Controllability properties of Grushin operators in dimension two

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VII Partial differential equations, optimal design and numerics

Thematic session on

Control and inverse problems for degenerate parabolic equations

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Outline

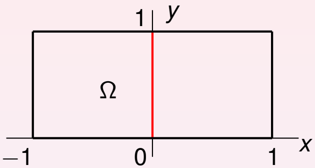
Generalized Grushin operators in dimension two

Controllability and observability

Extension to Grushin operators with singular potential

Generalized Grushin operators

$\Omega = (-1, 1) \times (0, 1)$
$T > 0$
$\Omega_T = (0, T) \times \Omega$



$\gamma > 0$

$$\begin{cases} \partial_t u - \underbrace{\left(\partial_x^2 u + |x|^{2\gamma} \partial_y^2 u\right)}_{G_\gamma u} = f & \text{in } \Omega_T \\ u(t, \pm 1, y) = 0 & 0 < y < 1 \\ u(t, x, 0) = 0 = u(t, x, 1) & -1 < x < 1 \\ u(0, x, y) = u_0(x, y) & (x, y) \in \Omega \end{cases}$$

- $u_0 \in L^2(\Omega)$ initial condition, $f \in L^2(\Omega_T)$ source term
- case $\gamma = 1$: M. Baouendi 1967, V. Grushin 1970-71

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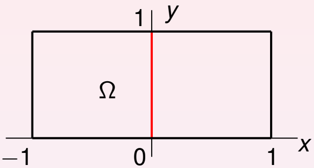

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Properties of Baouendi-Grushin operators

- sum of squares of vector fields

$$G_\gamma = \partial_x^2 + |x|^{2\gamma}\partial_y^2 = X_1^2 + X_2^2$$

- $\forall \gamma \in \mathbb{N}$: hypoellipticity

$$[X_1, X_2](x,y) = \begin{pmatrix} 0 \\ \gamma x^{\gamma-1} \end{pmatrix},\ [X_1,[X_1,X_2]](x,y) = \begin{pmatrix} 0 \\ \gamma(\gamma-1)x^{\gamma-2} \end{pmatrix},\ \dots$$

  satisfies Hörmander's condition $\forall \gamma \in \mathbb{N}$

- related to Laplace-Beltrami operator in almost riemannian structures (Boscain-Laurent 2013, Prandi-Rizzi-Seri 2017)

existence and uniqueness of solutions

$$\gamma > 0 \qquad \begin{cases} \partial_t u - \left(\partial_x^2 u + |x|^{2\gamma}\partial_y^2 u\right) = f & \text{in } \Omega_T \\ u(t,\pm 1, y) = 0\,,\ u(t,x,0) = 0 = u(t,x,1) & \text{on } \partial\Omega \\ u(0,x,y) = u_0(x,y) & (x,y) \in \Omega \end{cases}$$

$H = L^2(\Omega)$ and $V = \overline{C_0^\infty(\Omega)}$ with respect to

$$(f,g) = \int_\Omega \left(f_x g_x + |x|^{2\gamma} f_y g_y\right) dxdy$$

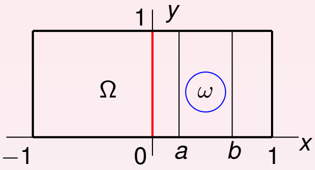
Well-posedness

$T > 0\,,\ u_0 \in L^2(\Omega)\,,\ f \in L^2(\Omega_T)$

$\Longrightarrow \exists!\ u \in C([0,T]; L^2(\Omega)) \cap L^2(0,T;V)\ :\ \forall t \in (0,T)\,,\ \phi \in C^2([0,T]\times\Omega)$

$$\int_\Omega [u(t)\phi(t) - u(0)\phi(0)] = \int_0^t \int_\Omega \left[u(\partial_t\phi + \partial_x^2\phi + |x|^{2\gamma}\partial_y^2\phi) + f\phi\right]$$

