Shape optimization under uncertainty

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Joint works

- J.C. Bellido, G. Buttazzo, B. Velichkov: *Worst-case shape optimization for the Dirichlet energy.* Nonlinear Analysis (2017).

- G. Buttazzo, B. Velichkov: *A shape optimal control problem and its probabilistic counterpart.* Paper submitted.

- G. Buttazzo, F. Maestre, B. Velichkov: *Optimal potentials for problems with changing sign data.* Paper in preparation.

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Shape optimal control problem:

- state equation is

$$-\Delta u = f \text{ in } \Omega, \qquad u \in H_0^1(\Omega);$$

- state variable is $u \in H_0^1(\mathbb{R}^d)$ (extended by zero outside Ω);
- control variable is the domain Ω;
- cost function is of the form

$$\int_\Omega j(x,u)\,dx$$

- class of admissible controls is

$$\mathcal{A} = \Big\{\Omega \subset D,\ |\Omega| \le m\Big\},$$

where D is a fixed bounded domain of $\mathbb{R}^d$.

The problem is to study the existence of an optimal domain; ther is a competition:

homogenization vs shape optimization

In general homogenization wins and no optimal domain exists, since minimizing sequences tend to create fine perforations (Cioranescu-Murat example) and optimal solution exist only in a suitable relaxed sense (capacitary measures introduced by Dal Maso-Mosco 1987). However, in some cases optimal shapes exist.

A first situation in which optimal shapes exist is when geometrical constraints are added to admissible controls, as for instance:

convexity, equi-Lipschitz condition, equi-bounded perimeter, uniform exterior cone condition, uniform capacity condition, uniform Wiener estimates, topological conditions (in dim. 2)...

that rule out the homogenization. In our case we only have the Lebesgue measure constraint $\{|\Omega| \le m\}$ which is not sufficient to provide enough compactness to enforce the existence of an optimal Ω.

Another case in which the existence of an optimal domain occurs is when the cost functional verifies a monotonicity condition.

Theorem [Buttazzo-Dal Maso (ARMA 1993)]
Let $F(\Omega)$ be such that:

- F *is* γ*-lower semicontinuous;*
- F *is decreasing for set inclusion.*

Then the shape optimization problem

$$\min\Big\{F(\Omega)\ :\ |\Omega|\le m\Big\}$$

admits a solution Ω_{opt}*, and* $|\Omega_{opt}|=m$*.*

Let us stress that the monotonicity condition above is rather restrictive and, even if some interesting problems (spectral optimization) verify it, in the linear quadratic case

$$F(u,\Omega)=\int_\Omega |u-u_0|^2\,dx$$

homogenization wins (i.e. no existence of Ω_{opt}).

We consider the case when the cost integrand j is linear; if R_Ω is the resolvent operator of the Dirichlet Laplacian in Ω, our problem can be rewritten as

$$\min\Big\{\int_\Omega h(x)R_\Omega(f)\,dx\ :\ |\Omega|\le m\Big\}.$$

Shape optimization under uncertainty on the right-hand side f; two possibilities:

• f is known with a given probability P on the space of data (stochastic optimization); we minimize the average cost

$$\mathcal{F}_{ave}(\Omega) = \int \Big[\int_\Omega h(x) R_\Omega(f)\, dx \Big] P(df)$$

in the admissible class $\{|\Omega| \le m\}$.

• Worst-case optimization: we optimize the worst case assuming the right-hand side f is known up to an error δ.

Worst case cost

$$\mathcal{F}_{wc}(\Omega) = \sup_{\|g\|_{L^2} \leq \delta} \left[\int_\Omega h(x) R_\Omega(f+g)\, dx \right]$$
$$= \sup_{\|g\|_{L^2} \leq \delta} \left[\int_\Omega R_\Omega(h)(f+g)\, dx \right]$$
$$= \int_\Omega R_\Omega(h) f\, dx + \delta \|R_\Omega(h)\|_{L^2}$$

Roughly speaking we are replacing the P-average by a supremum.

