Sturm-Liouville problem

$$X''(x) + \lambda^2 X(x) = 0,$$

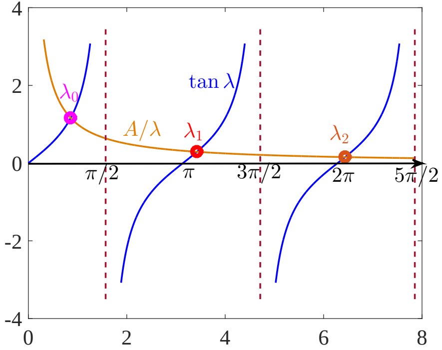
$$X'(0) = 0, \quad X'(1) + AX(1) = 0$$

$$X_i(x) = \frac{1}{\sqrt{\omega_i}} \cos \lambda_i x, \; i = 0, 1, ...,$$

$$\omega_i = \frac{1}{2} + \frac{\sin 2\lambda_i}{4\lambda_i},$$

$\lambda_i \in (\pi i, \pi i + \pi/2), \; i = 0, 1, ...$

satisfy $\lambda \tan \lambda = A$



Series solution: $R_{\alpha\beta}(x,s,t)$

$$\kappa = \frac{\alpha\beta}{\beta-\alpha},\ u_i^* = \int\limits_0^1 u^*(x)X_i(x)\mathrm{d}x$$

$$R_{\alpha\beta}(x,s,t) = \sum_{i=0}^{\infty} R_i(t)X_i(x)X_i(s),$$

$$R_i(t) = \frac{2\kappa\lambda_i^2 \exp\left(-2\lambda_i^2(t_f-t)\right)}{2\kappa\lambda_i^2 + 1 - \exp\left(-2\lambda_i^2(t_f-t)\right)}$$

Solvability condition:

$$2\kappa\lambda_i^2 + 1 - \exp\left(-2\lambda_i^2(t_f-t)\right) > 0\ \forall i = 0,1,...,\ \forall\ t \in [0,t_f] \Leftrightarrow \alpha < \beta$$

Series solution: $r_{\alpha\beta}(x,t)$, $\rho_{\alpha\beta}(t)$

$$\kappa = \frac{\alpha\beta}{\beta-\alpha}, \; u_i^* = \int_0^1 u^*(x) X_i(x) \mathrm{d}x$$

$$r_{\alpha\beta}(x,t) = \sum_{i=0}^{\infty} r_i(t) X_i(x)$$

$$\frac{dr_i}{dt} = \left(\lambda_i^2 + \frac{1}{\kappa} R_i(t)\right) r_i + 2\lambda_i^2 u_i^* R_i(t), \quad r_i(t_f) = 0$$

$$\rho_{\alpha\beta}(t) = -\int_t^{t_f} \left[\sum_{i=0|}^{\infty} \left(\lambda_i^2 u_i^* r_i(t) + \frac{1}{4\kappa} r_i^2(t) \right) \right] \mathrm{d}t$$

Series solution of equation

$$u_i^* = \int\limits_0^1 u^*(x)X_i(x)\mathrm{d}x, \quad q_i(t) = \int\limits_0^1 q(x,t)X_i(x)\mathrm{d}x, \quad g_i = \int\limits_0^1 g(x)X_i(x)\mathrm{d}x$$

$$u(x,t) = \sum_{i=1}^{\infty} u_i(t)X_i(x)$$

$$\frac{du_i}{dt} = -\left(\lambda_i^2 + \frac{1}{\alpha}R_i(t)\right)u_i + \frac{u_i^*}{\alpha}T_i(t) - \frac{1}{2\alpha}r_i(t) + q_i(t),$$

$$u_i(0) = g_i$$

Robust tracking problem solution

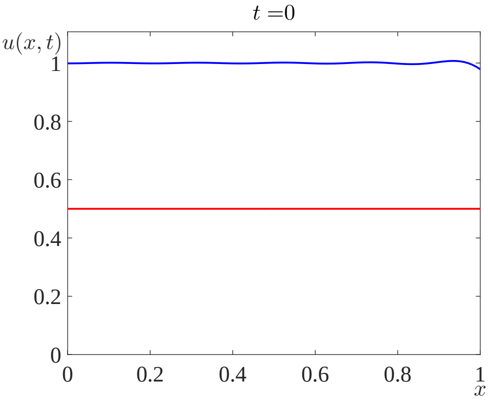
Theorem 1

For any $\zeta > 0$ *and for a given* $\nu > 0$*, there exist sufficiently small* $\beta^* = \beta^*(\zeta, \nu)$ *and* $\alpha^* = \alpha^*(\zeta, \nu) < \beta^*$ *such that:*
for all $\alpha < \alpha^*$, $\beta < \beta^*$, *the LQDG optimal strategy* $p^0_{\alpha\beta}(x, t, u)$ *guarantees*

$$G(u(\cdot)) = \int_0^1 [u(x, t_f) - u^*(x)]^2 \, \mathrm{d}x \leq \zeta,$$

Example

$t_f = 1,\ A = 1,\ q(x,t) = t\sin(10x),\ g(x) \equiv 1,\ u^*(x) \equiv 0.5,$
$\alpha = 0.001,\ \beta = 0.5,\ N = 10$



