Regularity of optimal ship forms based on Michell's wave resistance

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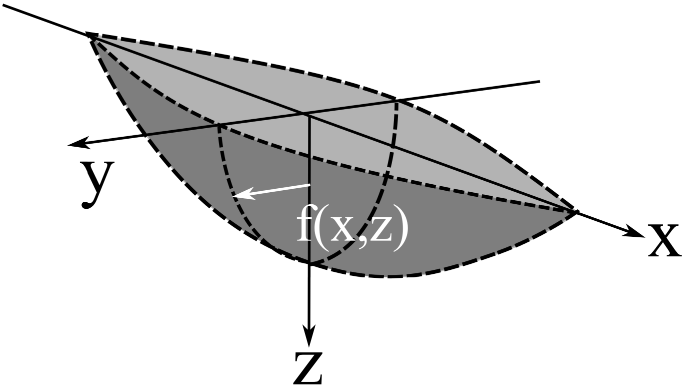
joint work with J. Dambrine

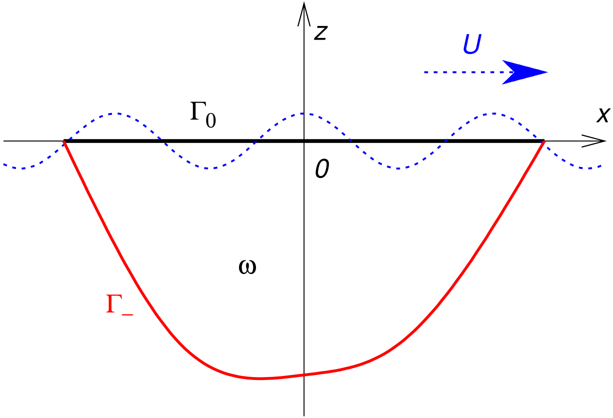
We use a simplified approach, where the resistance of water to the motion of a ship is represented as

$$R_{water} = R_{viscous} + R_{wave},$$

and R_{wave} is given by **Michell's formula (1898)**.


y
f(x,z)
x
z

The domain of parameters (x, z)



Consider a ship moving with constant velocity U on the surface of an unbounded fluid.

- coordinates xyz are fixed to the ship
- the xy-plane is the (undisturbed) water surface, z is vertically upward

The (half-)immerged hull surface is represented by a continuous nonnegative function

$$y = f(x, z) \geq 0, \quad (x, z) \in \omega,$$

with $f(x, z) = 0$ on Γ^- (= the boundary of ω under the surface)

- The fluid is incompressible, inviscid, the flow is irrotational
- A steady state has been reached
- Linearized theory (flow potential with linearized boundary conditions)
- Thin ship assumptions: $|\partial_x f| << 1$, $|\partial_z f| << 1$.

Michell's wave resistance is the **drag force** in this linearized model (recall d'Alembert's paradox !).

Experiments starting in the 1920's (**Wigley, Weinblum**): reasonable good agreement between Michell's theory and experiment (**Gotman'02**).

About the optimization problem for ω fixed

1st idea: finding a ship of minimal wave resistance among admissible functions $f : \omega \to \mathbf{R}_+$, for a constant speed U and a given volume V of the hull.

$f \mapsto R_{Michell}(f)$ is a positive semi-definite quadratic functional, **but** the problem above is **ill-posed** (**Sretensky'35, Krein'52**).

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Many authors proposed to add conditions and/or to work in finite dimension (**Weinblum'56, Kostyukov'68,. . .**)
Another approach, that we chose: add the **viscous resistance** which can be interpreted as a **regularization** (**Krein & Sizov'60 and '00, Lian-en'84, Michalski et al'87, Dambrine, P. & Rousseaux'15**)

