FLATNESS FOR A STRONGLY DEGENERATE 1-D PARABOLIC EQUATION

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ABSTRACT. We consider the degenerate equation

$$\partial_t f(t,x) - \partial_x \left(x^\alpha \partial_x f\right)(t,x) = 0,$$

on the unit interval $x \in (0,1)$, in the strongly degenerate case $\alpha \in [1,2)$ with adapted boundary conditions at $x = 0$ and boundary control at $x = 1$. We use the flatness approach to construct explicit controls in some Gevrey classes steering the solution from any initial datum $f_0 \in L^2(0,1)$ to zero in any time $T > 0$.

Keywords– partial differential equations; degenerate parabolic equation; boundary control; null-controllability; motion planning; flatness.

1. INTRODUCTION

We consider the following control system

$$\left\{\begin{array}{ll} \partial_t f(t,x) - \partial_x \left(x^\alpha \partial_x\right) f(t,x) = 0, & (t,x) \in (0,T) \times (0,1), \\ \left(x^\alpha \partial_x\right) f(t,x)|_{x=0} = 0, & t \in (0,T), \\ f(t,1) = u(t), & t \in (0,T), \\ f(0,x) = f_0(x), & x \in (0,1), \end{array}\right. \tag{1.1}$$

where the state is the solution $f(t,x)$ and the control is the function $u(t)$. The parameter $\alpha \in [1,2)$ is fixed through the whole article.

The aim of this work is to construct explicit controls u for the null-controllability of system (1.1) in finite time $T > 0$, using the flatness method.

1.1. Main result.

We will make use of the Gevrey class of functions.

DEFINITION 1.1. *Let $s \in \mathbb{R}^+$ and $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$. A function $h \in \mathscr{C}^\infty([t_1,t_2])$ is said to be Gevrey of order s if*

$$\exists M, R > 0 \text{ such that } \sup_{t_1 \le r \le t_2} \left| h^{(n)}(r) \right| \le \frac{M(n!)^s}{R^n}.$$

We then write $h \in \mathscr{G}^s([t_1,t_2])$.

Before stating the main result, we have to recall the notion of weak solutions of the inhomogeneous system (1.1).

DEFINITION 1.2 (Weak solutions). *Let $f_0 \in L^2(0,1)$, $T > 0$ and $u \in H^1(0,T)$. A weak solution of system (1.1) is a function $f \in \mathscr{C}^0([0,T]; L^2(0,1))$*

such that for every $t' \in [0,T]$ *and for every*

$$\psi \in \mathscr{C}^1([0,t'];L^2(0,1)) \cap \mathscr{C}^0([0,t'];H^2(0,1)) \tag{1.2}$$

such that

$$(x^\alpha \partial_x)\,\psi(t,x)|_{x=0} = \psi(t,1) = 0, \quad \forall t \in [0,t'], \tag{1.3}$$

one has

$$\int_0^{t'}\int_0^1 f(t,x)\left(\partial_t\psi + \partial_x(x^\alpha\partial_x\psi)\right)(t,x)\,\mathrm{d}t\,\mathrm{d}x$$
$$= \int_0^1 f(t',x)\psi(t',x)\,\mathrm{d}x - \int_0^1 f_0(x)\psi(0,x)\,\mathrm{d}x + \int_0^{t'} u(t)\partial_x\psi(t,1)\,\mathrm{d}t.$$

As we show in Section 2 (see Corollary 2.2), system (1.1) has a unique weak solution under suitable assumptions. Our main result is the following.

THEOREM 1.3. *Let* $f_0 \in L^2(0,1)$*,* $T > 0$*,* $\tau \in (0,T)$ *and* $s \in (1,2)$*. Then, there exists a flat output* $y \in \mathscr{G}^s([\tau,T])$ *such that the control*

$$u(t) = \begin{cases} 0, & \text{if } t \in [0,\tau], \\ \sum_{k=0}^{\infty} \frac{y^{(k)}(t)}{(2-\alpha)^{2k} k! \prod_{j=1}^{k}\left(j+\frac{\alpha-1}{2-\alpha}\right)}, & \text{if } t \in (\tau,T], \end{cases} \tag{1.4}$$

steers to zero at time T *the weak solution of system (1.1). Furthermore, the control* u *belongs to* $\mathscr{G}^s([0,T])$*.*

1.2. **Previous work.**

1.2.1. *Null-controllability.* The null-controllability of system

$$\begin{cases} \partial_t f(t,x) - \partial_x\left(x^\alpha\partial_x\right) f(t,x) = 1_\omega(x)v(t,x), & (t,x) \in (0,T)\times(0,1), \\ (x^\alpha\partial_x)\,f(t,x)|_{x=0} = 0, & t \in (0,T), \\ f(t,1) = 0, & t \in (0,T), \\ f(0,x) = f_0(x), & x \in (0,1), \end{cases}$$

where $\omega \subset (0,1)$, has been studied by P. Cannarsa, P. Martinez and J. Vancostenoble in [8]. Their strategy relies on appropriate Carleman estimates. To deal with the degeneracy at $\{x=0\}$, they use an adequate functional framework that we recall in Section 2, and Hardy-type inequalities.

The null-controllability of system (1.1) is a consequence of the internal null-controllability and the extension principle, since the control is located on $\{x=1\}$, away from the degeneracy. The interest of the present article is to provide explicit controls.

In the case of a control located on $\{x=0\}$, an approximate controllability result for $\alpha \in [0,1)$ has been proven by P. Cannarsa, J. Tort and M. Yamamoto in [10] using Carleman estimates. The exact controllability was later proven by M. Gueye in [13] again in the weakly degenerate case $\alpha \in [0,1)$ by using the transmutation method.

Other related one-dimensional problems have been treated: see [6, 7, 2], see [5] for a non-divergence setting, see [20] for a system with a singular potential. A multi-dimensional case has been studied in [9].

1.2.2. *The flatness method.* The main interest of the flatness method is to provide explicit controls. It has been developed for finite-dimensional systems (see [12]) and then generalised to some infinite-dimensional systems; see [17] for the heat equation on a cylindrical domain with boundary control, [18] for one-dimensional parabolic equations with varying coefficients and [19] for the one-dimensional Schrödinger equation. However, the strongly degenerate case $\alpha \in [1, 2)$ considered in Theorem 1.3 does not belong to the class concerned in [18]. Our goal is to adapt the flatness method to this case.

1.3. **Open questions and perspectives.** The flatness method may also be successful on similar equations, for instance in non-divergence form as in [5]. For the time being, this is an open problem.

1.4. **Structure of the article.** In Section 2 we recall a well-posedness result and the functional framework. In Section 3 we derive, thanks to an heuristic method, an explicit solution of system (1.1) consisting on a formal series development. We prove its convergence, provided that the corresponding flat output is in a Gevrey class. In Section 4 we discuss the spectral analysis of the associated stationary problem. In Section 5 we study the regularising effect of system (1.1) when $u = 0$. In Section 6 we construct an appropriate flat output steering the solution of (1.1) to zero, which concludes the proof of Theorem 1.3. Finally, we give in Appendices A and B a brief account of some results concerning the Gamma and Bessel functions needed in the proofs.

1.5. **Notation.** Since all the functions appearing in the article are real-valued, we omit any explicit mention by writing, for instance, $L^2(0, 1)$ instead of $L^2((0, 1); \mathbb{R})$. If $h \in \mathscr{C}^k([t_1, t_2])$, for some $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$ and $k \in \mathbb{N}^*$, we will denote by $h'(t)$ and $h''(t)$ its first and second derivatives and by $h^{(n)}(t)$, for every $n \in \mathbb{N}$, $2 < n \leq k$, the $n-$th derivative.

