Controllability of a nonlinear reaction-diffusion system

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The free system

Let $T > 0$ (final time), $N \in \mathbb{N}^*$ (the spatial dimension), Ω be a bounded, connected, open subset of $\mathbb{R}^N$ of class C^2, $Q := (0, T) \times \Omega$ and ω a nonempty open subset of Ω, $(d_1, d_2, d_3, d_4) \in (0, +\infty)^4$.

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We are interested in the following 4×4 reaction-diffusion system

$$\forall 1 \leq i \leq 4, \quad \begin{cases} \partial_t u_i - d_i \Delta u_i = (-1)^i (u_1 u_3 - u_2 u_4) & \text{in } (0, T) \times \Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (0, T) \times \partial\Omega, \\ u_i(0, .) = u_{i,0} & \text{in } \Omega, \end{cases}$$

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Global existence of weak solutions holds in any dimension.
The uniqueness is an open problem (for $N \geq 3$).

Controllability

$(u_1^*, u_2^*, u_3^*, u_4^*) \in [0, +\infty)^4$ such that

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$$\begin{cases} \partial_t u_i - d_i \Delta u_i = (-1)^i (u_1 u_3 - u_2 u_4) + h_i 1_\omega 1_{i \leq j} & \text{in } (0,T) \times \Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ u_i(0,.) = u_{i,0} & \text{in } \Omega. \end{cases} \tag{S}$$

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QUESTION : For every $u_0 \in L^\infty(\Omega)^4$, does there exist $(h_i)_{1 \leq i \leq j} \in L^\infty(Q)^j$ such that the solution u of (S) satisfies

$$\forall i \in \{1, 2, 3, 4\}, \; u_i(T, .) = u_i^*. \tag{1}$$

Two main results

$$\begin{cases} \partial_t u_i - d_i \Delta u_i = (-1)^i (u_1 u_3 - u_2 u_4) + h_i 1_\omega 1_{i \leq j} & \text{in } (0,T) \times \Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ u_i(0,.) = u_{i,0} & \text{in } \Omega. \end{cases} \tag{S}$$

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(S) *is **globally controllable** in $L^\infty(\Omega)$ with controls in $L^\infty(Q)$ **in large time and in small dimension ($N \leq 2$)**.*

The beginning : Null-controllability of the heat equation

Theorem (Lebeau-Robbiano (1995) and Fursikov-Imanuvilov (1996))

For every $u_0 \in L^2(\Omega)$*, there exists* $h \in L^2(Q)$ *such that the solution* u *of*

$$\left\{\begin{array}{ll} \partial_t u - \Delta u = h 1_\omega & \text{in } (0,T)\times\Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega, \end{array}\right.$$

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Proof.

1. Null-controllability $\Leftrightarrow$ Observability inequality for the adjoint system.
2. Carleman estimates $\Rightarrow$ Observability inequality for the adjoint system.

□

Null-controllability of linear parabolic *cascad* systems

Toy-system ($a_{ij} \in L^\infty(Q)$) :

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = a_{11} u_1 + a_{12} u_2 + a_{13} u_3 + h_1 1_\omega & \text{in } (0,T)\times\Omega, \\ \partial_t u_2 - d_2 \Delta u_2 = a_{21} u_1 + a_{22} u_2 + a_{23} u_3 & \text{in } (0,T)\times\Omega, \\ \partial_t u_3 - d_3 \Delta u_3 = \qquad\qquad a_{32} u_2 + a_{33} u_3 & \text{in } (0,T)\times\Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega. \end{cases} \tag{Ca}$$

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Proposition (Gonzalez-Burgos and de Teresa (2010))

If there exist $(t_1, t_2) \subset (0,T)$, $\omega_0 \subset \omega$ *and* $\varepsilon > 0$, *such that*

$$a.e.\ (t,x) \in (t_1,t_2)\times\omega_0, \ \ a_{21}(t,x) \geq \varepsilon, \ \ a_{32}(t,x) \geq \varepsilon,$$

then (Ca) *is null-controllable in* L^2.

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$$h_1 \overset{controls}{\rightsquigarrow} u_1 \overset{controls}{\rightsquigarrow} u_2 \overset{controls}{\rightsquigarrow} u_3.$$

Null-controllability of linear *crossed-diffusion* systems

Proposition (Guerrero (2007))

Let $(a_{11}, a_{12}, d) \in \mathbb{R}^3$. *Toy-system :*

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Then, the following conditions are equivalent.