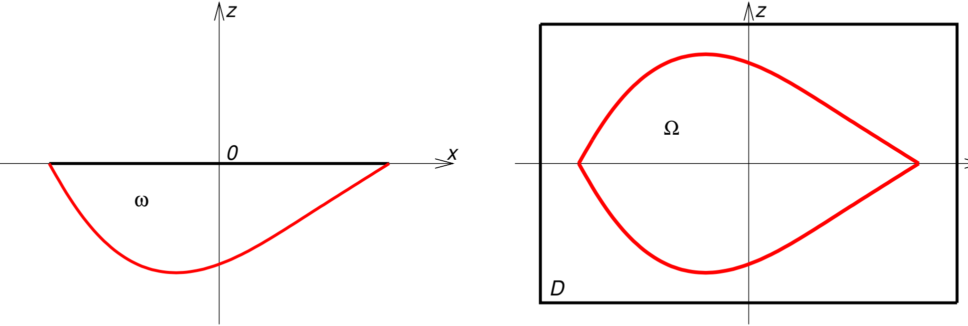

Figure: Symmetrization $z \mapsto -z$ and a possible bounding box

$f : \omega \to \mathbf{R}$ becomes $u : D \to \mathbf{R}$ with support Ω

The **normalized total resistance** is

$$J(u) = J_0(u) + J_{wave}(u), \quad (1)$$

where

$$J_0(u) = \int_D |\nabla u(x,z)|^2 dxdz \quad (2)$$

is the **normalized viscous resistance**, and

$$J_{wave}(u) = \int_D \int_D k(x,z,x',z')u(x,z)u(x',z')dxdzdx'dz' \geq 0 \quad (3)$$

is the **normalized wave resistance** functional. Here, $k : D \times D \to \mathbf{R}$ belongs to $L^q(D \times D)$ for some $q \in (1, +\infty]$ and satisfies the following symmetry assumptions:

$$k(x,z,x',z') = k(x',z',x,z) \quad (x,z,x',z') \in D \times D,$$

$$k(x,-z,x',z') = k(x,z,x',z') \quad (x,z,x',z') \in D \times D.$$

Formulation of the optimization problem

Let $V > 0$ (the volume of the hull) and $0 < a < |D|$ (the area of Ω).
Find an open and symmetric set $\Omega^\star$ such that

$$J(u_{\Omega^\star}) = \inf \{J(u_\Omega),\ \Omega \subset D \text{ open and symmetric},\ |\Omega| = a\}, \quad (4)$$

where u_Ω is uniquely defined by

$$J(u_\Omega) = \min \left\{ J(v),\ v \in H_0^1(\Omega),\ \check{v} = v,\ \int_\Omega v = V \right\}. \quad (5)$$

(We denote $\check{v}(x,z) = v(x,-z)$, $\forall (x,z) \in D$).
Two questions: *existence* **of** $\Omega^\star$ **and** *regularity* **of** $u_{\Omega^\star}$

Following a standard approach, we work with the space

$$\check{H} = \{u \in H_0^1(D),\ \check{u} = u \text{ a.e. in } D\},$$

which is a closed subspace of $H_0^1(D)$. For a function $u \in \check{H}$, we denote

$$\Omega_u = \{(x, z) \in D \ : \ u(x, z) \neq 0\}.$$

We also set $|\Omega_u|$ the area of Ω_u.

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We also set $|\Omega_u|$ the area of Ω_u. We define

$$C_V^a = \{v \in \check{H}\ :\ \int_D v dxdz = V,\ |\Omega_v| \leq a\},$$

and we reformulate the previous problem as follows:

$$(\mathcal{P}_V^a) \begin{cases} \text{Find } u \in C_V^a \text{ such that} \\ J(u) \leq J(v),\ \forall v \in C_V^a. \end{cases}$$

Theorem (Dambrine & P.)

Problem $(\mathcal{P}_V^a)$ has a solution u such that $J(u) < +\infty$.

Existence by considering a minimizing sequence in C_V^a.

Theorem (Dambrine & P.)

Let u solve problem $(\mathcal{P}_V^a)$ with $k \in L^q(D \times D)$, $q > 1$, and assume that u is nonnegative.

1. *If $q \in (1,2)$, then u is locally α-Hölder continuous on D with $\alpha = 2/q'$.*
2. *If $q = 2$, then u is locally α-Hölder continuous on D for all $\alpha < 1$.*
3. *If $q > 2$, then u is locally Lipschitz continuous on D.*

Method of **Alt & Caffarelli'81**, and adaptations to the Dirichlet energy (+ symmetry + integral kernel k):

- Penalized version of the problem (isoperimetric inequality used)
- elliptic estimates and measure-theoretic arguments

Michell's wave resistance kernel reads

$$k_\nu(x,z,x',z') = \frac{4\nu^4}{\pi C_F(\nu)} K(\nu(x-x'), \nu(|z|+|z'|)), \qquad (6)$$

with $\nu = g/U^2$ (g=gravity and U=speed of ship), and

$$K(X,Z) = \int_1^\infty e^{-\lambda^2 Z} \cos(\lambda X) \frac{\lambda^4}{\sqrt{\lambda^2-1}} d\lambda. \qquad (7)$$

Proposition

Michell's normalized wave resistance kernel k_ν (6) belongs to $L^q(D \times D)$ for all $1 \leq q < 5/4$. Moreover, if D contains an open disc centered on the x-axis, then k_ν does not belong to $L^{5/4}(D \times D)$.