If $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ are two real-valued functions and $\mu \in \overline{\mathbb{R}}$, we will write $h_1 \sim h_2$ as $x \to \mu$ to denote that $\lim_{t\to\mu} \frac{h_1(t)}{h_2(t)} = 1$.

We will denote by $\langle \cdot, \cdot \rangle$ the inner product in $L^2(0, 1)$.

2. Well-posedness

We consider, for $T > 0$ and $f_0 \in L^2(0, 1)$, the following system

$$
\left\{
\begin{array}{ll}
\partial_t f(t, x) - \partial_x (x^\alpha \partial_x) f(t, x) = h(t, x), & (t, x) \in (0, T) \times (0, 1), \\
(x^\alpha \partial_x) f(t, x)|_{x=0} = 0, & t \in (0, T), \\
f(t, 1) = 0, & t \in (0, T), \\
f(0, x) = f_0(x), & x \in (0, 1).
\end{array}
\right.
\tag{2.5}
$$

We recall below a well-posedness result for system (2.5) proven originally in [7]. The strategy of the proof consists in a semigroup approach and the introduction of adequate weighted Sobolev spaces, that we recall below. We refer to [7, 4] for further details.

We introduce the weighted Sobolev space

$$H^1_\alpha(0,1) := \Big\{ f \in L^2(0,1);\ f \text{ is loc. absolutely continuous on } (0,1], \\ x^{\frac{\alpha}{2}} f' \in L^2(0,1) \text{ and } f(1) = 0 \Big\},$$

endowed with the norm

$$\|f\|^2_{H^1_\alpha(0,1)} := \|f\|^2_{L^2(0,1)} + \|x^{\frac{\alpha}{2}} f'\|^2_{L^2(0,1)}, \quad \forall f \in H^1_\alpha(0,1).$$

We remark that $H^1_\alpha(0,1)$ is a Hilbert space with the scalar product

$$\langle f, g\rangle_{H^1_\alpha} := \int_0^1 f(x)g(x)\,\mathrm{d}x + \int_0^1 x^\alpha f'(x) g'(x)\,\mathrm{d}x, \quad \forall f, g \in H^1_\alpha(0,1). \tag{2.6}$$

PROPOSITION 2.1 ([7], Proposition 3.2 and Theorem 3.1)**.** *Let*

$$\begin{cases} D(A) := \left\{ f \in H^1_\alpha(0,1);\ x^\alpha f' \in H^1(0,1) \right\}, \\ Af := -(x^\alpha f')'. \end{cases} \tag{2.7}$$

Then, $A : D(A) \to L^2(0,1)$ is a closed self-adjoint positive operator with dense domain. As a consequence, A is the infinitesimal generator of a strongly continuous semigroup, and for any $f_0 \in L^2(0,1)$, and $h \in L^2((0,T)\times(0,1))$ there exists a unique weak solution of system (2.5), i.e., a function $f \in \mathscr{C}^0([0,T]; L^2(0,1)) \cap L^2(0,T; H^1_\alpha(0,1))$ such that

$$f(t) = S(t)f_0 + \int_0^t S(t-s)h(s)\,\mathrm{d}s, \quad \text{in } L^2(0,1), \quad \forall t \in [0,T].$$

As a consequence, using classical arguments (see for instance [11, Section 2.5.3]), we deduce the following result.

COROLLARY 2.2. *Let $T > 0$, $f_0 \in L^2(0,1)$ and $u \in H^1(0,T)$. Then, system (1.1) has a unique weak solution (see Definition 1.2).*

Proof. Let $f_0 \in L^2(0,1)$, $u \in H^1(0,T)$ and

$$\theta(x) := x^2, \quad x \in [0,1].$$

We consider the system

$$\begin{cases} (\partial_t - \partial_x (x^\alpha \partial_x))\, g(t,x) = H(t,x), & (t,x) \in (0,T)\times(0,1), \\ (x^\alpha \partial_x)\, g(t,x)|_{x=0} = 0, & t \in (0,T), \\ g(t,1) = 0, & t \in (0,T), \\ g(0,x) = f_0(x) - u(0)\theta(x), & x \in (0,1), \end{cases}$$

with

$$H(t,x) := -u'(t)\theta(x) - u(t)A\theta(x), \quad \forall (t,x) \in (0,T)\times(0,1).$$

Since $H \in L^2((0,T)\times(0,1))$, by Proposition 2.1 there exists a unique weak solution $g \in \mathscr{C}^0([0,T];L^2(0,1)) \cap L^2(0,T;H^1_\alpha(0,1))$ of this problem. We set

$$f(t,x) := g(t,x) + u(t)\theta(x).$$

Then, using the integral formulation associated to g, one shows that f is a weak solution of system (1.1) in the sense of Definition 1.2.

The uniqueness follows since, if f_1 and f_2 are weak solutions of (1.1), then $f_1 - f_2$ is the unique weak solution of system (2.5) with $h \equiv 0$, and then by Proposition 2.1, $f_1 - f_2 = 0$. □

3. Explicit solution

3.1. **Heuristics.**

We consider the following formal expansion

$$f(t,x) = \sum_{k=0}^{\infty} c_{2k}(t)\left(x^{1-\frac{\alpha}{2}}\right)^{2k}, \quad \forall (t,x) \in (0,T)\times(0,1).$$

where $(c_{2k}(t))_{k\in\mathbb{N}}$ is a sequence of real numbers. We formally have

$$\begin{aligned}\partial_x\left(x^\alpha \partial_x f\right)(t,x) &= \sum_{k=0}^{\infty} c_{2(k+1)}(t)(2-\alpha)^2(k+1)\left[k+1+\frac{\alpha-1}{2-\alpha}\right]\left(x^{1-\frac{\alpha}{2}}\right)^{2k}, \\ \partial_t f(t,x) &= \sum_{k=0}^{\infty} c'_{2k}(t)\left(x^{1-\frac{\alpha}{2}}\right)^{2k}.\end{aligned}$$

If f solves (1.1), then the following recurrence relation holds

$$c_{2(k+1)}(t) = \frac{c'_{2k}(t)}{(2-\alpha)^2(k+1)\left(k+1+\frac{\alpha-1}{2-\alpha}\right)}, \quad \forall k \in \mathbb{N}.$$

Choosing a flat output $c_0(t) := y(t)$ and iterating, we readily have

$$c_{2k}(t) = \frac{y^{(k)}(t)}{(2-\alpha)^{2k}k!\prod_{j=1}^{k}\left(j+\frac{\alpha-1}{2-\alpha}\right)}, \quad \forall t \in (0,T),\ \forall k \in \mathbb{N}.$$

This gives a formal solution of (1.1),

$$f(t,x) = \sum_{k=0}^{\infty} \frac{y^{(k)}(t)\left(x^{1-\frac{\alpha}{2}}\right)^{2k}}{(2-\alpha)^{2k}k!\prod_{j=1}^{k}\left(j+\frac{\alpha-1}{2-\alpha}\right)}, \tag{3.8}$$

and a control given by $u(t) = f(t,1)$, which is

$$u(t) = \sum_{k=0}^{\infty} \frac{y^{(k)}(t)}{(2-\alpha)^{2k}k!\prod_{j=1}^{k}\left(j+\frac{\alpha-1}{2-\alpha}\right)}. \tag{3.9}$$

3.2. **Pointwise solutions.** The goal of this section is to introduce a notion of pointwise solution of system (1.1) to give a sense to the heuristics made in the previous section.