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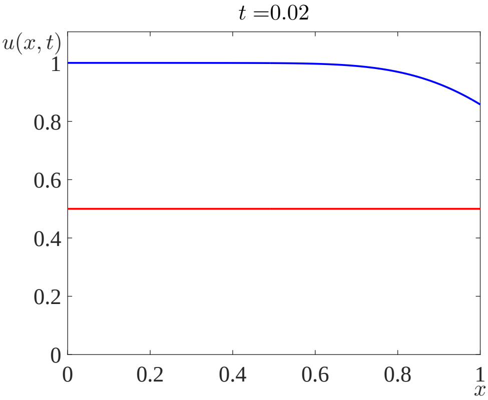

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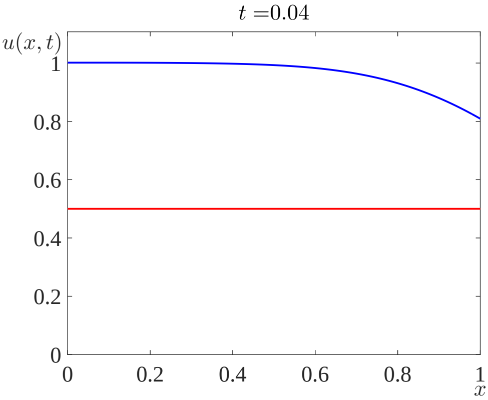

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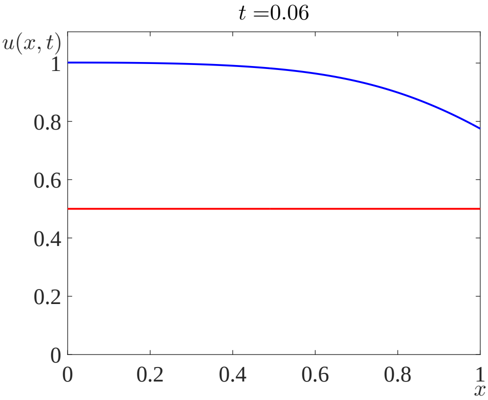

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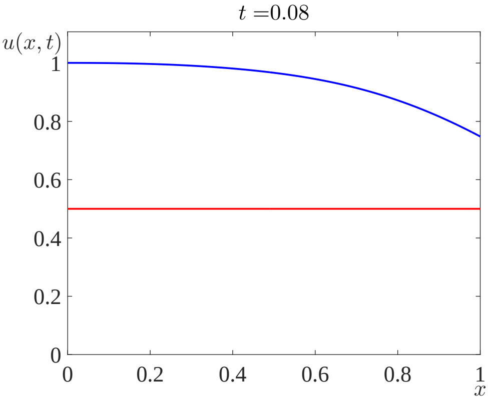

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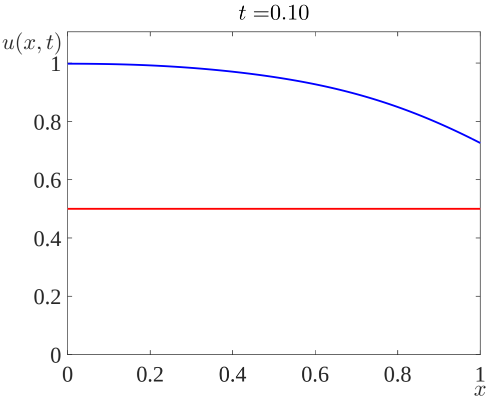

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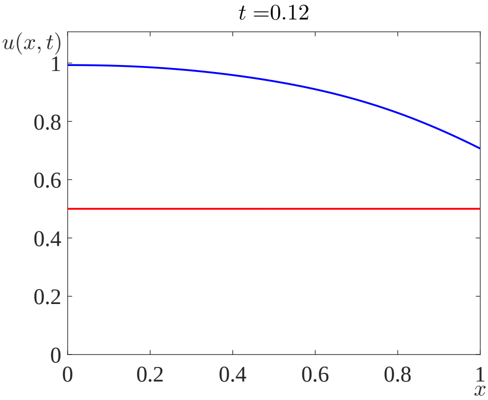

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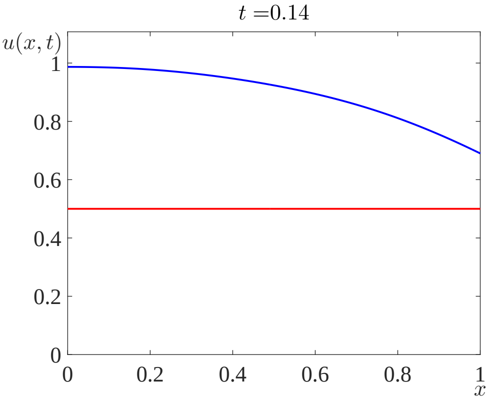