controllability



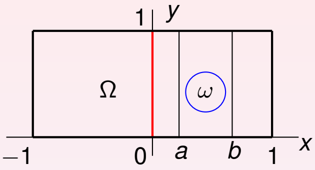
$$u^f \longleftrightarrow \begin{cases} \partial_t u - \left(\partial_x^2 u + |x|^{2\gamma}\partial_y^2 u\right) = \chi_\omega(x,y) f(t,x,y) \\ u(t,\pm 1, y) = 0\,,\ u(t,x,0) = 0 = u(t,x,1) \\ u(0,x,y) = u_0(x,y) \end{cases} \qquad (G)$$

- $u_0 \in L^2(\Omega)$, $\quad f \in L^2(\Omega_T)$ control
- $\omega \subset (a,b) \times (0,1)$ with $0 < a < b < 1$

want to study

- approximate controllability in time $T > 0$
- null controllability in time $T > 0$

controllability



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References on generalized Grushin operator

- Beauchard – Cannarsa – G. (2014)
  $\boxed{\partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u = \chi_\omega(x,y) f(t,x,y)}$ $(\#)$
  positive and negative controllability results, depending on γ

- Beauchard – Cannarsa – Yamamoto (2014) inverse source problem $\partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u = f(t,x,y)R(t,x)$

- Beauchard – Miller – Morancey (2015)
  sharp minimum time for $\gamma = 1$ in $(\#)$ with control in symmetric strip $\omega = (-b,-a)\times(a,b)$, $a > 0$

- Beauchard – Dardé – Ervedoza (2017)
  sharp minimum time for $\gamma = 1$ in $(\#)$ with $\omega = (a,b)$, $a > 0$

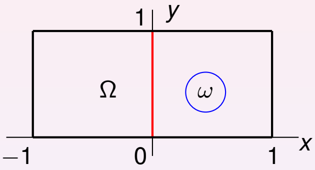
- Koenig (2017)
  lack of null controllability under some geometric conditions

- Anh, Toi (2013): multi-dimensional case

approximate controllability

approximate controllability $\Longleftrightarrow$ unique continuation

$$\begin{cases} \partial_t p - \partial_x^2 p - |x|^{2\gamma}\partial_y^2 p = 0 & (t,x,y) \in (0,T)\times\Omega \\ p(t,x,y) = 0 & (t,x,y)\in(0,T)\times\partial\Omega \end{cases} \tag{1}$$



Garofalo (1993): unique continuation for elliptic operator

$$A = \partial_x^2 + |x|^{2\gamma}\partial_y^2$$

for parabolic operators:

Proposition (Beauchard-Cannarsa-G.)

Let $T > 0$, $\gamma > 0$, let $\omega \subset (0,1)\times(0,1)$, and let p be a solution of (1). If $p \equiv 0$ on $(0,T)\times\omega$, then $p \equiv 0$ on $(0,T)\times\Omega$.

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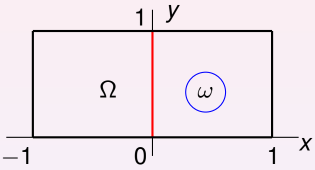

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Sketch of the proof

Main tools:

(1) Fourier decomposition: v solution to

$$\begin{cases} \partial_t v - \partial_x^2 v - |x|^{2\gamma}\partial_y^2 v = 0 & (t,x,y)\in(0,T)\times\Omega \\ v(t,\pm 1,y)=0\,,\ v(t,x,0)=0=v(t,x,1) & t\in(0,T) \\ v(0,x,y)=v_0(x,y) & (x,y)\in\Omega \end{cases} \qquad (G^*)$$

$$v(t,x,y)=\sum_{n=1}^{\infty} v_n(t,x)e_n(y) \quad \text{with} \quad e_n(y):=\sqrt{2}\sin(n\pi y)$$

where $v_n(t,x):=\int_0^1 v(t,x,y)e_n(y)dy$ satisfies

$$\begin{cases} \partial_t v_n - \partial_x^2 v_n + (n\pi)^2|x|^{2\gamma} v_n = 0 & (t,x)\in(0,T)\times(-1,1) \\ v_n(t,\pm 1)=0 & t\in(0,T) \\ v_n(0,x)=v_{0,n}(x) & x\in(-1,1) \end{cases} \qquad (G_n^*)$$

$$\int_\Omega |v(T,x,y)|^2 dxdy = \sum_{n=1}^{\infty}\int_{-1}^{1}|v_n(T,x)|^2 dx$$