We define

$$\mathscr{C}^2_\alpha(0,1) := \left\{ f \in \mathscr{C}^0([0,1]) \cap \mathscr{C}^2((0,1)) \text{ such that } x^\alpha f'(x) \in \mathscr{C}^0([0,1)) \right\}.$$

DEFINITION 3.1 (Pointwise solution)**.** *Let $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$. Let $f_{t_1} \in \mathscr{C}^0(0,1)$ and $u \in \mathscr{C}^0([t_1,t_2])$. A pointwise solution of system*

$$\begin{cases} \partial_t f(t,x) - \partial_x\left(x^\alpha \partial_x f\right)(t,x) = 0, & (t,x) \in (t_1,t_2)\times(0,1), \\ x^\alpha \partial_x f(t,x)|_{x=0} = 0, & t \in (t_1,t_2), \\ f(t,1) = u(t), & t \in (t_1,t_2), \\ f(t_1,x) = f_{t_1}(x), & x \in (0,1), \end{cases} \tag{3.10}$$

is a function $f \in \mathscr{C}^0([t_1,t_2]\times[0,1]) \cap \mathscr{C}^1((t_1,t_2)\times(0,1))$ such that

(1) $f(t,\cdot) \in \mathscr{C}^2_\alpha(0,1)$, $\forall t \in (t_1,t_2)$,
(2) $\partial_t f - \partial_x(x^\alpha \partial_x f) = 0$ *pointwisely in* $(t_1,t_2)\times(0,1)$,
(3) $\lim_{x\to 0^+} x^\alpha \partial_x f(t,x) = 0$, $\forall t \in (t_1,t_2)$,
(4) $f(t,1) = u(t)$, $\forall t \in (t_1,t_2)$,
(5) $f(t_1,x) = f_{t_1}(x)$, $\forall x \in (0,1)$.

REMARK 3.2. *The usual energy argument proves that, given $u \in \mathscr{C}^0([t_1,t_2])$, the pointwise solution of system (3.10) is unique. We observe that, changing parameters adequately in Definition 1.2 a pointwise solution of (3.10) is also a weak solution.*

3.3. **Convergence.** The goal of this section is the proof of the following result.

PROPOSITION 3.3. *Let $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2$. If $y \in \mathscr{G}^s([t_1,t_2])$ for some $s \in (0,2)$, then*

(1) *the control u given by (3.9) is well defined and belongs to $\mathscr{G}^s([t_1,t_2])$,*
(2) *the function given by (3.8) is a pointwise solution (see Definition 3.1) of system (3.10) in $(t_1,t_2)\times(0,1)$ with u given by (3.9) and initial datum*

$$f_{t_1}(x) := \sum_{k=1}^{\infty} \frac{y^{(k)}(t_1)\left(x^{1-\frac{\alpha}{2}}\right)^{2k}}{(2-\alpha)^{2k} k! \prod_{j=1}^{k}\left(j + \frac{\alpha-1}{2-\alpha}\right)}, \quad \forall x \in [0,1].$$

Proof. Let $M, R > 0$ be such that $|y^{(n)}(t)| \le \frac{M n!^s}{R^n}$, for any $n \in \mathbb{N}$, $t \in [t_1,t_2]$.

Step 1: We prove that u is well defined and belongs to $\mathscr{C}^\infty([t_1,t_2])$.

For any $t \in [t_1,t_2]$, $k \in \mathbb{N}^*$, we have, as $\frac{\alpha-1}{2-\alpha} \ge 0$,

$$\frac{|y^{(k)}(t)|}{(2-\alpha)^{2k} k! \prod_{j=1}^{k}\left(j + \frac{\alpha-1}{2-\alpha}\right)} \le \frac{M k!^s}{R^k (2-\alpha)^{2k} k!^2} = \frac{M}{R^k(2-\alpha)^{2k} k!^{2-s}}.$$

Hence, the series in (3.9) converges uniformly w.r.t. $t \in [t_1, t_2]$ and $u \in \mathscr{C}^0([t_1, t_2])$. Furthermore, for any $n \in \mathbb{N}^*$, the function $\xi_{n,k}(t) := \frac{y^{(k+n)}(t)}{(2-\alpha)^{2k} k! \prod_{j=1}^{k}\left(j+\frac{\alpha-1}{2-\alpha}\right)}$ satisfies

$$|\xi_{n,k}(t)| \leq \frac{M(k+n)!^s}{R^{n+k}(2-\alpha)^{2k} k!^2}, \quad \forall t \in [t_1, t_2],\ k, n \in \mathbb{N}.$$

Thus, $\sum_k \xi_{n,k}(t)$ converges uniformly w.r.t $t \in [t_1, t_2]$. Whence, $u \in \mathscr{C}^\infty([t_1, t_2])$ and for every $n \in \mathbb{N}$, $t \in [t_1, t_2]$, $u^{(n)}(t) = \sum_{k=0}^{\infty} \xi_{n,k}(t)$.

Step 2: We prove that u is Gevrey of order s.

Let $n \in \mathbb{N}$. We deduce from last inequality that

$$\begin{aligned} \left|u^{(n)}(t)\right| &\leq \sum_{k=0}^{\infty} \frac{M(k+n)!^s}{R^{n+k}(2-\alpha)^{2k} k!^2} \\ &\leq M\left[\sum_{k=0}^{\infty} \frac{1}{(k!)^{2-s}}\left(\frac{2^s}{R(2-\alpha)^2}\right)^k\right]\left(\frac{2^s}{R}\right)^n n!^s, \end{aligned} \tag{3.11}$$

where we have used (A.41). The D'Alembert criterium for entire series shows that, whenever $s \in (0,2)$, the series above converges, which shows that $u \in \mathscr{G}^s([t_1, t_2])$.

Step 3: We show that the function f given by (3.8) is well defined and $f \in \mathscr{C}^0([t_1, t_2] \times [0,1]) \cap \mathscr{C}^1((t_1, t_2) \times (0,1))$.

Let, for every $k \in \mathbb{N}$,

$$f_k(t,x) := \frac{y^{(k)}(t)\left(x^{1-\frac{\alpha}{2}}\right)^{2k}}{(2-\alpha)^{2k} k! \prod_{j=1}^{k}\left(j+\frac{\alpha-1}{2-\alpha}\right)}, \quad \forall (t,x) \in [t_1, t_2] \times [0,1].$$

Then,

$$|f_k(t,x)| \leq \frac{M}{k!^{2-s}}\left(\frac{1}{R(2-\alpha)}\right)^k, \quad \forall (t,x) \in [t_1, t_2] \times [0,1].$$

This proves that $\sum_k f_k$ converges uniformly w.r.t. $(t,x) \in [t_1, t_2] \times [0,1]$. Thus, $f \in \mathscr{C}^0([t_1, t_2] \times [0,1])$.