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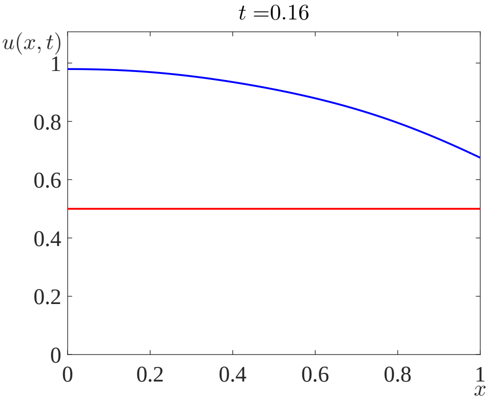

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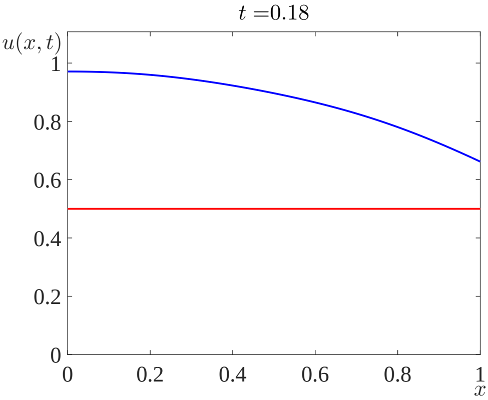

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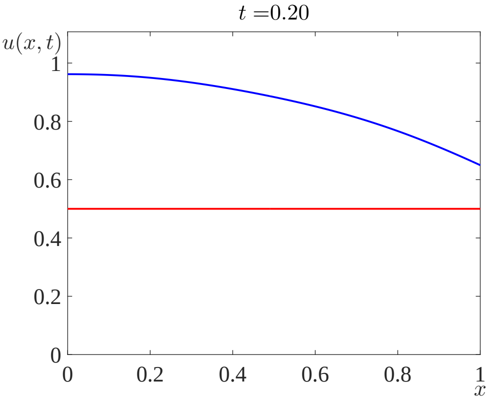

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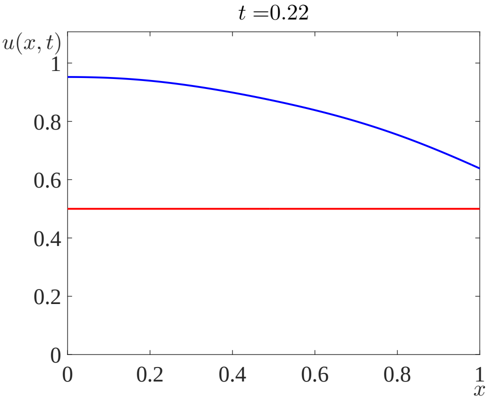

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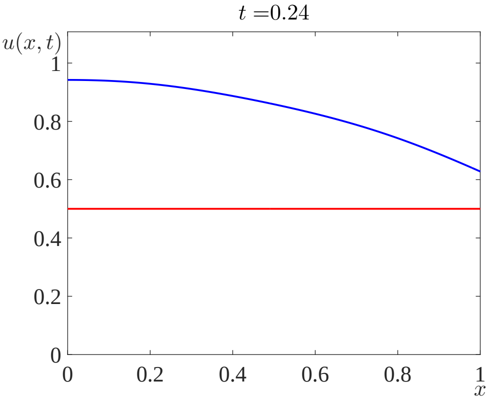

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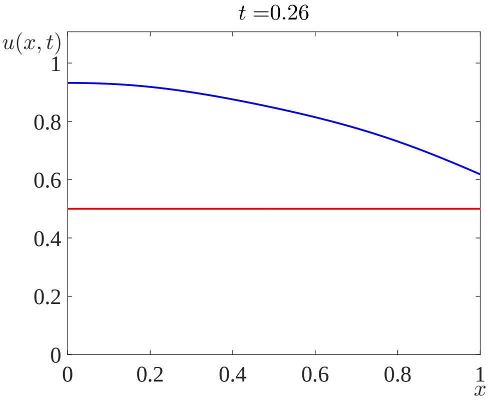

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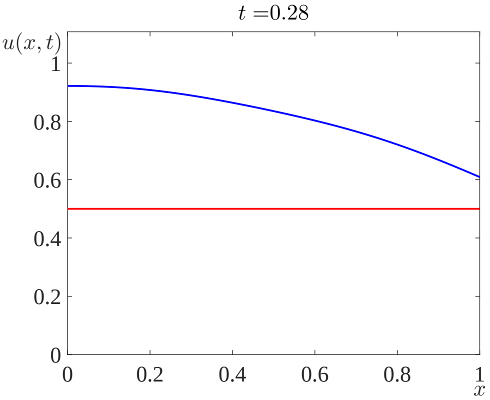