1. *System* (CD) *is null-controllable in* $\left\{ u_0 \in L^2(\Omega)^2 \; ; \int_\Omega u_{2,0} = 0 \right\}$.
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Remark

The case of non constant coupling coefficients is not totally solved.

Null-controllability of *semilinear* parabolic systems - linearization in 0

$f_1, f_2 \in C^\infty(\mathbb{R}^2; \mathbb{R})$.

$$\left\{\begin{array}{ll} \partial_t u_1 - d_1 \Delta u_1 = f_1(u_1, u_2) + h_1 1_\omega & \text{in } (0,T)\times\Omega, \\ \partial_t u_2 - d_2 \Delta u_2 = f_2(u_1, u_2) & \text{in } (0,T)\times\Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,.) = u_0 & \text{in } \Omega., \end{array}\right. \qquad \text{(NL)}$$

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Proposition (Ammar Khodja, Benabdallah, Dupaix (2006) - Coron, Guerrero, Rosier (2010))

If $\frac{\partial f_2}{\partial u_1}(0,0) \neq 0$, *then* (NL) *is locally controllable in* $L^\infty(\Omega)$ *with control in* $L^\infty(Q)$.

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Proof.

1. The linearized system in $(0,0)$ is null-controllable in L^2.

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2. The linearized system in $(0,0)$ is null-controllable in L^∞.

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Proof.

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2. The linearized system in $(0,0)$ is null-controllable in L^∞.
3. Fixed-point argument in L^∞.

Null-controllability of *semilinear* parabolic systems - *return method*

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STRATEGY : linearization around a **non trivial trajectory** $(\overline{u_1}, \overline{u_2}, \overline{h_1})$ of the nonlinear system which goes from 0 to 0. This procedure is called the *return method* and was introduced by Coron for Euler equations.

Proposition (Ammar Khodja, Benabdallah, Dupaix (2006) - Coron, Guerrero, Rosier (2010))

We assume that there exist $(t_1,t_2) \subset (0,T)$, $\omega_0 \subset \omega$ *and* $\varepsilon > 0$ *such that* $\frac{\partial f_2}{\partial u_1}(\overline{u_1},\overline{u_2}) \geq \varepsilon$ *on* $(t_1,t_2)\times\omega_0$. *Then, then* (NL) *is locally controllable in* $L^\infty(\Omega)$ *with control in* $L^\infty(Q)$.

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Example : $f_2(u_1,u_2) = u_1^3 + Ru_2$ $(R\in\mathbb{R})$ (Coron, Guerrero, Rosier (2010))

Restatements of the system

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Trajectory of the nonlinear controlled system

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$$\left\{\begin{array}{ll} \partial_t u_i - d_i \Delta u_i = (-1)^i (u_1 u_3 - u_2 u_4) + h_i 1_\omega 1_{i \leq j} & \text{in } (0,T) \times \Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ u_i(0,.) = u_{i,0} & \text{in } \Omega. \end{array}\right. \tag{S}$$

$$Y = L^2(0,T; H^1(\Omega)) \cap W^{1,2}(0,T; (H^1(\Omega))').$$

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Definition

*For $u_0 \in L^\infty(\Omega)^4$, $((u_i)_{1\leq i\leq 4}, (h_i)_{1\leq i\leq j})$ is a **trajectory** of (S) if*

1. $((u_i)_{1\leq i\leq 4}, (h_i)_{1\leq i\leq j}) \in (Y \cap L^\infty(Q))^4 \times L^\infty(Q)^j$,
2. $(u_i)_{1\leq i\leq 4}$ *is the (unique) solution of* (S).