We observe that $\exists k_0 = k_0(\alpha) \in \mathbb{N}^*$ such that $(2-\alpha)\, k_0 \geq 1$. Then, for every $k \geq k_0$, $f_k(t,\cdot) \in \mathscr{C}^1([0,1])$ and

$$\begin{aligned} |\partial_x f_k(t,x)| &= \left|\frac{y^{(k)}(t) 2k\left(1-\frac{\alpha}{2}\right) x^{-\frac{\alpha}{2}}\left(x^{1-\frac{\alpha}{2}}\right)^{2k-1}}{(2-\alpha)^{2k} k! \prod_{j=1}^{k}\left(j+\frac{\alpha-1}{2-\alpha}\right)}\right| \\ &\leq 2M\left(1-\frac{\alpha}{2}\right)\frac{k}{k!^{2-s}}\left(\frac{1}{R(2-\alpha)^2}\right)^k, \quad \forall x \in [0,1], \end{aligned}$$

since $\left(1-\frac{\alpha}{2}\right)(2k-1)-\frac{\alpha}{2} \geq 0$. This proves that $\sum_{k \geq k_0} \partial_x f_k$ converges uniformly w.r.t. $(t,x) \in [t_1, t_2] \times [0,1]$. Thus, $f(t,\cdot) \in$

$\mathscr{C}^1((0,1])$ for every $t \in [t_1, t_2]$. Note that f may not be differentiable w.r.t. x at $x = 0$ because of the finite number of terms $\sum_{k=0}^{k_0} \partial_x f_k$. Moreover, $\partial_x f(t,x) = \sum_{k=0}^{\infty} \partial_x f_k(t,x)$ for every $(t,x) \in (t_1,t_2) \times (0,1)$.

A similar argument shows that, for every $x \in (0,1)$, $f(\cdot, x) \in \mathscr{C}^1(t_1,t_2)$ and

$$\partial_t f(t,x) = \sum_{k=0}^{\infty} \partial_t f_k(t,x), \quad \forall (t,x) \in (t_1,t_2) \times (0,1). \tag{3.12}$$

Finally, since the partial derivatives of f exist and are continuous in $(t_1,t_2) \times (0,1)$, $f \in \mathscr{C}^1((t_1,t_2) \times (0,1))$.

Step 4: We show that $f(t,\cdot) \in \mathscr{C}^2_\alpha(0,1)$, for every $t \in (t_1,t_2)$.

Let $k_1 = k_1(\alpha) \in \mathbb{N}^*$ such that $k_1(2-\alpha) \geq 2$. Working as in Step 3, we see that $\sum_{k \geq k_1} \partial_x^2 f_k$ converges uniformly w.r.t. $(t,x) \in (t_1,t_2) \times (0,1)$. Thus, $f(t,\cdot) \in \mathscr{C}^2(0,1)$, $\forall t \in (t_1,t_2)$. Furthermore,

$$\partial_x \left(x^\alpha \partial_x f\right)(t,x) = \sum_{k=1}^{\infty} \frac{y^{(k)}(t) \left(x^{1-\frac{\alpha}{2}}\right)^{2(k-1)}}{(2-\alpha)^{2(k-1)}(k-1)! \prod_{j=1}^{k-1} \left(j + \frac{\alpha-1}{2-\alpha}\right)}. \tag{3.13}$$

for every $(t,x) \in (t_1,t_2) \times (0,1)$. From Step 3, we obtain

$$\begin{aligned} |x^\alpha \partial_x f(t,x)| &= \left| \sum_{k=1}^{\infty} \frac{y^{(k)}(t) 2k \left(1-\frac{\alpha}{2}\right) x^{2k\left(1-\frac{\alpha}{2}\right)+\alpha-1}}{(2-\alpha)^{2k} k! \prod_{j=1}^{k} \left(j+\frac{\alpha-1}{2-\alpha}\right)} \right| \\ &\leq 2M \left(1-\frac{\alpha}{2}\right) \sum_{k=1}^{\infty} \left[\frac{k}{k!^{2-s}} \left(\frac{1}{R(2-\alpha)^2} \right)^k \right] x, \end{aligned}$$

for all $(t,x) \in (t_1,t_2) \times (0,1)$, which implies, since $\alpha \in [1,2)$, that

$$x^\alpha \partial_x f(t,x) \to 0, \quad \text{as } x \to 0^+.$$

Therefore, $f(t,\cdot) \in \mathscr{C}^2_\alpha$, for every $t \in (t_1,t_2)$.

Step 5: According to (3.12) and (3.13), an straightforward computation shows that the equation in (3.10) is satisfied.

□

4. Spectral Analysis

The goal of this section is to give the explicit expression of the eigenfunctions and eigenvalues of the spectral problem

$$\begin{cases} A\varphi(x) = \lambda \varphi(x), & x \in (0,1), \\ (x^\alpha \varphi')\,|_{x=0} = \varphi(1) = 0, \end{cases} \tag{4.14}$$

where A is given by (2.7). We will make use of several results about Bessel functions recalled in Appendix B. Form now on, we use the notation

$$\nu := \frac{\alpha - 1}{2-\alpha}. \tag{4.15}$$

PROPOSITION 4.1. *Let*

$$\varphi_k(x) = \frac{\sqrt{2-\alpha}}{|J_{\nu+1}(j_{\nu,k})|} x^{\frac{1-\alpha}{2}} J_\nu\left(j_{\nu,k} x^{1-\frac{\alpha}{2}}\right), \quad \forall x \in (0,1),\ k \in \mathbb{N}^*. \tag{4.16}$$

Then,

(1) $\varphi_k \in D(A)$, $\forall k \in \mathbb{N}^*$,
(2) φ_k *satisfies* (4.14) *with*

$$\lambda_k := \left(1 - \frac{\alpha}{2}\right)^2 j_{\nu,k}^2, \quad \forall k \in \mathbb{N}^*, \tag{4.17}$$

(3) $(\varphi_k)_{k\in\mathbb{N}^*}$ *is a Hilbert basis of* $L^2(0,1)$,
(4) *for every* $f_0 \in L^2(0,1)$ *the solution of* (2.5) *with* $h = 0$ *writes*

$$f(t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \langle f_0, \varphi_k \rangle \varphi_k \quad \textit{in } L^2(0,1),\ \forall t \in [0,T]. \tag{4.18}$$

Proof. We will note for simplicity $b_k := \frac{\sqrt{2-\alpha}}{|J_{\nu+1}(j_{\nu,k})|}$ and $\tilde{\varphi}_k := \frac{1}{b_k}\varphi_k$, for every $k \in \mathbb{N}^*$.

Step 1: We prove that $\varphi_k \in D(A)$, for every $k \in \mathbb{N}^*$ and that $A\varphi_k - \lambda_k \varphi_k = 0$.

Let $k \in \mathbb{N}^*$. We observe that $\varphi_k \in \mathscr{C}^\infty((0,1]) \cap \mathscr{C}^0([0,1])$, for any $k \in \mathbb{N}^*$ and $x \in (0,1)$. We have

$$\tilde{\varphi}_k'(x) = \frac{1-\alpha}{2} x^{-\frac{1+\alpha}{2}} J_\nu(j_{\nu,k} x^{1-\frac{\alpha}{2}}) + j_{\nu,k}\left(1 - \frac{\alpha}{2}\right) x^{\frac{1}{2}-\alpha} J_\nu'(j_{\nu,k} x^{1-\frac{\alpha}{2}}). \tag{4.19}$$

Whence, using (B.48) and Lemma B.3, we deduce

$$x^{\frac{\alpha}{2}} \tilde{\varphi}_k' = (1-\alpha) \underset{x\to 0^+}{O}\left(x^{\frac{\alpha}{2}-1}\right) + \underset{x\to 0^+}{O}\left(x^{1-\frac{\alpha}{2}}\right).$$