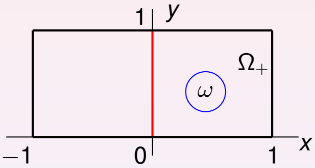
$$\int_{\omega=(a,b)\times(0,1)} |v(t,x,y)|^2 dxdy = \sum_{n=1}^{\infty}\int_a^b |v_n(t,x)|^2 dx$$

Sketch of the proof (cnt)

(2) unique continuation in 1−D for the uniformly parabolic equation satisfied by the Fourier coefficients v_n

$\omega \subset \Omega_+ = (0,1) \times (0,1)$

$$v \equiv 0 \quad (0,T) \times \omega \quad \Longrightarrow \quad v \equiv 0 \quad (0,T) \times \Omega_+$$



$$v(t,x,y) = \sum_{n=1}^{\infty} v_n(t,x) e_n(y) \quad \Longrightarrow \quad v_n \equiv 0 \quad (0,T) \times (0,1) \quad \forall n \geq 1$$

with

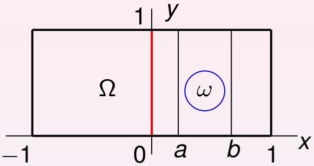
$$\begin{cases} \partial_t v_n - \partial_x^2 v_n + (n\pi)^2 |x|^{2\gamma} v_n = 0 & (t,x) \in (0,T) \times (-1,1) \\ v_n(t,\pm 1) = 0 & t \in (0,T) \end{cases}$$

then

$$v_n \equiv 0 \quad (0,T) \times (-1,1) \quad \forall n \geq 1 \quad \Longrightarrow \quad v \equiv 0 \quad (0,T) \times \Omega$$

null controllability: $0 < \gamma < 1$ and $\gamma = 1$

$$\begin{cases} \partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u = \chi_\omega(x,y) f(t,x,y) \\ u(t,\pm 1, y) = 0\,, \; u(t,x,0) = 0 = u(t,x,1) \\ u(0,x,y) = u_0(x,y) \end{cases} \qquad (G)$$



Theorem (Beauchard-Cannarsa-G.)

- $0 < \gamma < 1$ $\Longrightarrow$ (G) *null controllable* $\forall T > 0$
- $\gamma = 1$ & $\omega = (a,b) \times (0,1)$ $\Longrightarrow$ (G) *null controllable* $\forall\, T > T^* \geqslant a^2/2$

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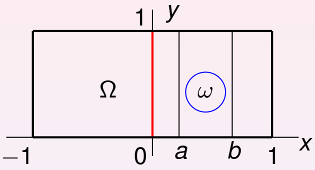

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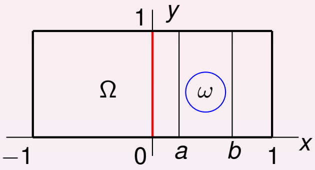

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Main idea

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observable in $[0,T]\times\omega$ iff $\exists C_T > 0$ such that $\forall v_0 \in L^2(\Omega)$

$$\int_\Omega |v(T,x,y)|^2 dxdy \le C_T \int_0^T \int_\omega |v(t,x,y)|^2 dxdy \qquad (O)$$

observability for (G^*) in ω $\iff$ uniform observability for (G_n^*) in (a,b)

$$\int_{-1}^{1} |v_n(T,x)|^2 dx \le C \int_0^T \int_a^b |v_n(t,x)|^2 dxdt \qquad \forall n \ge 1$$