Monotonicity is lost, since the two terms behave in a different way.

Results

- for the stochastic case, there exists a solution Ω_{opt} (Buttazzo-Velichkov arxiv 2017) but the measure constraint could be not saturated, i.e. in general $|\Omega_{opt}| \leq 1$.

- for the worst case, there exists a solution Ω_{opt} provided the error δ is small enough (Bellido-Buttazzo-Velichkov Nonlinear Anal. 2017).

A numerical example

$$D = [0,1] \times [0,1], \quad p = 2, \quad \delta = 0.25$$

$$f = \begin{cases} 1 \text{ on } [0, \frac{1}{2}] \times [0,1] \\ 2 \text{ on } [\frac{1}{2}, 1] \times [0,1] \end{cases}$$

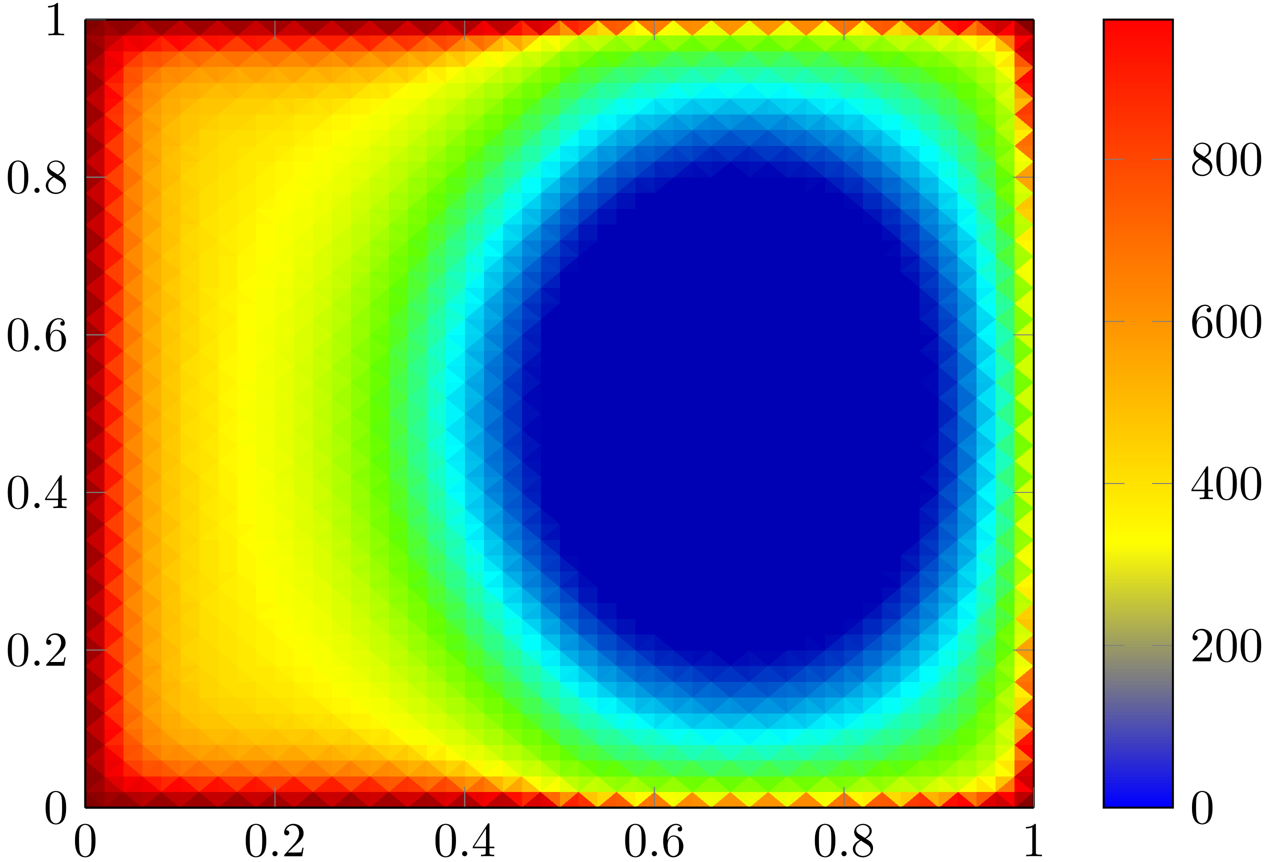
It is numerically convenient to simulate a domain Ω by a potential $V(x)$ taking the value 0 in Ω and $+\infty$ outside. The measure $|\Omega|$ is then simulated through the quantity