It follows that $x^{\frac{\alpha}{2}} \varphi_n' \in L^2(0,1)$. Thus $\varphi_k \in H^1_\alpha(0,1)$. Moreover, from (4.19), a direct computation shows

$$\begin{aligned} (x^\alpha \tilde{\varphi}_k')' &= -\left(\frac{1-\alpha}{2}\right)^2 x^{\frac{\alpha-3}{2}} J_\nu(j_{\nu,k} x^{1-\frac{\alpha}{2}}) \\ &\quad + \left(1 - \frac{\alpha}{2}\right)^2 j_{\nu,k} x^{-\frac{1}{2}} J_\nu'(j_{\nu,k} x^{1-\frac{\alpha}{2}}) \\ &\quad + \left(1 - \frac{\alpha}{2}\right)^2 j_{\nu,k}^2 x^{\frac{1-\alpha}{2}} J_\nu''(j_{\nu,k} x^{1-\frac{\alpha}{2}}). \end{aligned} \tag{4.20}$$

Then, evaluating equation (B.47) at $z = j_{\nu,k} x^{1-\frac{\alpha}{2}}$ and multiplying by $x^{\frac{\alpha-3}{2}}$, it follows

$$\begin{aligned} & j_{\nu,k}^2 x^{\frac{1-\alpha}{2}} J_\nu''(j_{\nu,k} x^{1-\frac{\alpha}{2}}) \\ &\quad = -j_{\nu,k} x^{-\frac{1}{2}} J_\nu'(j_{\nu,k} x^{1-\frac{\alpha}{2}}) - j_{\nu,k}^2 x^{\frac{1-\alpha}{2}} J_\nu(j_{\nu,k} x^{1-\frac{\alpha}{2}}) \\ &\qquad + \left(\frac{\alpha-1}{2-\alpha}\right)^2 x^{\frac{\alpha-3}{2}} J_\nu(j_{\nu,k} x^{1-\frac{\alpha}{2}}). \end{aligned}$$

Substituting in (4.20), this gives

$$-\left(x^\alpha \tilde{\varphi}_k'\right)' = \left(1-\frac{\alpha}{2}\right)^2 j_{\nu,k}^2 x^{\frac{1-\alpha}{2}} J_\nu(j_{\nu,k}x^{1-\frac{\alpha}{2}}) = \lambda_k \tilde{\varphi}_k.$$

Then, we readily have $(x^\alpha \tilde{\varphi}_k')' \in H^1_\alpha(0,1) \subset L^2(0,1)$. Thus, $\varphi_k \in D(A)$. Moreover, $A\varphi_k = \lambda_k \varphi_k$.

Step 2: We check the boundary condition of (4.14) at $x = 0$.

We observe first that the case $\alpha = 1$ is straightforward. From (4.19), (B.48) and Lemma B.3, we have

$$|x^\alpha \tilde{\varphi}_n'(x)| = \underset{x\to 0^+}{O}\left(x^{\alpha-1}\right).$$

Then, it follows that $\lim_{x\to 0^+} x^\alpha \tilde{\varphi}_n'(x) = 0$. This shows, combined with Step 1, that φ_k satisfies (4.14).

Step 3: We prove that $(\varphi_k)_{k\in\mathbb{N}^*}$ is an orthonormal family in $L^2(0,1)$.

Let $n, m \in \mathbb{N}^*$. Then, changing variables and using (B.46), we get

$$\begin{aligned}
&\int_0^1 \varphi_n(x)\varphi_m(x)\,\mathrm{d}x \\
&= (2-\alpha)\int_0^1 x^{1-\alpha}\frac{J_\nu(j_{\nu,n}x^{1-\frac{\alpha}{2}})}{|J_{\nu+1}(j_{\nu,n})|}\frac{J_\nu(j_{\nu,m}x^{1-\frac{\alpha}{2}})}{|J_{\nu+1}(j_{\nu,m})|}\,\mathrm{d}x \\
&= \frac{2}{|J_{\nu+1}(j_{\nu,n})||J_{\nu+1}(j_{\nu,m})|}\int_0^1 yJ_\nu(j_{\nu,n}y)J_\nu(j_{\nu,m}y)\,\mathrm{d}y = \delta_{n,m},
\end{aligned}$$

where $\delta_{n,m}$ stands for the Kronecker delta.

Step 4: We prove that $(\varphi_k)_{k\in\mathbb{N}^*}$ is a Hilbert basis of $L^2(0,1)$ by checking the Bessel equality. Let $f \in L^2(0,1)$ and let

$$a_k := \int_0^1 f(x)\varphi_k(x)\,\mathrm{d}x, \quad \forall k \in \mathbb{N}^*. \tag{4.21}$$

Then, using Lemma B.1 and changing variables twice, we get

$$\begin{aligned}
\sum_{k=1}^\infty |a_k|^2 &= \sum_{k=1}^\infty \left|\int_0^1 f(x)\frac{\sqrt{2-\alpha}}{|J_{\nu+1}(j_{\nu,k})|}x^{\frac{1-\alpha}{2}}J_\nu\left(j_{\nu,k}x^{1-\frac{\alpha}{2}}\right)\,\mathrm{d}x\right|^2 \\
&= \frac{2}{2-\alpha}\sum_{k=1}^\infty\left|\int_0^1 y^{\frac{\alpha-1}{2-\alpha}+\frac{1}{2}}f(y^{\frac{2}{2-\alpha}})\frac{\sqrt{2y}}{|J_{\nu+1}(j_{\nu,k})|}J_\nu(j_{\nu,k}y)\,\mathrm{d}y\right|^2 \\
&= \frac{2}{2-\alpha}\int_0^1 y^{\frac{2(\alpha-1)}{2-\alpha}+1}\left|f(y^{\frac{2}{2-\alpha}})\right|^2\,\mathrm{d}y \\
&= \int_0^1 |f(z)|^2\,\mathrm{d}z = \|f\|^2_{L^2(0,1)}.
\end{aligned}$$

Step 5: Finally, (4.18) is a consequence of [3, Theorem 8.2.3, pp.237–240].

□

5. Regularising effect

We use the orthonormal basis obtained in Proposition 4.1 and some properties of Bessel functions to quantify the smoothing of the solution of system (1.1) when $u \equiv 0$.

PROPOSITION 5.1. *Let $f_0 \in L^2(0,1)$, $T>0$ and let $f \in \mathscr{C}^0([0,T];L^2(0,1))$ be the unique weak solution of system (2.5) when $h=0$, according to Proposition 2.1. Then, there exists $Y \in \mathscr{C}^\infty((0,T])$ such that for every $\sigma \in (0,T)$,*

$$Y \in \mathscr{G}^1([\sigma,T])$$

and

$$f(t,x) = \sum_{n=0}^{\infty} \frac{Y^{(n)}(t)\left(x^{1-\frac{\alpha}{2}}\right)^{2n}}{(2-\alpha)^{2n} n! \prod_{j=1}^{n}\left(j+\frac{\alpha-1}{2-\alpha}\right)}, \quad \forall (t,x) \in [\sigma,T]\times[0,1]. \tag{5.22}$$

Moreover, f solves system (3.10) pointwisely (see Definition 3.1) in $(\sigma,T)\times(0,1)$ with $u=0$ and initial datum $f_\sigma(x)=f(\sigma,x)$.