*Moreover, $((u_i)_{1\leq i\leq 4}, (h_i)_{1\leq i\leq j})$ is a **trajectory** of (S) reaching $(u_i^*)_{1\leq i\leq 4}$ if*

$$\forall i \in \{1, \dots, 4\}, \; u_i(T,.) = u_i^*.$$

Invariant quantities of the nonlinear dynamics : 2 controls

$((u_i)_{1\leq i\leq 4}, (h_i)_{1\leq i\leq j})$ is a **trajectory** of (S) reaching $(u_i^*)_{1\leq i\leq 4}$.

$$\frac{d}{dt}\left(\int_\Omega (u_3(t,x)+u_4(t,x))dx\right) = 0 \text{ for a.e. } 0\leq t\leq T.$$

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Moreover,

$$\begin{cases} \partial_t(u_3+u_4) - d_4\Delta(u_3+u_4) = (d_3-d_4)\Delta u_3 & \text{in } (0,T)\times\Omega, \\ \frac{\partial(u_3+u_4)}{\partial n} = 0 & \text{on } (0,T)\times\partial\Omega. \end{cases}$$

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If $d_3 = d_4$, then the **backward uniqueness for the heat equation** gives that

$$\forall t \in [0,T],\ (u_3+u_4)(t,.) = (u_3+u_4)(T,.) = u_3^* + u_4^*.$$

Consequently,

$$\Big(d_3 = d_4\Big) \Rightarrow \Big(u_{3,0} + u_{4,0} = u_3^* + u_4^*\Big). \qquad \text{(SMC-3,4)}$$

Invariant quantities of the nonlinear dynamics : 1 control

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$$\Big(k \neq l \in \{2,3,4\},\ d_k = d_l\Big) \Rightarrow \Big(u_{k,0} + u_{l,0} = u_k^* + u_l^*\Big). \qquad \text{(SMC-k,l)}$$

Backward uniqueness

Theorem (BU - Bardos, Tartar (1971))

Let $k \in \mathbb{N}^*$, $D = diag(d_1, \dots, d_k)$ *where* $d_i \in (0, +\infty)$, $C \in \mathcal{M}_k(L^\infty(Q))$, $\zeta_0 \in L^\infty(\Omega)^k$. *Let* $\zeta \in Y^k$ *be the solution of*

$$\begin{cases} \zeta_t - D\Delta\zeta = C(t,x)\zeta & \text{in } (0,T) \times \Omega, \\ \frac{\partial \zeta}{\partial n} = 0 & \text{on } (0,T) \times \partial\Omega, \\ \zeta(0,.) = \zeta_0 & \text{in } \Omega. \end{cases}$$

If $\zeta(T,.) = 0$, *then for every* $t \in [0,T]$, $\zeta(t,.) = 0$.

More restrictive conditions on the initial data : 2 controls

$((u_i),(h_i))$ a trajectory of (S) reaching (u_i^*) such that $(u_3^*, u_4^*) = (0,0)$.

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Conversely, $u_0 \in L^\infty(\Omega)^4$ such that $(u_{3,0},u_{4,0}) = (0,0)$, (S) becomes

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To sum up : Constraints on the initial condition

3 controls : We introduce

$$X_{3,(d_i),(u_i^*)} = L^\infty(\Omega)^4.$$

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$d_3 = d_4$	$u_{3,0} + u_{4,0} = u_3^* + u_4^*$
$d_3 \neq d_4$	$\frac{1}{\lvert\Omega\rvert}\int_\Omega (u_{3,0} + u_{4,0}) = u_3^* + u_4^*$

$$X_{2,(d_i),(u_i^*)} := \{u_0 \in L^\infty(\Omega)^4 \;;\; u_0 \text{ satisfies the associated condition}\}.$$

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$d_2 = d_4, \; d_2 \neq d_3$	$u_{2,0} - u_{4,0} = u_2^* - u_4^*, \; \frac{1}{\lvert\Omega\rvert} \int_\Omega (u_{3,0} + u_{4,0}) = u_3^* + u_4^*$
$d_2 \neq d_3, \; d_3 \neq d_4, \; d_2 \neq d_4$	$\frac{1}{\lvert\Omega\rvert} \int_\Omega (u_{2,0} + u_{3,0}) = u_2^* + u_3^*, \; \frac{1}{\lvert\Omega\rvert} \int_\Omega (u_{3,0} + u_{4,0}) = u_3^* + u_4^*$

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Local controllability result

Theorem

For every $j \in \{1, 2, 3\}$*, for every* $(u_i^*) \in (\mathbb{R}^+)^4$ *such that* $u_1^* u_3^* = u_2^* u_4^*$*,* (S) *is* ***locally controllable*** *in* $X_{j,(d_i),(u_i^*)}$ *with controls in* $L^\infty(Q)^j$*.*

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Remark

This local controllability result can be improved in a global controllability result for easy cases :