$$\int_D e^{-\alpha V(x)}\, dx \quad \text{with } \alpha \text{ small.}$$

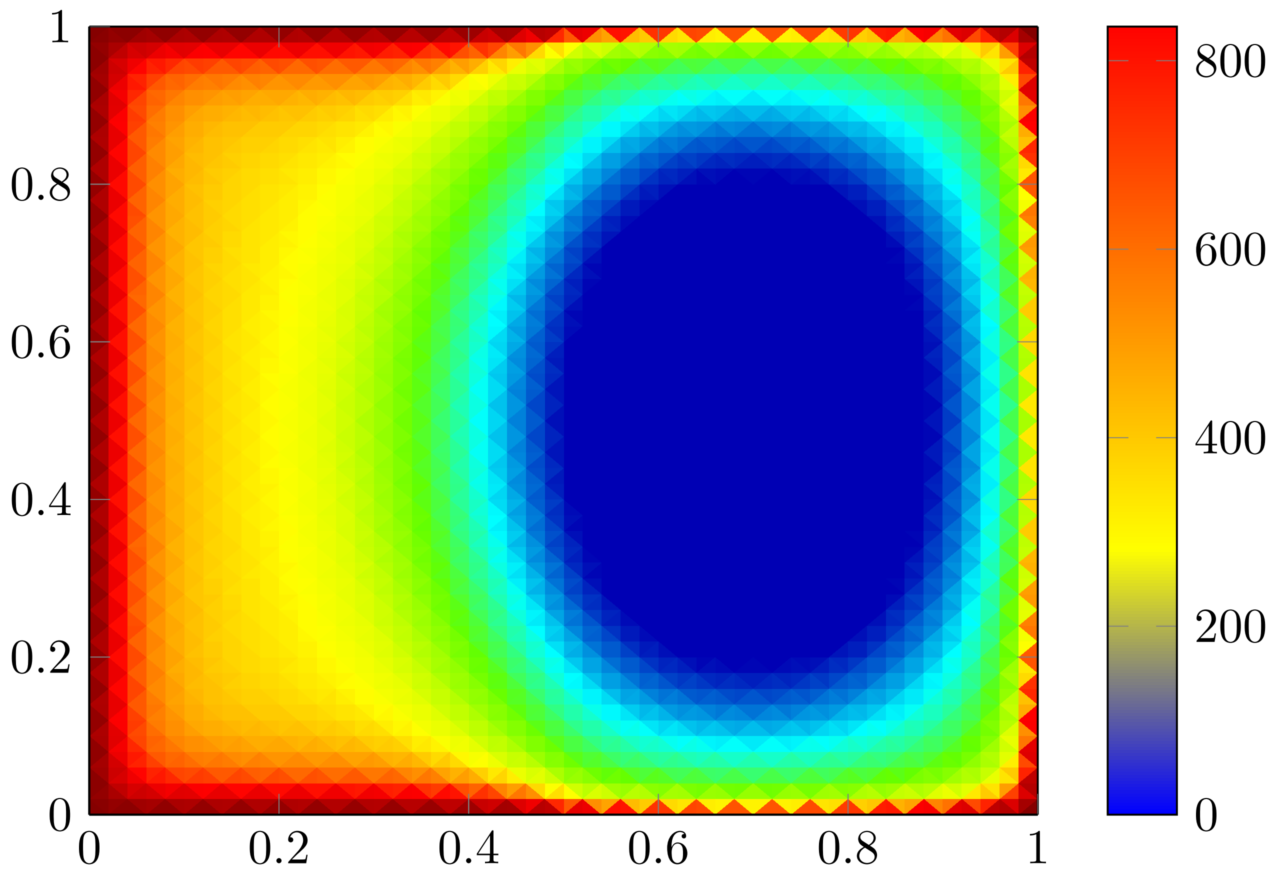
More precisely this approximation has to be stated in terms on Γ-convergence, proved in [BGRV, JEP 2014].