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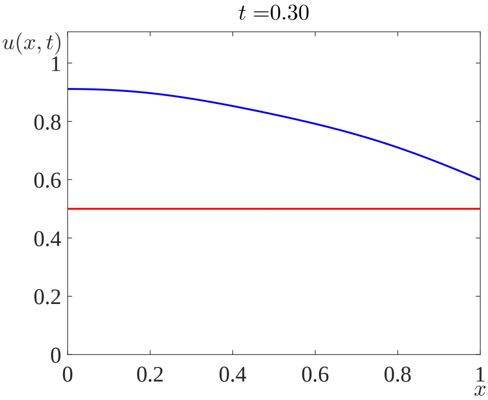

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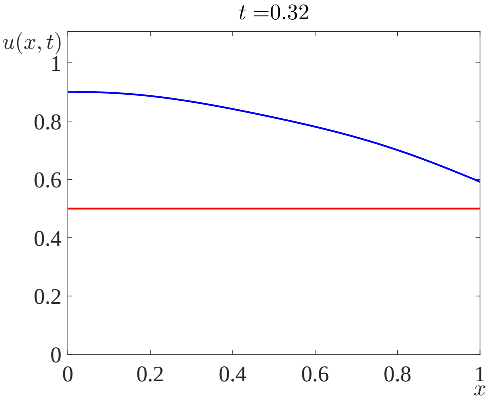

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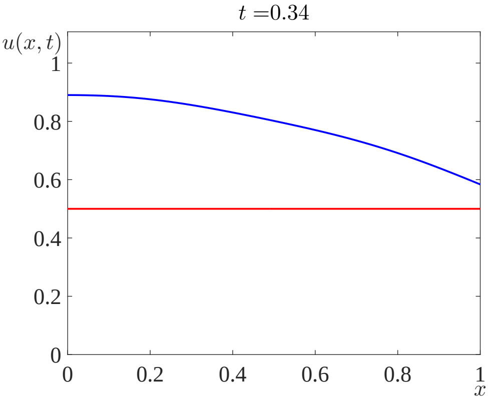

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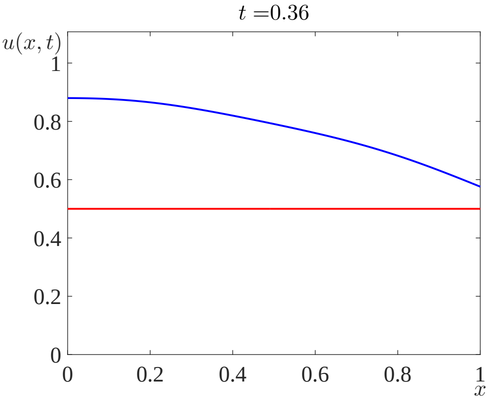

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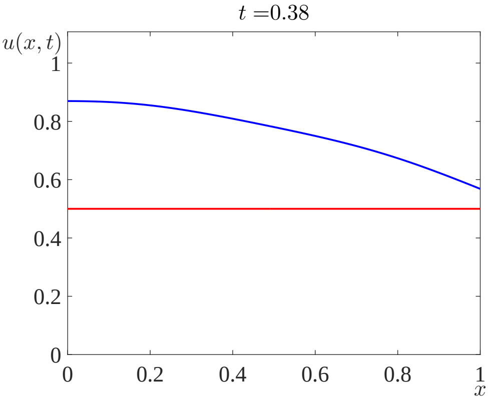

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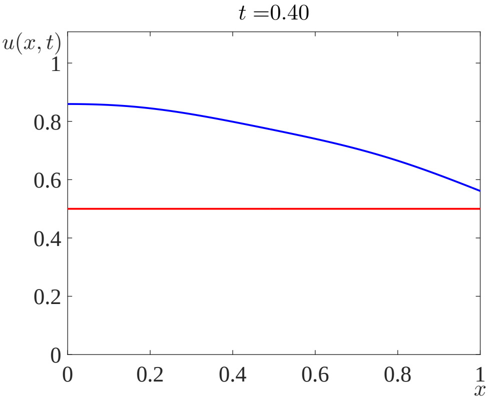

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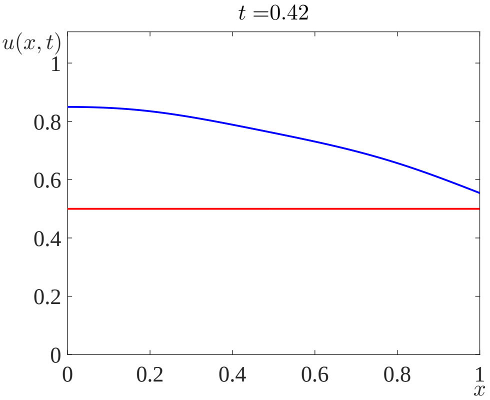

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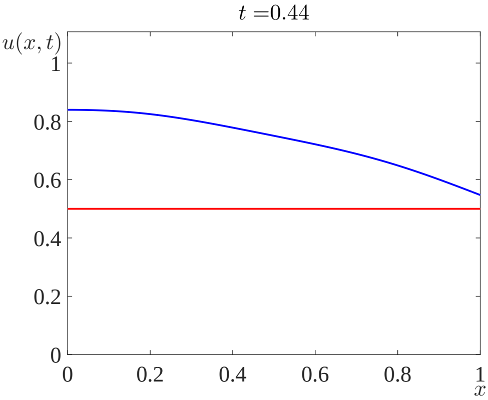