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Sketch of the proof ($\gamma \in (0,1)$ and $\gamma = 1$)

- Fourier decomposition and reduction to (UO) for (G_n^*) in (a,b)
- growth rate of the first eigenvalue of the 1−D operator

$$G_n\varphi := -\varphi'' + (n\pi)^2|x|^{2\gamma}\varphi$$

$$\lambda_n = \min \sigma(G_n) = \min_{v\in H_0^1(-1,1)\setminus\{0\}} \frac{\int_{-1}^1 \left[|v'|^2 + (n\pi)^2|x|^{2\gamma}v^2\right] dx}{\int_{-1}^1 v^2 dx}$$

Lemma (dissipation speed)

$\forall\gamma > 0 \quad \exists\, c^*(\gamma) \geq c_*(\gamma) > 0$ *such that* $\boxed{c_* n^{\frac{2}{1+\gamma}} \leq \lambda_n \leq c^* n^{\frac{2}{1+\gamma}}}$

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Sketch of the proof ($\gamma \in (0,1)$ and $\gamma = 1$)

- Carleman estimate with suitable weights

For $n \in \mathbb{N}^*$, we introduce the operator

$$\mathcal{P}_n g := \frac{\partial g}{\partial t} - \frac{\partial^2 g}{\partial x^2} + (n\pi)^2 |x|^{2\gamma} g.$$

Proposition

Let $\gamma \in (0,1]$. Then $\exists \beta \in C^1([-1,1];\mathbb{R}_+^*)$ and $\mathcal{C}_1, \mathcal{C}_2 > 0$ s. t. $\forall n \in \mathbb{N}^*$, $T > 0$, $g \in C^0([0,T];L^2(-1,1)) \cap L^2(0,T;H_0^1(-1,1))$ holds

$$\mathcal{C}_1 \int_0^T \int_{-1}^1 \left(\frac{M}{t(T-t)} \left| \frac{\partial g}{\partial x}(t,x) \right|^2 + \frac{M^3}{(t(T-t))^3} \left| g(t,x) \right|^2 \right) e^{-\frac{M\beta(x)}{t(T-t)}} dxdt$$
$$\leqslant \int_0^T \int_{-1}^1 |\mathcal{P}_n g|^2 e^{-\frac{M\beta(x)}{t(T-t)}} dxdt + \int_0^T \int_a^b \frac{M^3}{(t(T-t))^3} |g(t,x)|^2 e^{-\frac{M\beta(x)}{t(T-t)}} dxdt$$

where $M := \mathcal{C}_2 \max\{T + T^2; nT^2\}$.

proof of uniform observability $\gamma \in (0,1)$

note for $t \in (T/3, 2T/3)$

$$\frac{4}{T^2} \leqslant \frac{1}{t(T-t)} \leqslant \frac{9}{2T^2} \quad \text{and} \quad \int_{-1}^{1} v_n(T,x)^2 dx \leqslant e^{-\frac{2}{3}\lambda_n T} \int_{-1}^{1} v_n(t,x)^2 dx$$

by Carleman estimate and dissipation speed

$$\int_{-1}^{1} v_n(T,x)^2 dx \le c_0 T^2 e^{c_1 \frac{M_n}{T^2} - c_2 n^{\frac{2}{1+\gamma}} T} \int_0^T \int_a^b v_n(t,x)^2 dxdt$$

for some constants c_0, c_1, c_2 (independent of n, T and v_n)

- $\boxed{n < 1 + \frac{1}{T}}$ since $M_n = \mathcal{C}(T + T^2)$

$$\int_{-1}^{1} v_n(T,x)^2 dx \leqslant \mathcal{C}_0 T^2 e^{\mathcal{C}_1 \left(1+\frac{1}{T}\right)} \int_0^T \int_a^b v_n(t,x)^2 dxdt$$