1. $(u_3^*, u_4^*) = (0, 0)$ *for* 2 *controls,*
2. $u_3^* = 0$ *for* 1 *control.*

Global controllability result - small dimension - large time

Theorem

*We assume that $N = 1$ or 2 ($N =$ dimension). Let $j \in \{1, 2, 3\}$ and $(u_i^*)_{1 \leq i \leq 4} \in (\mathbb{R}^+)^4$ be such that $u_1^* u_3^* = u_2^* u_4^*$. Then, for every $u_0 \in X_{j,(d_i),(u_i^*)}$ satisfying **a condition of positivity**, there exists $T^* > 0$ (sufficiently large) and $h^j \in L^\infty((0, T^*) \times \Omega)^j$ such that the solution u of*

$$\begin{cases} \partial_t u_i - d_i \Delta u_i = (-1)^i (u_1 u_3 - u_2 u_4) + h_i 1_\omega 1_{i \leq j} & \text{in } (0, T^*) \times \Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (0, T^*) \times \partial\Omega, \\ u_i(0, .) = u_{i,0} & \text{in } \Omega. \end{cases}$$

satisfies

$$u(T^*, .) = u^*.$$

Sketch of proof of the local controllability result - $j = 3$

LINEARIZATION of (S) around (u_i^*) :

$$\begin{cases} \partial_t u_i - d_i \Delta u_i = (-1)^i (u_3^* u_1 - u_4^* u_2 + u_1^* u_3 - u_2^* u_4) + h_i 1_\omega 1_{i \leq 3}, \\ \frac{\partial u_i}{\partial n} = 0, \\ u_i(0,.) = u_{i,0}. \end{cases}$$

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$$h_1 \overset{controls}{\rightsquigarrow} u_1, h_2 \overset{controls}{\rightsquigarrow} u_2, h_3 \overset{controls}{\rightsquigarrow} u_3.$$

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And u_4 ? $\partial_t u_4 - d_4 \Delta u_4 = u_3^* u_1 - u_4^* u_2 + u_1^* u_3 - u_2^* u_4$.

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And u_4 ? $\partial_t u_4 - d_4 \Delta u_4 = u_3^* u_1 - u_4^* u_2 + u_1^* u_3 - u_2^* u_4$.
First case : $(u_1^*, u_3^*, u_4^*) \neq (0,0,0)$. For example, $u_3^* \neq 0$, then
$u_1 \overset{controls}{\rightsquigarrow} u_4$.

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And u_4 ? $\partial_t u_4 - d_4 \Delta u_4 = \cancel{u_3^* u_1 + u_4^* u_2 + u_1^* u_3} - u_2^* u_4$.

First case : $(u_1^*, u_3^*, u_4^*) \neq (0,0,0)$**.** For example, $u_3^* \neq 0$, then
$u_1 \overset{controls}{\rightsquigarrow} u_4$.

Second case : $(u_1^*, u_3^*, u_4^*) = (0,0,0)$**, *return method.*** Linearization around a **non trivial trajectory of** (S) which comes from $(0, u_2^*, 0, 0)$ and goes to $(0, u_2^*, 0, 0)$:

Sketch of proof of the local controllability result - $j = 3$

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$$\overline{u_3} \in C^\infty(\overline{Q}),\ \overline{u_3} \geq 0,\ \overline{u_3} \neq 0,\ supp(\overline{u_3}) \subset (0,T) \times \omega,\ \overline{h_3} = \partial_t \overline{u_3} - d_3 \Delta \overline{u_3}.$$

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Then, $\partial_t u_4 - d_4 \Delta u_4 = \overline{u_3} u_1 - u_2^* u_4 \text{ in } (0,T) \times \Omega$.

As $\overline{u_3} \neq 0$, $u_1 \overset{controls}{\rightsquigarrow} u_4$.