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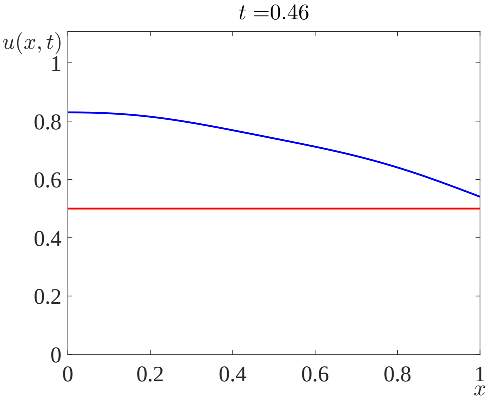

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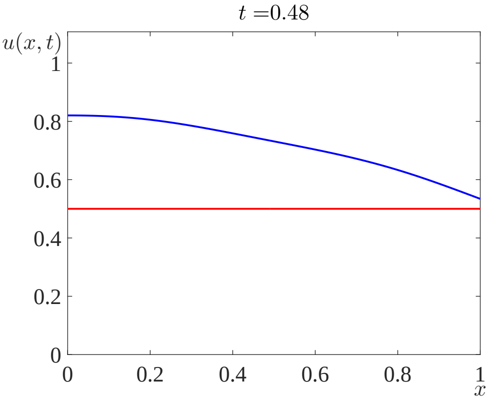

Example

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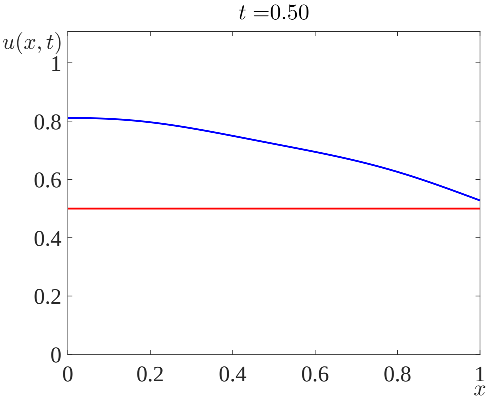

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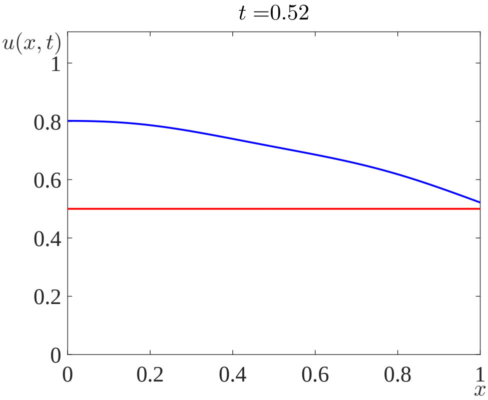

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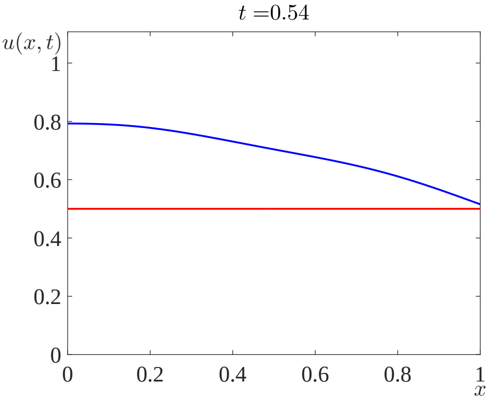

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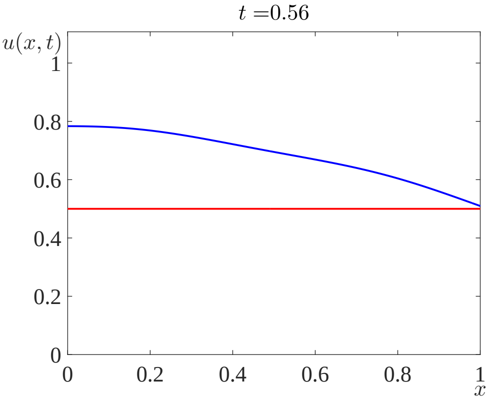

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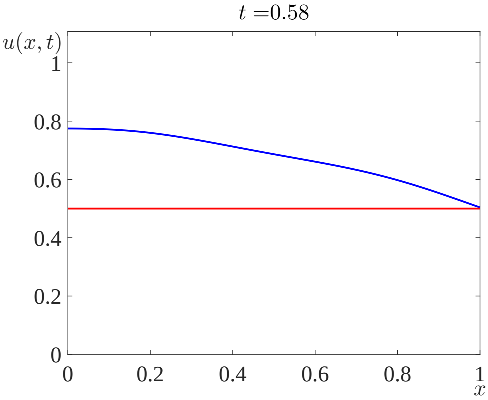