Proof. Let ν be given by (4.15) and a_k as in (4.21). Let $\sigma \in (0,T)$ be fixed but arbitrary. Let $t \in [\sigma,T]$ be fixed. By (4.18) and (B.43), we have, for a.e. $x \in [0,1]$,

$$\begin{aligned}
f(t,x) &= \sum_{k=1}^{\infty} e^{-\lambda_k t}\frac{a_k\sqrt{2-\alpha}}{|J_{\nu+1}(j_{\nu,k})|}x^{\frac{1-\alpha}{2}} J_\nu\left(j_{\nu,n}x^{1-\frac{\alpha}{2}}\right)\\
&= \sum_{k=1}^{\infty} e^{-\lambda_k t}\frac{a_k\sqrt{2-\alpha}}{|J_{\nu+1}(j_{\nu,k})|}x^{\frac{1-\alpha}{2}} \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+1+\nu)}\left(\frac{j_{\nu,k}x^{1-\frac{\alpha}{2}}}{2}\right)^{2n+\nu}\\
&= \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} B_{n,k}(t,x),
\end{aligned} \tag{5.23}$$

where, for every $(n,k) \in \mathbb{N}\times\mathbb{N}^*$,

$$B_{n,k}(t,x) := e^{-\lambda_k t} b_k \frac{(-1)^n j_{\nu,k}^{2n+\nu}}{n!\Gamma(n+1+\nu)2^{2n+\nu}} \frac{\left(x^{1-\frac{\alpha}{2}}\right)^{2n}}{|J_{\nu+1}(j_{\nu,k})|},$$

and $b_k := a_k\sqrt{2-\alpha}$, $\forall k \in \mathbb{N}^*$.

Step 1: We show that

$$\sum_{n=0}^{\infty}\left(\sum_{k=1}^{\infty}|B_{n,k}(t,x)|\right) < \infty, \quad \forall x \in [0,1]. \tag{5.24}$$

Indeed, since $\lambda_k > 0$, we have for every $(n,k) \in \mathbb{N}\times\mathbb{N}^*$ and $x \in [0,1]$,

$$|B_{n,k}(t,x)| \le \frac{|b_k| j_{\nu,k}^{2n+\nu} e^{-\lambda_k \sigma}}{2^{2n+\nu} n!\Gamma(n+1+\nu)|J_{\nu+1}(j_{\nu,k})|}$$

$$\leq \frac{C_1|b_k|e^{-\lambda_k\sigma}j_{\nu,k}^{2n+\nu+\frac{1}{2}}}{2^{2n}n!\Gamma(n+1+\nu)}, \tag{5.25}$$

for a constant $C_1 > 0$, using Lemma B.4.

We fix $n \in \mathbb{N}$ and we define the function $h_n^\alpha \in \mathscr{C}^\infty(\mathbb{R}^+;\mathbb{R}^+)$ by

$$h_n^\alpha(x) := e^{-\left(1-\frac{\alpha}{2}\right)^2 x^2\sigma} x^{2n+\nu+\frac{1}{2}}, \quad \forall x \in [0,+\infty),$$

which satisfies that

$$\frac{\mathrm{d}}{\mathrm{d}x}h_n^\alpha(x) > 0,\ \forall x \in (0, N_n^\alpha) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}h_n^\alpha(x) < 0,\ \forall x \in (N_n^\alpha, \infty), \tag{5.26}$$

where $N_n^\alpha := \frac{2}{2-\alpha}\sqrt{\frac{1}{\sigma}\left(n+\frac{\alpha}{4(2-\alpha)}\right)}$. Hence, from (5.25) and (4.17),

$$\sum_{k=1}^{\infty}|B_{n,k}(t,x)| \leq \frac{C_1 \sup_k |b_k|}{2^{2n}n!\Gamma(n+1+\nu)}\sum_{k=1}^{\infty}h_n^\alpha(j_{\nu,k}) \tag{5.27}$$

Introducing $K_n^\alpha := \sup\{k \in \mathbb{N}^*; j_{\nu,k} \leq N_n^\alpha\}$, we write

$$\sum_{k=1}^{\infty}h_n^\alpha(j_{\nu,k}) = h_n^\alpha(j_{\nu,K_n^\alpha}) + h_n^\alpha(j_{\nu,K_n^\alpha+1}) + \sum_{k\in\mathbb{N}^*-\{K_n^\alpha,K_n^\alpha+1\}} h_n^\alpha(j_{\nu,k}) \tag{5.28}$$

On one hand, we have

$$\begin{aligned} h_n^\alpha(j_{\nu,K_n^\alpha}) + h_n^\alpha(j_{\nu,K_n^\alpha+1}) &\leq 2h_n^\alpha(N_n^\alpha) \\ &\leq 2e^{-\left(n+\frac{\alpha}{4(2-\alpha)}\right)}\left(n+\frac{\alpha}{4(2-\alpha)}\right)^{n+\frac{\alpha}{4(2-\alpha)}}\left[\frac{1}{\sigma}\left(\frac{2}{2-\alpha}\right)^2\right]^{n+\frac{\alpha}{4(2-\alpha)}} \\ &\leq C_2\Gamma\left(n+\frac{\alpha}{4(2-\alpha)}+\frac{1}{2}\right)\left[\frac{1}{\sigma}\left(\frac{2}{2-\alpha}\right)^2\right]^{n+\frac{\alpha}{4(2-\alpha)}}, \end{aligned} \tag{5.29}$$

for a constant $C_2 > 0$, using Lemma A.1 with $a = 1, b = \frac{1}{2}$. On the other hand, using (5.26), we write

$$\begin{aligned} &\sum_{k\in\mathbb{N}^*-\{K_n^\alpha,K_n^\alpha+1\}} h_n^\alpha(j_{\nu,k}) \leq \\ &\leq \sum_{k=1}^{K_n^\alpha-1}\frac{1}{j_{\nu,k+1}-j_{\nu,k}}\int_{j_{\nu,k}}^{j_{\nu,k+1}} h_n^\alpha(x)\,\mathrm{d}x + \sum_{K_n^\alpha+1}^{\infty}\frac{1}{j_{\nu,k}-j_{\nu,k-1}}\int_{j_{\nu,k-1}}^{j_{\nu,k}} h_n^\alpha(x)\,\mathrm{d}x \\ &\leq \sup_{k\in\mathbb{N}^*}\left\{\frac{1}{j_{\nu,k+1}-j_{\nu,k}}\right\}\left(\int_{j_{\nu,1}}^{j_{\nu,K_n^\alpha}} h_n^\alpha(x)\,\mathrm{d}x + \int_{j_{\nu,K_n+1}^\alpha}^{\infty} h_n^\alpha(x)\,\mathrm{d}x\right) \\ &\leq C_3\int_0^\infty h_n^\alpha(x)\,\mathrm{d}x, \end{aligned}$$

for a constant $C_3 > 0$, using (B.45). Moreover, we have

$$\begin{aligned}\int_0^\infty h_n^\alpha(x)\,\mathrm{d}x &= \int_0^\infty e^{-\left(1-\frac{\alpha}{2}\right)^2 x^2\sigma} x^{2n+\frac{\alpha}{2(2-\alpha)}}\,\mathrm{d}x \\ &= \int_0^\infty e^{-t}\left(\frac{2}{2-\alpha}\sqrt{\frac{t}{\sigma}}\right)^{2n+\frac{\alpha}{2(2-\alpha)}} \frac{1}{2\sqrt{\sigma t}}\left(\frac{2}{2-\alpha}\right)\mathrm{d}t \\ &= \frac{1}{2}\left[\frac{1}{\sqrt{\sigma}}\left(\frac{2}{2-\alpha}\right)\right]^{2n+\frac{\alpha}{2(2-\alpha)}+1}\int_0^\infty e^{-t}t^{n+\frac{\alpha}{4(2-\alpha)}-\frac{1}{2}}\,\mathrm{d}t \\ &= \frac{1}{2}\left[\frac{1}{\sqrt{\sigma}}\left(\frac{2}{2-\alpha}\right)\right]^{2n+\frac{\alpha}{2(2-\alpha)}+1}\Gamma\left(n+\frac{\alpha}{4(2-\alpha)}+\frac{1}{2}\right),\end{aligned}$$