The simulation has been made by J.C. Bellido using:

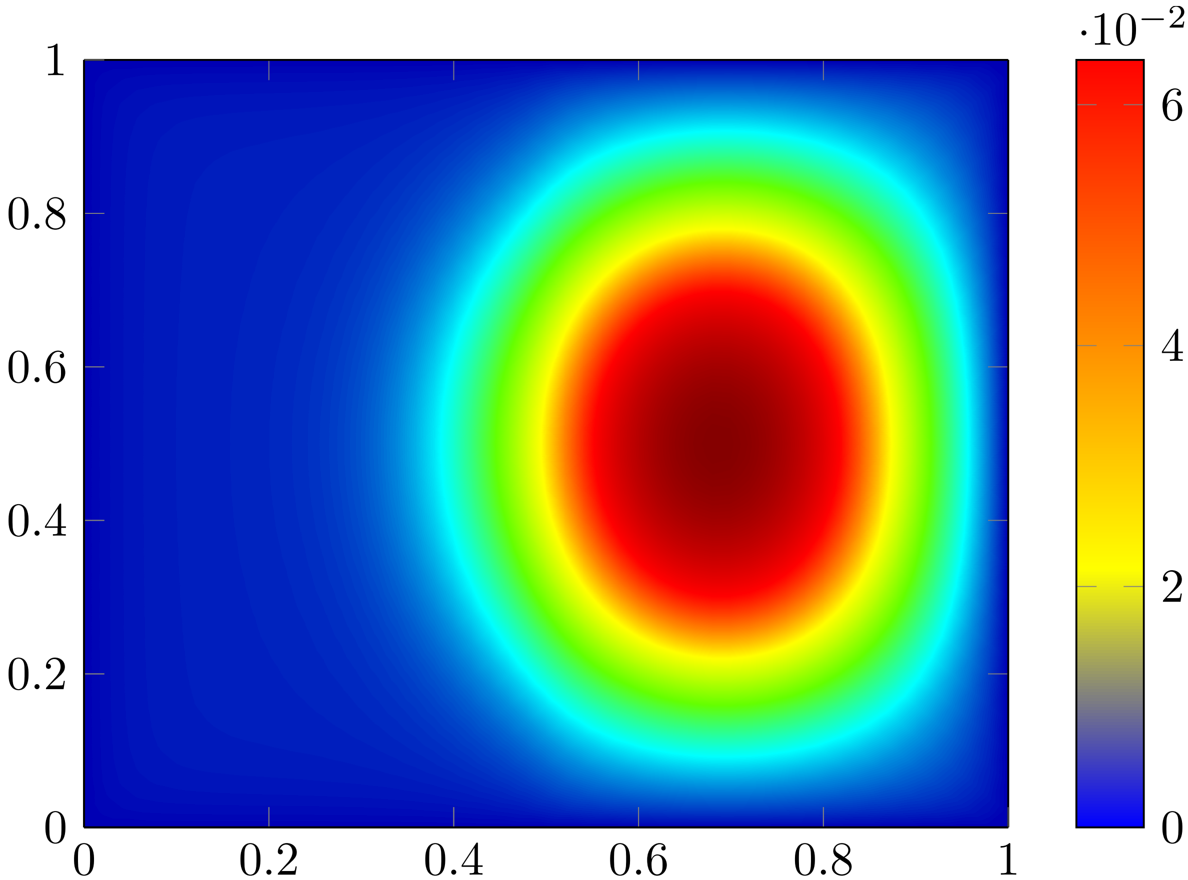
- `FreeFEM++`
- the *Method of Moving Asymptotes* (a kind of gradient method widely used for Topology and Structural Optimization problems)
- a mesh of 50×50 elements.