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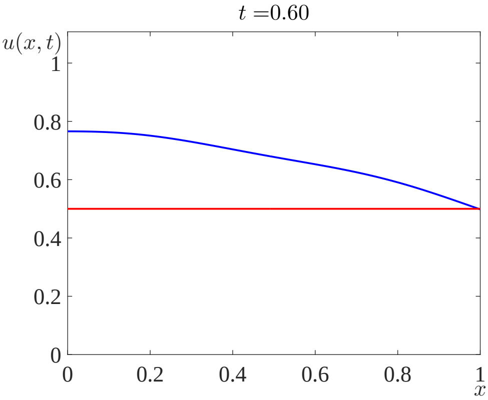

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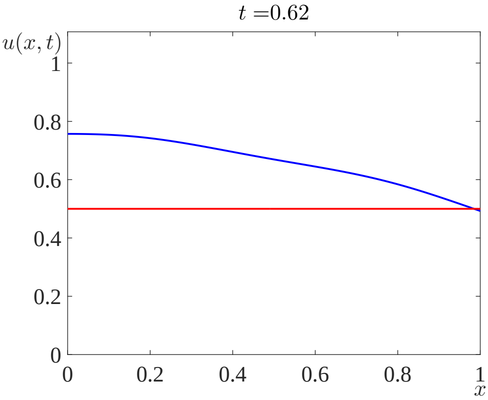

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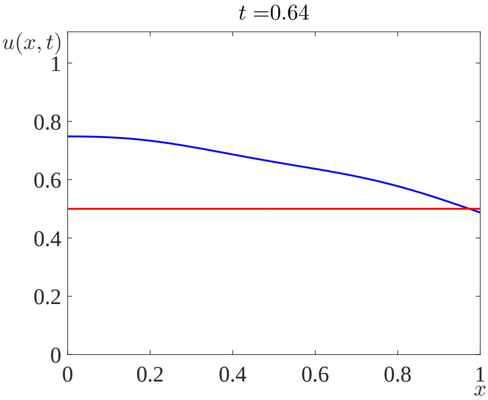

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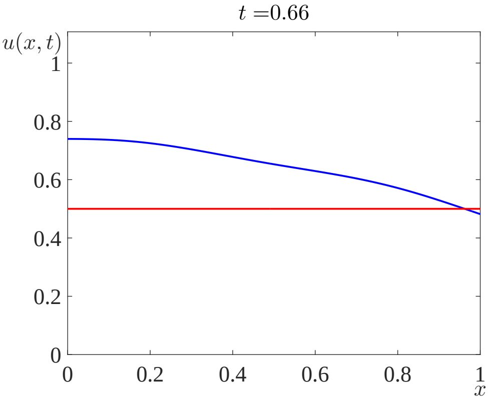

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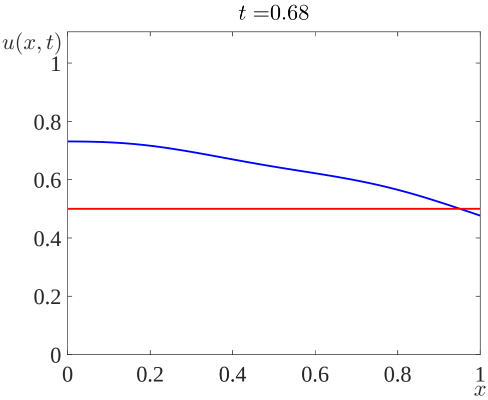

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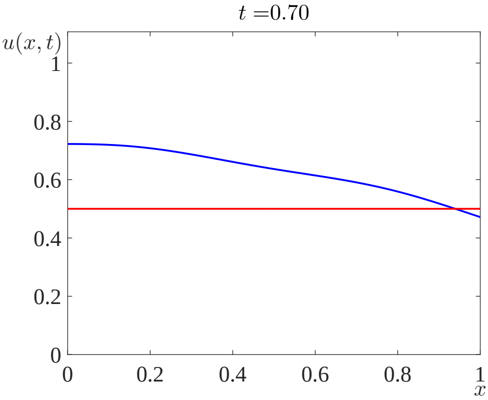

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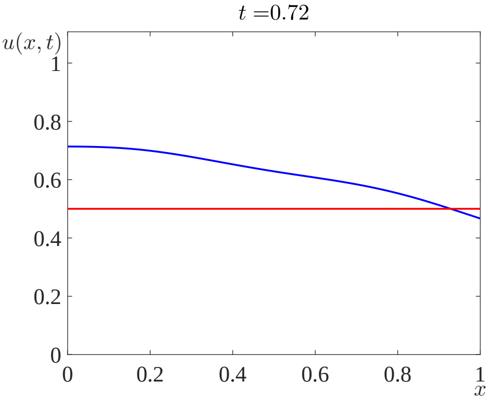