- $\boxed{n \ge 1 + \frac{1}{T}}$ since $M_n = \mathcal{C} n T^2$ maximizing $x \mapsto c_1 \mathcal{C} x - c_2 x^{\frac{2}{1+\gamma}} T$ we obtain

$$\int_{-1}^{1} v_n(T,x)^2 dx \le \mathcal{C}_2 T^2 e^{c_3 T^{-\frac{1+\gamma}{1-\gamma}}} \int_0^T \int_a^b v_n(t,x)^2 dxdt$$

optimality: the case of $\gamma > 1$ and $\gamma = 1$

$$\begin{cases} \partial_t v - \partial_x^2 v - |x|^{2\gamma}\partial_y^2 v = 0 & (0,T)\times\Omega \\ v(t,\pm 1,y) = 0\,,\ v(t,x,0) = 0 = v(t,x,1) & t\in(0,T) \\ v(0,x,y) = v_0(x,y) & (x,y)\in\Omega \end{cases} \qquad (G^*)$$

Theorem (BCG)

- $\gamma > 1 \implies (G^*)$ *not observable*
- $\gamma = 1 \implies \exists T^* \geqslant a^2/2$ *such that* (G^*) *not observable* $\forall\, T < T^*$

$$\begin{cases} \partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u = \chi_\omega(x,y) f(t,x,y) \\ u(t,\pm 1,y) = 0\,,\ u(t,x,0) = 0 = u(t,x,1) \\ u(0,x,y) = u_0(x,y) \end{cases} \qquad (G)$$

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Lack of Null Controllability ($\gamma > 1$ and $\gamma = 1$)

Main steps of the proof:

- disprove uniform observability for (G_n^*)
- comparison argument
- explicit supersolution $W_n \in C^2([x_n, 1], \mathbb{R})$ of

$$\begin{cases} -W_n''(x) + [(n\pi)^2 x^{2\gamma} - \lambda_n] W_n(x) \geqslant 0\,, & x \in (x_n, 1)\,, \\ W_n(1) \geqslant 0\,, \\ W_n'(x_n) < -\sqrt{x_n}\lambda_n\,, \end{cases}$$

  where $x_n := \left(\frac{\lambda_n}{(n\pi)^2}\right)^{\frac{1}{2\gamma}}$ for every $n \in \mathbb{N}^*$

- dissipation speed of the first eigenvalue of the 1−D operators G_n

references on Grushin with singular potential

- Boscain – Laurent (2013) Laplace-Beltrami on a 2D compact manifold showing that solution of

$$\boxed{\partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u + \frac{c(\gamma)}{x^2}u = 0} \qquad (\gamma \geq 1, x \in \mathbb{R}, y \in \mathbb{T})$$

is supported in $\mathbb{R}_+ \times \mathbb{T}$ if so is $u(0)$

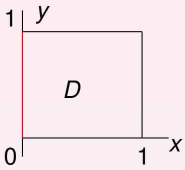
- Cannarsa – G. (2013) positive controllability result for

$$\partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u + \frac{\lambda}{x^2}u = 0 \qquad \gamma > 0, x, y \in (0,1), \lambda > -\frac{1}{4}$$

- Morancey (2015) approximate controllability result for $x \in (-1,1)$, $\gamma > 0$, $\lambda \in \left(-\frac{1}{4}, \frac{3}{4}\right)$
- Anh, Toi (2016): multi-dimensional case

Grushin operator with singular potential

$D = (0,1) \times (0,1)$
$T > 0$
$D_T = (0,T) \times D$



$\gamma > 0 \quad \& \quad \lambda \in \mathbb{R}$:

$$\begin{cases} \partial_t u - (\partial_x^2 u + |x|^{2\gamma} \partial_y^2 u) - \frac{\lambda}{x^2} u = f & \text{in } D_T \\ u(x,y,t) = 0 & \text{on } \partial D \times (0,T)\,, \\ u(x,y,0) = u_0(x,y) & (x,y) \in D\,, \end{cases} \qquad (GruSingPot)$$