Theorem

Let u be a solution of problem $(\mathcal{P}_V^a)$. If u is nonnegative, then u is locally α-Hölder continuous on D for all $\alpha \in (0, 2/5)$.

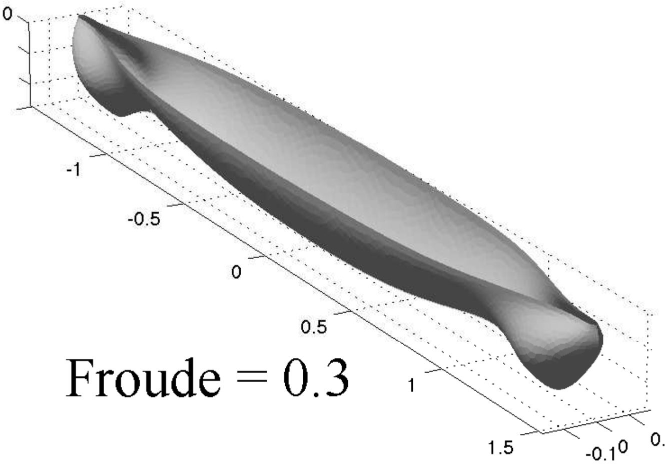
Theorem (Improved regularity below the water/air interface)

Let the assumptions of Theorem 3.2 be satisfied. Then u is locally Lipschitz continous on $D^\star = \{(x, z) \in D \ : z \neq 0\} = D \cap (\mathbf{R} \times \mathbf{R}^\star)$ (where $\mathbf{R}^\star = \mathbf{R} \setminus \{0\}$).

Using analycity, we also proved

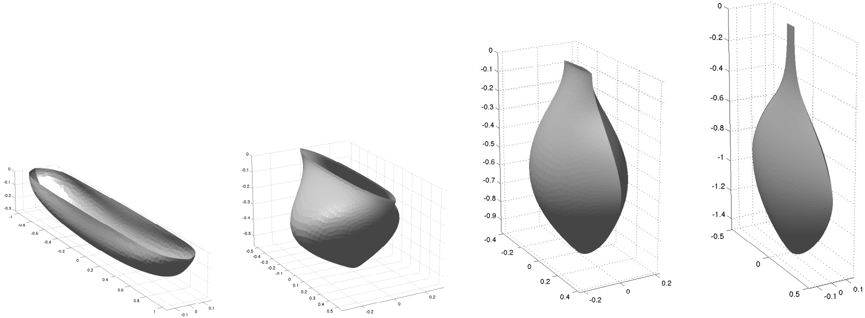
Theorem

Let u solve problem $(\mathcal{P}_V^a)$. If $D^+ = \{(x,z) \in D \ : \ z > 0\}$ is connected, then the constraint $|\Omega_u| \leq a$ is saturated, and so $|\Omega_u| = a$.



An optimized form for a variable domain (algorithm from **Allaire's book**)

The bulbous bow of “Harmony of the Seas” (2015)



A minimizing sequence for $Fr = 0.75$

Work in progress

- More robust numerical simulations
- Dependence with respect to the speed U

Some open questions

- Study regularity of u with the positivity condition, i.e. consider problem

$$(\mathcal{P}_V^{a,+})\begin{cases}\text{Find } u \in C_V^{a,+} \text{ such that}\\ J(u) \leq J(v),\ \forall v \in C_V^{a,+},\end{cases}$$

where

$$C_V^{a,+} = \{v \in \check{H}\ : v \geq 0 \text{ a. e. in } D,\ \int_D v dxdz = V,\ |\Omega_v| \leq a\};$$

- Prove existence without the bounding box D ? (at least for some values of U).

Some open questions (continued)

- Study regularity of $u \in H^1(\mathbf{R}^d)$ which solves

$$\begin{cases} u \in K^{a,+} \\ \mathcal{F}(u) \leq \mathcal{F}(v), \ \forall v \in K^{a,+}, \end{cases}$$

where

$$\mathcal{F}(v) = \frac{1}{2}\int_{\mathbf{R}^d} |\nabla v|^2 dx - \int_{\mathbf{R}^d} fv dx$$

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$$K^{a,+} = \{v \in H^1(\mathbf{R}^d) \ : \ v \geq 0 \text{ a.e. in } \mathbf{R}^d, \ |\Omega_v| \leq a\}.$$

Here, $f \in C_c^0(\mathbf{R}^d)$ (for instance).

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Thank you for your attention !