where we have used (A.38) with $p = n+\frac{\alpha}{4(2-\alpha)}+\frac{1}{2}$. Hence, combining this with (5.28) and (5.29), we get

$$\sum_{k=1}^\infty h_n^\alpha(j_{\nu,k}) \le \left(C_2+\frac{C_3}{\sqrt{\sigma}(2-\alpha)}\right)\left[\frac{1}{\sqrt{\sigma}}\left(\frac{2}{2-\alpha}\right)\right]^{2n+\frac{\alpha}{2(2-\alpha)}}\Gamma\left(n+\frac{\alpha}{4(2-\alpha)}+\frac{1}{2}\right),$$

which, according to (5.27), implies

$$\sum_{k=1}^\infty |B_{n,k}(t,x)| \le C_4\left[\frac{1}{\sqrt{\sigma}}\left(\frac{2}{2-\alpha}\right)\right]^{2n+\frac{\alpha}{2(2-\alpha)}}\frac{\Gamma\left(n+\frac{\alpha}{4(2-\alpha)}+\frac{1}{2}\right)}{2^{2n}n!\Gamma\left(n+\nu+1\right)}.$$

Henceforth, the D'Alembert criterium for entire series gives (5.24).

Step 2: We find $Y \in \mathscr{G}^1([\sigma,T])$ such that (5.22) holds.

Thanks to Fubini's theorem, (5.23) and (A.39), we may write

$$f(t,x) = \sum_{n=0}^\infty \frac{y_n(t)\left(x^{1-\frac{\alpha}{2}}\right)^{2n}}{(2-\alpha)^{2n}n!\prod_{j=1}^n(j+\nu)},$$

where, for every $n \in \mathbb{N}$,

$$y_n(t) := \frac{(-1)^n\sqrt{2-\alpha}\left(1-\frac{\alpha}{2}\right)^{2n}}{2^\nu\Gamma\left(\frac{1}{2-\alpha}\right)}\sum_{k=1}^\infty a_k e^{-\lambda_k t}\frac{j_{\nu,k}^{2n+\nu}}{|J_{\nu+1}(j_{\nu,k})|}, \quad \forall t\in[\sigma,T],$$

and ν is given by (4.15). Putting

$$Y(t) := \frac{\sqrt{2-\alpha}}{2^\nu\Gamma\left(\frac{1}{2-\alpha}\right)}\sum_{k=1}^\infty \frac{a_k j_{\nu,k}^\nu}{|J_{\nu+1}(j_{\nu,k})|}e^{-\left(1-\frac{\alpha}{2}\right)^2 j_{\nu,k}^2 t}, \quad t\in[\sigma,T], \tag{5.30}$$

we have that, since $\sigma > 0$, Y is analytic in $[\sigma,T]$. Moreover,

$$Y^{(n)}(t) = y_n(t), \quad \forall t\in[\sigma,T],\ \forall n\in\mathbb{N}.$$

Hence, we obtain (5.22) with this choice. Since $\sigma \in (0,T)$ is arbitrary, we have in addition that $Y \in \mathscr{C}^\infty((0,T])$.

Furthermore, applying Proposition 3.3 to (5.22) with $t_1 = \sigma$ and $t_2 = T$, we deduce that f solves (1.1) pointwisely in $(\sigma, T) \times (0,1)$ with $u = 0$ and $f_\sigma(x) = f(\sigma, x)$.

□

6. Construction of the control

Let $s \in \mathbb{R}$ with $s > 1$. The function (see [17, Section 2] and [21, Theorem 11.2, p.48])

$$\phi_s(t) := \begin{cases} 1, & \text{if } t \leq 0, \\ \dfrac{e^{-(1-t)^{-\frac{1}{s-1}}}}{e^{-(1-t)^{-\frac{1}{s-1}}} + e^{-t^{-\frac{1}{s-1}}}}, & \text{if } 0 < t < 1, \\ 0, & \text{if } t \geq 1, \end{cases} \tag{6.31}$$

belongs to $\mathscr{G}^s([0,1])$ and satisfies

$$\phi_s(0) = 1,\ \phi_s(1) = 0, \quad \phi_s^{(i)}(0) = \phi_s^{(i)}(1) = 0,\ \forall i \in \mathbb{N}^*. \tag{6.32}$$

Proof of Theorem 1.3. Let $f_0 \in L^2(0,1)$, $T > 0$. Let f and Y be given by Proposition 5.1.

We pick $\tau \in (0,T)$, $s \in (1,2)$ and we set the flat output

$$y(t) := \phi_s\left(\frac{t-\tau}{T-\tau}\right) Y(t), \quad \forall t \in (0,T],$$

which belongs to $\mathscr{C}^\infty(0,T)$. Moreover, for every $\sigma \in (0,T)$, $y \in \mathscr{G}^s([\sigma,T])$, as it is a product of two functions in $\mathscr{G}^s([\sigma,T])$. We define accordingly the function

$$\tilde{f}(t,x) := \sum_{k=1}^{\infty} \frac{y^{(n)}(t)\left(x^{1-\frac{\alpha}{2}}\right)^{2n}}{(2-\alpha)^{2n} n! \prod_{j=1}^{n}\left(j + \frac{\alpha-1}{2-\alpha}\right)}, \quad \forall (t,x) \in (0,T] \times [0,1],$$

and the control

$$u(t) = \begin{cases} 0, & t \in [0,\tau], \\ \tilde{f}(t,1), & t \in (\tau,T]. \end{cases} \tag{6.33}$$

Since $y \in \mathscr{G}^s([\sigma,T])$ for some $s \in (1,2)$, Proposition 3.3 shows that

$$\begin{array}{c} \forall \sigma \in (0,T),\ \tilde{f} \text{ is the pointwise solution of (3.10) with} \\ t_1 = \sigma,\ t_2 = T,\ f_{t_1} = f(\sigma,\cdot) \text{ and (6.33).} \end{array} \tag{6.34}$$

As a consequence of (6.32), we have

$$\begin{aligned} y(t) &= Y(t),\ \forall t \in (0,\tau], \\ y(T) &= 0. \end{aligned} \tag{6.35}$$

Whence, $\tilde{f}(t,x) = f(t,x)$, for every $(t,x) \in (0,\tau) \times (0,1)$. Thus, as $f \in \mathscr{C}^0([0,T]; L^2(0,1))$, we deduce

$$\tilde{f} \in \mathscr{C}^0([0,T]; L^2(0,1)), \tag{6.36}$$

$$\tilde{f}(0) = f_0 \quad \text{in } L^2(0,1). \tag{6.37}$$

We have to check that $\tilde{f}$ is the weak solution of system (1.1) on $(0,T)$. To do so, and according to Definition 1.2, let $t' \in (0,T)$ and let ψ satisfying (1.2) and (1.3). Then, by (6.34) and since a pointwise solution is a weak solution (see Remark 3.2), we have, for every $\sigma > 0$,

$$\int_\sigma^{t'} \int_0^1 \tilde{f}(t,x) \left(\partial_t \psi + \partial_x (x^\alpha \partial_x \psi)\right)(t,x) \, \mathrm{d}t \, \mathrm{d}x$$

$$= \int_0^1 \tilde{f}(t',x)\psi(t',x) \, \mathrm{d}x - \int_0^1 \tilde{f}(\sigma,x)\psi(\sigma,x) \, \mathrm{d}x + \int_\sigma^{t'} u(t) \left(x^\alpha \partial_x \psi\right)(t,1) \, \mathrm{d}t.$$

Then, from (6.33), (6.36), (6.37) and (1.2), taking $\sigma \to 0^+$, we get the conclusion.