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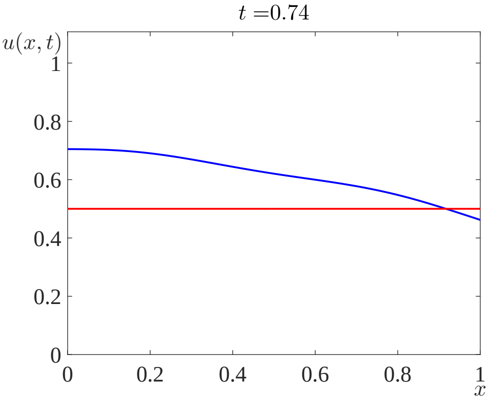

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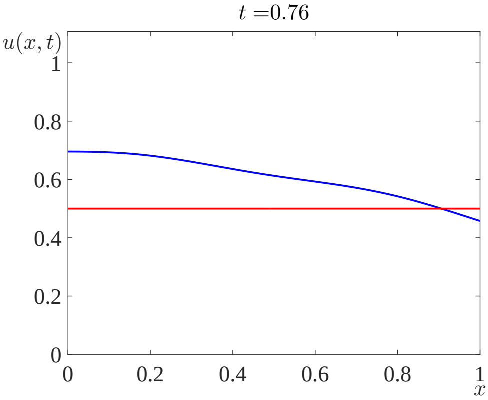

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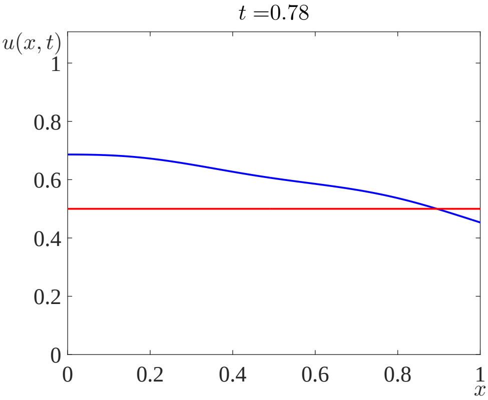

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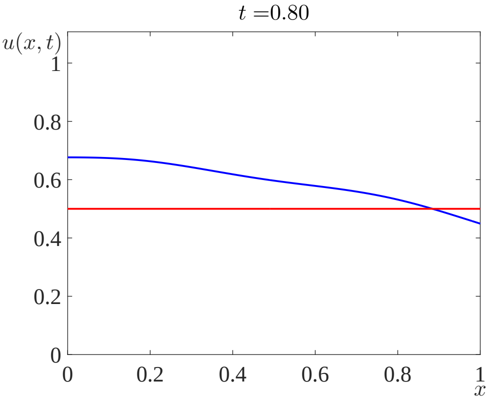

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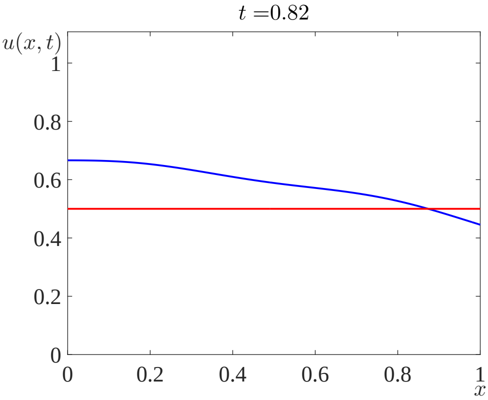

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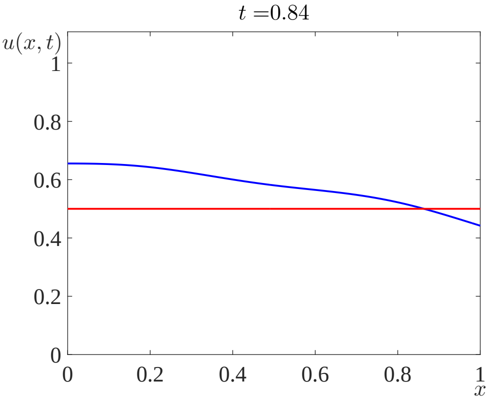

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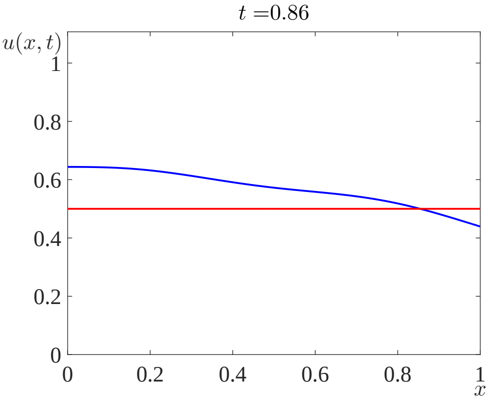

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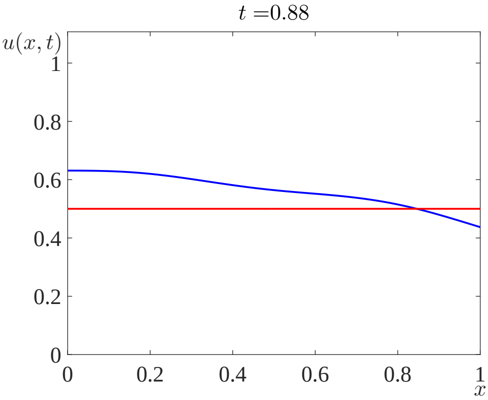

