A quantitative Fattorini-Hautus test: the minimal null control time problem in the parabolic setting

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"*Controllability of parabolic equations : new results and open problems*"

Abstract linear control problem

$$\begin{cases} y'(t) = Ay(t) + Bu(t), \quad t \in (0,T), \\ y(0) = y_0, \end{cases} \tag{S}$$

where

- A generates a C^0-semigroup on the Hilbert space $(H, \|\cdot\|)$,
- the space of controls is the separable Hilbert space $(U, \|\cdot\|_U)$,
- the control operator $B \in \mathcal{L}(U, D(A^*)')$ is assumed to be admissible

$$\exists K_T > 0, \quad \text{such that} \quad \int_0^T \|B^* e^{tA^*} z\|_U^2 \, \mathrm{d}t \leq K_T \|z\|^2, \quad \forall z \in D(A^*).$$

Let $T > 0$. For any $y_0 \in H$ and any $u \in L^2(0,T;U)$, a solution $y \in C^0([0,T],H)$ is defined by

$$\langle y(t), z\rangle - \langle y_0, e^{tA^*} z\rangle = \int_0^t \langle u(\tau), B^* e^{(t-\tau)A^*} z\rangle_U \, \mathrm{d}\tau, \quad \forall t \in [0,T], \, \forall z \in H.$$

Wellposedness

For any $y_0 \in H$ and any $u \in L^2(0,T;U)$, there exists a unique solution $y \in C^0([0,T],H)$. Moreover, there exists $C > 0$ such that for any y_0, u, the solution satisfies

$$\|y(t)\| \leq C \left(\|y_0\| + \|u\|_{L^2(0,T;U)} \right), \quad \forall t \in [0,T].$$

Approximate controllability/unique continuation

- **Approximate controllability in time T:** for any y_0, $y_1 \in H$, for any $\varepsilon > 0$, there exists $u \in L^2(0,T;U)$ such that $\|y(T) - y_1\| \le \varepsilon$.

- Let $y_0 = 0$ and define the input-to-state map

$$\begin{array}{rcl} L_T : L^2(0,T;U) & \to & H \\ u & \mapsto & y(T) \end{array}$$

$$\begin{aligned} \text{Approximate controllability in time } T &\iff \overline{L_T\big(L^2(0,T;U)\big)} = H \\ &\iff \operatorname{Ker} L_T^* = \{0\}. \end{aligned}$$

For any $z \in H$,

$$L_T^* z : t \in (0,T) \mapsto B^* e^{(T-t)A^*} z.$$

- **Unique continuation:**

$$\left(\begin{cases} \partial_t z(t) + A^* z(t) = 0 \\ z(T) = z^T \end{cases} \quad + \quad B^* z(t) = 0, \ \forall t \in (0,T) \right) \quad \Longrightarrow \quad z^T = 0.$$

Some necessary conditions for approximate controllability

- Observation of eigenvectors.

$$A^*\varphi_k = \lambda_k\varphi_k \quad \Longrightarrow \quad B^*\varphi_k \neq 0.$$

ex: pointwise control.

- $\dim U = 1$: simple eigenvalues.

$$\begin{cases} A^*\varphi_{k,1} = \lambda_k\varphi_{k,1} \\ A^*\varphi_{k,2} = \lambda_k\varphi_{k,2} \end{cases} \quad \Longrightarrow \quad \dim \mathrm{Span}(\varphi_{k,1}, \varphi_{k,2}) = 1.$$

ex: a single boundary control in 1D.

Exact controllability

- **Exact controllability in time T:** for any $y_0, y_1 \in H$, there exists $u \in L^2(0,T;U)$ such that $y(T) = y_1$.

- Let $y_0 = 0$ and recall

$$\begin{array}{rcl} L_T : L^2(0,T;U) & \to & H \\ u & \mapsto & y(T) \end{array}$$

$$\begin{aligned} \text{Exact controllability in time } T &\iff L_T\big(L^2(0,T;U)\big) = H \\ &\iff \exists c > 0;\ \|L_T^* z\|_{L^2(0,T;U)} \geq c\|z\|,\ \ \forall z \in H. \end{aligned}$$

- **Observability in time T:**

$$\int_0^T \|B^* z(t)\|_U^2 \,\mathrm{d}t \geq c\|z^T\|^2, \quad \forall z^T \in D(A^*),$$

where

$$\begin{cases} \partial_t z(t) + A^* z(t) = 0 \\ z(T) = z^T \end{cases}$$

Remark: observability = quantitative unique continuation.

Regularization is an obstacle to exact controllability.

- **Controllability to trajectories in time T:** for any $y_0, \overline{y_0} \in H$, for any $\overline{u} \in L^2(0,T;U)$, there exists $u \in L^2(0,T;U)$ such that $y(T) = \overline{y}(T)$ where

$$\begin{cases} \overline{y}'(t) = A\overline{y}(t) + B\overline{u}(t), & t \in (0,T), \\ \overline{y}(0) = \overline{y_0}. \end{cases}$$

- **Null controllability in time T:** for any $y_0 \in H$, there exists $u \in L^2(0,T;U)$ such that $y(T) = 0$.

Controllability to trajectories in time T $\iff$ Null controllability in time T.

