A (short) Tour on techniques for exact controllability for entropy solutions

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Benasque, August 23, 2017

Outline

1. Generalized Characteristics
2. Vanishing Viscosity
3. Wave Front Tracking
4. Lyapunov Functionals

The idea

- Classical characteristics.
- Generalized characteristics
- Extremal backward characteristics.

Some articles

- Ancona-Marson (1998) : Reachable states, half-line, 0 initial condition.
- Horsin (1998) : sufficient conditions on interval for Burgers.

Advantages and Limitations

- A posteriori analysis.
- Very precise description.
- Only 1d.
- Much more difficult with non convex flux.
- Much more difficult with system.

The idea

- Shock selection and viscosity.
- Boundary layer.

Some articles

- Glass-Guerrero (2007) : Burgers constant target.
- Imanuvilov-Puel (2009) : 2D Burgers.
- Leautaud (2012) : More general flux.

Advantages and limitations

- Work with smoother functions.
- Real boundary.
- Uniform controllability (good and bad).
- More difficult with system (Bianchini Bressan Ancona).
- Reachable states?

The idea

- Riemann problem.
- Piecewise constant approximations.
- Non physical fronts.
- Backward solver.

Some Articles

- Bressan-Coclite (2002), counter-example, no linear test.
- Ancona-Coclite (2005), Temple systems.
- Glass (2007, 2014) , Euler isentropic and non-isentropic.
- Li Tatsien- Lei Yu (2016), Linearly degenerate systems.

Advantages and limitations

- Any scalar flux.
- Work with system.
- Numerically implementable (at least theoretically).
- No black box, even for obvious results.
- Reachable states.

The case of the transport equation

$$\begin{aligned} \partial_t y + c\partial_x y &= 0, \qquad (t,x) \in (0,T) \times (0,L) \\ y(t,0) &= 0, \qquad t \in (0,T). \end{aligned}$$

Using the method of characteristics :

$$y(t,x) = \begin{cases} y_0(x-ct) & \text{if } x > ct, \\ 0 & \text{otherwise.} \end{cases}$$

For $t \geq L$, $y(t,.) = 0$.

A Family of Lyapunov Functionals

- For $\nu > 0$:

$$J_\nu(t) := \int_0^L y^2(t,x)e^{-\nu x}dx.$$

- Formally at least :

$$\begin{aligned}\dot{J}_\nu(t) &= \int_0^L 2y_t(t,x)y(t,x)e^{-\nu x}dx \\ &= \int_0^L -2cy_x(t,x)y(t,x)e^{-\nu x}dx \\ &= [-cy^2(t,x)e^{-\nu x}]_0^L - c\nu J_\nu(t) \\ &\leq -c\nu J_\nu(t).\end{aligned}$$

- Using Gronwall :

$$J_\nu(t) \leq e^{-c\nu t}J_\nu(0).$$

Return on the L^2 norm

- Norm equivalence

$$\forall t \geq 0, \qquad e^{-\nu L}||y(t,.)||^2_{L^2(0,L)} \leq J_\nu(t) \leq ||y(t,.)||^2_{L^2(0,L)}.$$

- Inequality on L^2

$$||y(t,.)||^2_{L^2(0,L)} \leq e^{-\nu c(t-\frac{L}{c})}||y_0||^2_{L^2(0,L)},$$

- For $t \geq \frac{L}{c}$, letting $\nu \to +\infty$ we get $y(t,.) = 0$.

A simple geometric condition

Definition

*Let Ω be an smooth open set of $\mathbb{R}^d$ and I be a segment of $\mathbb{R}$. We say that they satisfy **the replacement condition in time** $T > 0$ if there exists a vector $w \in \mathbb{R}^d$ and a positive number c such that*

$$L := \sup_{x\in\Omega}\langle w|x\rangle - \inf_{x\in\Omega}\langle w|x\rangle < +\infty.$$

$$\forall u \in I, \qquad \langle f'(u)|w\rangle \geq c,$$

and $T = \frac{L}{c}$.

and its simple corresponding result

Theorem (Donadello, P.)

Let $v \in L^\infty((0,+\infty)\times\Omega)$ be an entropy solution to

$$\partial_t u + \mathrm{div}(f(u)) = 0$$

and u_0 be a function in $L^\infty(\Omega)$.
Suppose that both u_0 and v take value in a segment I such that Ω, I and f satisfy the replacement condition in time T. Then for any times T_1 and T_2 greater than T we have an entropy solution u of the previous equation satisfying

$$u(0,x) = u_0(x), \qquad u(T_1,x) = v(T_2,x) \qquad \text{for almost all } x \in \Omega.$$

THANK YOU FOR YOUR ATTENTION