- $u_0 \in L^2(D)$ initial condition
- $f \in L^2(D_T)$ control function

Unique continuation and approximate controllability

Consider the adjoint system

$$\begin{cases} \partial_t g - \partial_x^2 g - |x|^{2\gamma}\partial_y^2 g - \frac{\lambda}{x^2} g = 0 & \text{in } D \times (0,T)\,, \\ g(x,y,t) = 0 & \text{on } \partial D \times (0,T)\,, \\ g(x,y,0) = g_0(x,y) \in L^2(D)\,. \end{cases} \qquad (AdjGrPot)$$

Proposition (Cannarsa-G.)

Let $T > 0$, $\gamma > 0$, $\lambda < 1/4$, ω an open subset of $(0,1) \times (0,1)$, let $g \in C([0,T]; H) \cap L^2(0,T; W)$ be a weak solution of system $(AdjGrPot)$.

If $g \equiv 0$ on $\omega \times (0,T)$, then $g \equiv 0$ on $\Omega \times (0,T)$.

Null Controllability in large times for $\gamma = 1$

Theorem (Cannarsa–G.)

Let $\omega = (a, b) \times (0, 1)$ *for some* $0 < a < b \leqslant 1$ *and* $\lambda < 1/4$. *Then there exists* $T^* > 0$ *such that for every* $T > T^*$ *system*

$$\begin{cases} \partial_t u - (\partial_x^2 u + |x|^2 \partial_y^2 u) - \frac{\lambda}{x^2} u = f & \text{in } D_T \\ u(x, y, t) = 0 & \text{on } \partial D \times (0, T),\ 0 < y < 1, \\ u(x, y, 0) = u_0(x, y) & (x, y) \in D, \end{cases}$$

is null controllable in time T.

Equivalent to the observability in large times from ω for the adjoint system (*AdjGrPot*)

Sketch of the proof ($\gamma = 1$)

- Hardy's inequality and improved Hardy-Poincaré's inequality
- Fourier decomposition and reduction to (UO) for the $1-$D adjoint systems in (a, b)
- revisited growth rate of the first eigenvalue μ_n of the $1-$D operator

$$A_n\varphi := -\varphi'' + \left[(n\pi)^2|x|^{2\gamma} - \frac{\lambda}{x^2}\right]\varphi$$

$\forall\gamma > 0, \lambda < 1/4, \quad \exists\ C^*(\gamma) \geq C_*(\gamma) > 0 \quad$ such that

$$\boxed{C_* n^{\frac{2}{1+\gamma}} \leq \mu_n \leq C^* n^{\frac{2}{1+\gamma}}}$$

- Carleman estimate with a suitable spatial weight $\beta(x) := \frac{2-x^2}{4}$, where $M := C_2 \max(T^{k/2} + T^{2k}, T^{2k}n)$.

Outlook

- $\boxed{\partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u = \chi_\omega(x,y) f(x,y,t)}$
  - 2−D Carleman estimate for the Grushin operator?
  - null controllability for $\gamma = 1$ and more general ω (counterexample by Koenig (2017))
  - sharp estimate of T^* for $\gamma = 1$ ($T^* = a^2/2$ proved by Beauchard–Miller–Morancey (2015) and by Beauchard–Dardé–Ervedoza)
- $\boxed{\partial_t u - \partial_x^2 u - |x|^{2\gamma}\partial_y^2 u + \frac{c}{x^2} u = \chi_\omega(x,y) f(x,y,t)}$

when $x \in (0,1)$:

  - null controllability should hold for $0 < \gamma < 1$;
  - null controllability should fail in the case $\gamma > 1$ or in the case $\gamma = 1$ with $T < T^*$;

in the case $x \in (-1,1)$: both degeneracy of the diffusion coefficient and singularity of the potential term in the interior, approximate controllability proved by M. Morancey
null controllability completely open

Thank you
for your attention!