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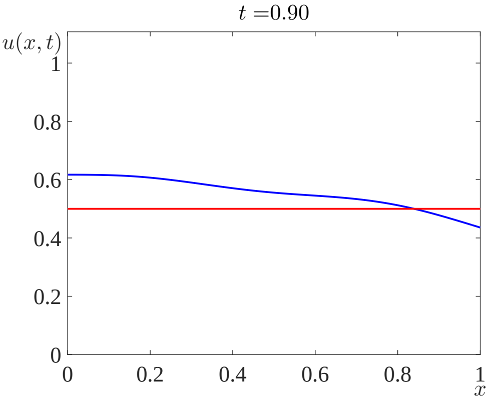

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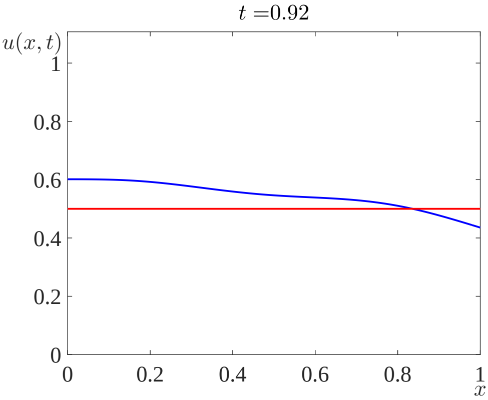

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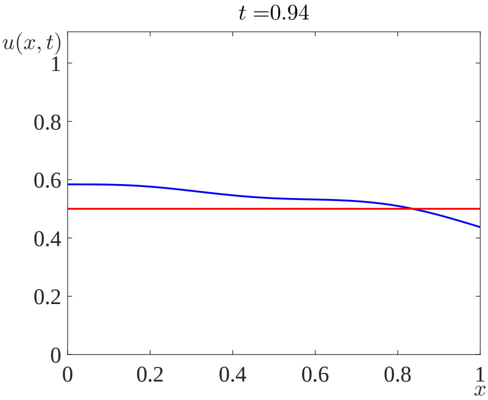

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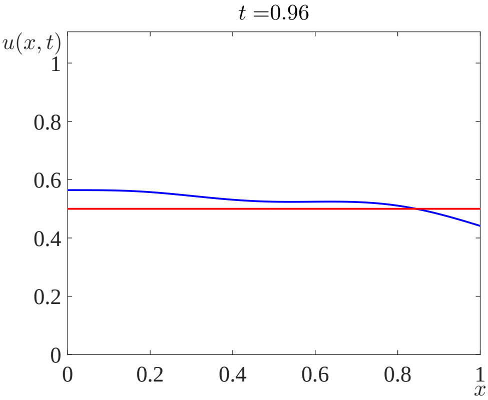

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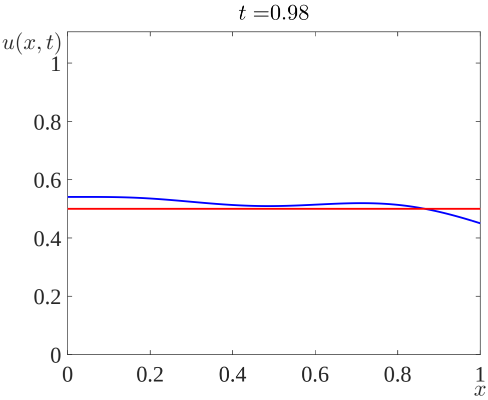

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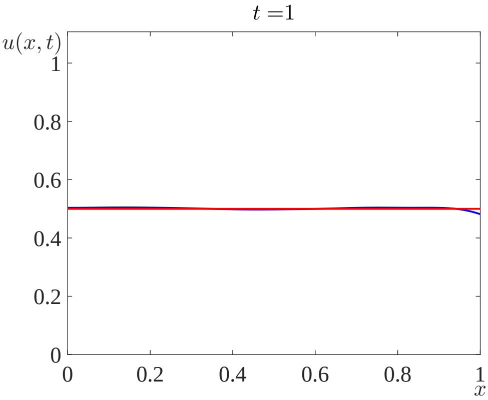

Possible generalizations

- More complicated tracking functional

$$G(u(\cdot)) =$$

$$\sum_{k=1}^{K} \int_{t_k}^{t_{k+1}} \int_{0}^{1} \left[u(x,t) - u_k^*(x,t)\right] dxdt + \sum_{m=1}^{M} \int_{0}^{1} \left[u(x,t_m) - u_m^*(x)\right] dx$$

- Other linear equations:

$$\frac{\partial u}{\partial t} = \mathcal{A}u + p(x,t) + q(x,t)$$

- Dimension > 1

Thank you

for your

attention!

Questions?