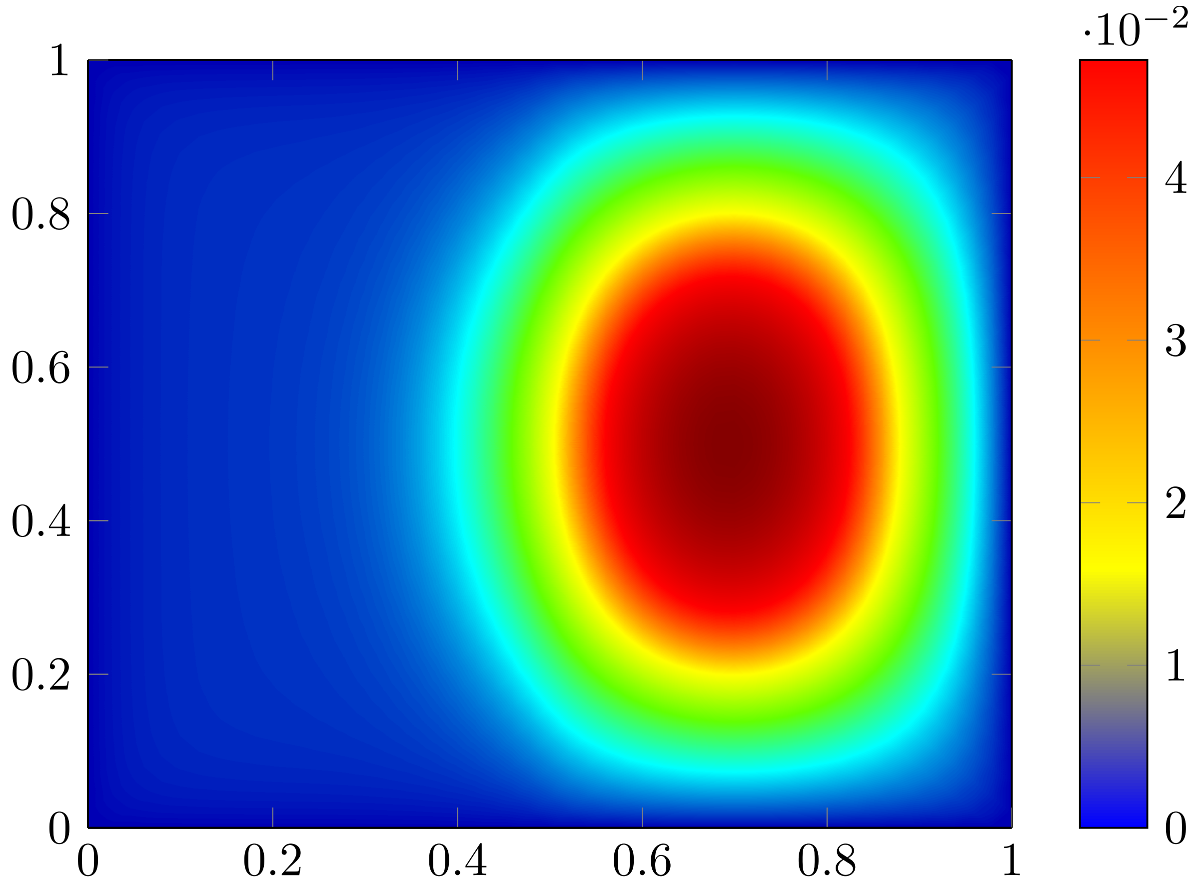
Optimal potential for the unperturbed case



Results for the perturbed case with $\delta = 0.25$



Optimal state for the unperturbed case



Optimal state for the perturbed case with $\delta = 0.25$