Finally, by construction (6.35) implies that $\tilde{f}(T,x) = 0$, for every $x \in (0,1)$.

□

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Appendix A. Some properties of the Gamma function

For any $p \in \mathbb{R}^+$, the Gamma function is defined (see [1, 6.1.1, p.254]) by

$$\Gamma(p) := \int_0^\infty e^{-t} t^{p-1} \, \mathrm{d}t, \tag{A.38}$$

which is a monotone increasing function on $(0,\infty)$. Furthermore, (see [1, 6.1.15, p.256])

$$\Gamma(x+1) = x\Gamma(x), \quad \forall x \in (0,\infty). \tag{A.39}$$

We have the following asymptotics of the Gamma function.

LEMMA A.1 ([1], 6.1.39)**.** *Let $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$. Then,*

$$\Gamma(ax+b) \underset{x\to\infty}{\sim} \sqrt{2\pi} e^{-ax} (ax)^{ax+b-\frac{1}{2}}. \tag{A.40}$$

We show an inequality used in Proposition 3.3.

LEMMA A.2.

$$(n+k)! \leq 2^{k+n} n! k!, \quad \forall n,k \in \mathbb{N}. \tag{A.41}$$

Proof. Let us observe first that

$$(2n)! \leq 2^{2n} n!^2, \quad \forall n \in \mathbb{N}. \tag{A.42}$$

This inequality follows by induction, since, for every $n \in \mathbb{N}$,

$$\begin{aligned} (2(n+1))! &= (2n)!(2n+1)(2n+2) \\ &\leq (2n)! 2^2 (n+1)^2 \leq 2^{2(n+1)} (n+1)!. \end{aligned}$$

To show (A.41), we assume, w.l.o.g., that $n < k$. Then, using (A.42),

$$\begin{aligned}(n+k)! &= (2n)!\prod_{j=1}^{k-n}(2n+j)\\ &\leq (2n)!2^{k-n}\prod_{j=1}^{k-n}(n+j) \leq 2^{n+k}n!k!.\end{aligned}$$

□

Appendix B. Some properties of Bessel functions

Let $\nu \in \mathbb{R}$. The Bessel function of order ν of the first kind is ([1, 9.1.10, p.360])

$$J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)}\left(\frac{z}{2}\right)^{2n+\nu}, \quad \forall z \in [0,\infty). \tag{B.43}$$

We denote by $\{j_{\nu,n}\}_{n\in\mathbb{N}^*}$ the increasing sequence of zeros of J_ν, which are real for any $\nu \geq 0$ and enjoy the following properties (see [1, 9.5.2, p.370] and [15, Proposition 7.8, p.135]).

$$\nu < j_{\nu,n} < j_{\nu,n+1},\ \forall n \in \mathbb{N}^*, \tag{B.44}$$

$$j_{\nu,n+1} - j_{\nu,n} \to \pi, \quad \text{as } n \to \infty. \tag{B.45}$$

We also have the integral formula ([1, 11.4.5, p.485])

$$\int_0^1 yJ_\nu(j_{\nu,n}y)J_\nu(j_{\nu,m}y)\,\mathrm{d}y = \frac{1}{2}|J_{\nu+1}(j_{\nu,n})|^2\delta_{n,m}, \quad \forall n,m \in \mathbb{N}^*. \tag{B.46}$$

This allows to show the following.

LEMMA B.1. [14, p.40] *Let $\nu \geq 0$. The family $\{w_n\}_{n\in\mathbb{N}^*}$ defined by*

$$w_n(z) := \frac{\sqrt{2z}}{|J_{\nu+1}(j_{\nu,n})|}J_\nu(j_{\nu,n}z), \quad \forall z \in (0,1),$$

is an orthonormal basis of $L^2(0,1)$. In particular, if $f \in L^2(0,1)$ and $d_n := \int_0^1 f(z)w_n(z)\,\mathrm{d}z$, $\forall n \in \mathbb{N}^*$, then $\|f\|^2_{L^2(0,1)} = \sum_{n=1}^{\infty}|d_n|^2$.

We recall that $\forall \nu \in \mathbb{R}$, the Bessel function J_ν satisfies the following differential equation (see [1, 9.1.1, p.358])

$$z^2J_\nu''(z) + zJ_\nu'(z) + (z^2-\nu^2)J_\nu(z) = 0, \quad \forall z \in (0,+\infty), \tag{B.47}$$

and the recurrence relation (see [1, 9.1.27, p.361]),

$$2J_\nu'(z) = J_{\nu-1}(z) + J_{\nu+1}(z), \quad \forall z \in (0,+\infty). \tag{B.48}$$

Asymptotic behaviour. We recall the asymptotic behaviour of J_ν for large arguments and near zero.

LEMMA B.2. [15, Lemma 7.2, p.129] *For any* $\nu \in \mathbb{R}$,

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + \underset{z\to\infty}{O}\left(\frac{1}{z\sqrt{z}}\right).$$

LEMMA B.3. [1, 9.1.7, p.360] *For any* $\nu \in \mathbb{R} \setminus \{-\mathbb{N}^*\}$,

$$J_\nu(z) \underset{z\to 0}{\sim} \frac{z^\nu}{2^\nu \Gamma(\nu+1)}.$$

The following asymptotic result is important in the proof of Proposition 5.1. We give the proof for the sake of completeness.

LEMMA B.4. *Let* $\nu \in \mathbb{R}^+$. *Then,*

$$\sqrt{j_{\nu,k}}|J_{\nu+1}(j_{\nu,k})| = \sqrt{\frac{2}{\pi}} + \underset{k\to\infty}{O}\left(\frac{1}{j_{\nu,k}}\right). \tag{B.49}$$

In particular, there exists a constant $C_1 > 0$ *such that for all* $k \in \mathbb{N}^*$,

$$\frac{1}{|J_{\nu+1}(j_{\nu,k})|} \leq C_1\sqrt{j_{\nu,k}}.$$

Proof. Using Lemma B.2, for $\nu+1$ and $x = j_{\nu,k}$,

$$\begin{aligned}\sqrt{j_{\nu,k}}|J_{\nu+1}(j_{\nu,k})| &= \sqrt{\frac{2}{\pi}}\left|\cos\left(j_{\nu,k} - \frac{\pi(\nu+1)}{2} - \frac{\pi}{4}\right)\right| + \underset{k\to\infty}{O}\left(\frac{1}{j_{\nu,k}}\right) \\ &= \sqrt{\frac{2}{\pi}}\left|\sin\left(j_{\nu,k} - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)\right| + \underset{k\to\infty}{O}\left(\frac{1}{j_{\nu,k}}\right).\end{aligned}$$

Using again Lemma B.2 with ν and $x = j_{\nu,k}$, we have that

$$\cos\left(j_{\nu,k} - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) = \underset{k\to\infty}{O}\left(\frac{1}{j_{\nu,k}}\right),$$

which gives

$$\left|\sin\left(j_{\nu,k} - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)\right| = \sqrt{1 + \underset{k\to\infty}{O}\left(\frac{1}{j_{\nu,k}^2}\right)} = 1 + \underset{k\to\infty}{O}\left(\frac{1}{j_{\nu,k}}\right)$$

and then (B.49).

□

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