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(S) reduces to

$$\forall 1 \leq i \leq 3, \begin{cases} \partial_t u_i - d_i \Delta u_i = (-1)^i (u_1 u_3 - u_2(m - u_3)) + h_i 1_\omega 1_{i \leq 2}, \\ \frac{\partial u_i}{\partial n} = 0, \\ u_i(0, .) = u_{i,0}. \end{cases}$$

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Otherwise, $u_3^* = 0$, then $m = u_4^*$.

$$(m \neq 0) \Rightarrow (u_2 \overset{controls}{\rightsquigarrow} u_3).$$

Proof of the local controllability result - $j = 2$, $d_3 \neq d_4$

We can assume that $(u_3^*, u_4^*) \neq (0,0)$.

$$\boxed{(u_1, u_2, u_3, u_4)(T,.) = (u_1^*, u_2^*, u_3^*, u_4^*)}$$

if and only if

$$\boxed{(u_1, u_2, u_3, u_3 + u_4)(T,.) = (u_1^*, u_2^*, u_3^*, u_3^* + u_4^*)}\ .$$

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Local controllability result - $j = 1$, (d_2, d_3, d_4) different

We can find α, β and $\gamma \neq 0$ such that

$$\boxed{(u_1, u_2, u_3, u_4)(T, .) = (u_1^*, u_2^*, u_3^*, u_4^*)}$$

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and the LINEARIZED-system satisfied by
$(v_1, v_2, v_3, v_4) := (u_1, u_2, u_2 + u_3, \alpha u_2 + \beta u_3 + \gamma u_4)$ gives

$$h_1 \overset{controls}{\rightsquigarrow} v_1,$$

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Global controllability result - two steps

Theorem

We assume that $N = 1$ or 2 ($N =$ dimension). Let $j \in \{1, 2, 3\}$ and $(u_i^*)_{1 \leq i \leq 4} \in (\mathbb{R}^+)^4$ be such that $u_1^* u_3^* = u_2^* u_4^*$. Then, for every $u_0 \in X_{j,(d_i),(u_i^*)}$ satisfying ***a condition of positivity****, there exists $T^* > 0$ (sufficiently large) and $h^j \in L^\infty((0, T^*) \times \Omega)^j$ such that the solution u of*

$$\left\{ \begin{array}{ll} \partial_t u_i - d_i \Delta u_i = (-1)^i (u_1 u_3 - u_2 u_4) + h_i 1_\omega 1_{i \leq j} & \text{in } (0, T^*) \times \Omega, \\ \frac{\partial u_i}{\partial n} = 0 & \text{on } (0, T^*) \times \partial\Omega, \\ u_i(0, .) = u_{i,0} & \text{in } \Omega. \end{array} \right.$$

satisfies

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Proof.

1. Let evolve the system **without control** : convergence to (z_i), a particular stationary constant state.
2. Use **the local controllability result** together with a connectivity-compactness argument to link (z_i) and (u_i^*).

Global controllability result - first step

Proposition (Desvillettes-Fellner (2014), Pierre-Suzuki-Yamada-Zou (2016))

$1 \leq N \leq 2$.

The solution $u \in L^\infty((0,\infty)\times\Omega)^4$ *of*

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satisfies

$$\lim_{T\to+\infty} \|u(T,.) - z\|_{L^\infty(\Omega)^4} = 0,$$

where $z \in (\mathbb{R}^{+,*})^4$ *satisfies* $z_1 z_3 = z_2 z_4$ *and a* ***positivity condition***.

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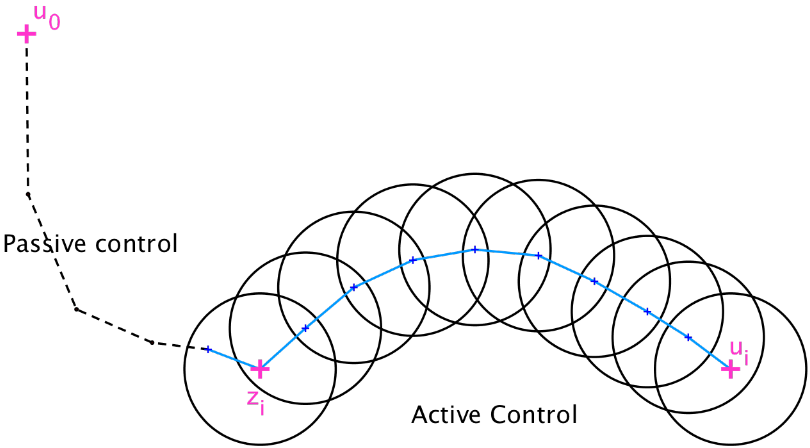
Remark

For $N \geq 3$, *" the " solution* $u \in L^1((0,\infty)\times\Omega)$ *only converges in* $L^1(\Omega)$ *to* z.

Second step



Aknowledgements

Thank you for your attention.