- Let $y_0 \in H$. Then

$$y(T) = e^{TA} y_0 + L_T u.$$

$$\begin{aligned} \text{Null controllability in time } T &\iff \forall y_0 \in H,\ \exists u \in L^2(0,T;U);\ L_T u = -e^{TA} y_0 \\ &\iff e^{TA}(H) \subset L_T\big(L^2(0,T;U)\big) \\ &\iff \exists c > 0;\ \|L_T^* z\|_{L^2(0,T;U)} \geq c \|e^{TA^*} z\|,\ \ \forall z \in H. \end{aligned}$$

- **(Final time) Observability in time T:**

$$\int_0^T \|B^* z(t)\|_U^2 \,\mathrm{d}t \geq c \|z(0)\|^2, \quad \forall z^T \in D(A^*),$$

where

$$\begin{cases} \partial_t z(t) + A^* z(t) = 0 \\ z(T) = z^T \end{cases}$$

Hautus test: characterization of controllability in finite dimension

- **Finite dimensional setting:** $H = \mathbb{R}^n$, $U = \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$.
  Equivalence between exact, null and approximate controllability in time T.

- **Hautus test:** M.L.J. Hautus (1969).

$$\text{controllability in time } T \quad \Longleftrightarrow \quad \operatorname{Ker}(A^* - \lambda) \cap \operatorname{Ker} B^* = \{0\}, \quad \forall \lambda \in \mathbb{C}.$$

Necessary condition : $\dim \operatorname{Ker}(A^* - \lambda) \leq m$ for all $\lambda \in \mathbb{C}$.

Fattorini criterion: a sufficient condition for approximate controllability

- Setting: assume moreover that A^* has compact resolvent and the root vectors of A^* form a complete sequence in H. Assume that $C : D(C) \subset H \to U$ is A^*-bounded i.e.
$$\|Cz\| \leq \alpha\|z\| + \beta\|A^*z\|, \quad \forall z \in D(A^*).$$
- **Fattorini criterion:** H.O. Fattorini (1966).

$$\Big(z \in D(A^*) \text{ and } Ce^{tA^*}z = 0, \text{ for a.e. } t \in (0, +\infty)\Big) \Longrightarrow z = 0,$$
$$\Longleftrightarrow$$
$$\operatorname{Ker}(A^* - \lambda) \cap \operatorname{Ker} C = \{0\}, \quad \forall \lambda \in C.$$

- Thus, if B^* is A^*-bounded and moreover the semigroup generated by A is analytic,
$$\text{Approximate controllability} \quad \Longleftrightarrow \quad \operatorname{Ker}(A^* - \lambda) \cap \operatorname{Ker} B^* = \{0\}, \quad \forall \lambda \in \mathbb{C}.$$

Question: quantitative Fattorini-Hautus test for null controllability ?

A necessary inequality

T. Duyckaerts & L. Miller (2012). Assume that A generates a C^0-semigroup on H and that B is admissible.

Null controllability in time $T \implies \exists C_T > 0; \quad \forall y \in D(A^*), \forall \lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) > 0$,

$$\|y\|^2 \leq C_T e^{2\operatorname{Re}(\lambda)T} \left(\frac{\|(A^* + \lambda)y\|^2}{\operatorname{Re}(\lambda)^2} + \frac{\|B^* y\|_U^2}{\operatorname{Re}(\lambda)} \right).$$

Necessary conditions for null controllability in time T:

- Sufficient observation of eigenvectors (depending on time !)

$$-A^* \varphi_k = \lambda_k \varphi_k \quad \implies \quad \|B^* \varphi_k\|_U \geq C_T \sqrt{\operatorname{Re}(\lambda_k)} e^{-\operatorname{Re}(\lambda_k)T}.$$

- $\dim U = 1$: distance between eigenvalues

$$\begin{cases} -A^* \varphi_k = \lambda_k \varphi_k \\ -A^* \varphi_j = \lambda_j \varphi_j \end{cases} + \varphi_k \perp \varphi_j \quad \implies \quad |\lambda_k - \lambda_j| \geq C_T \lambda_k e^{-\operatorname{Re}(\lambda_k)T}.$$

Parabolic setting

- Infinite speed of propagation;
- Null controllability in arbitrary time;
- No restriction on the control domain (internal control, boundary control).

H.O. Fattorini & D.L. Russell (1971),
A.V. Fursikov & O.Y. Imanuvilov (1996)
and G. Lebeau & L. Robbiano (1995).

Hyperbolic setting

- Finite speed of propagation;
- Exact controllability only in large time;
- Geometric condition on the control domain (internal control, boundary control).

C. Bardos, G. Lebeau & J. Rauch (1992).

However, there are examples of parabolic control problems exhibiting a **minimal time for null controllability** and/or a geometric condtion...

Let

$$T_0 = \inf\Big\{T > 0\,;\ \exists C_T > 0; \forall y \in D(A^*), \forall \lambda \in \mathbb{C} \text{ with } \operatorname{Re}(\lambda) > 0,$$
$$\|y\|^2 \leq C_T e^{2T\operatorname{Re}(\lambda)} \left(\frac{\|(A^* + \lambda)y\|^2}{\operatorname{Re}(\lambda)^2} + \frac{\|B^* y\|_U^2}{\operatorname{Re}(\lambda)} \right) \Big\} \qquad (*)$$

and $T_0 = +\infty$ when the previous set is empty.

With a pointwise control

S. Dolecki (1973)

$$\begin{cases} \partial_t y - \partial_{xx} y = \delta_{x=x_0} u(t), & t \in (0,T), x \in (0,1), \\ y(t,0) = y(t,1) = 0. \end{cases}$$

Notations: $\varphi_k(x) = \sqrt{2}\sin(k\pi x)$, $\lambda_k = k^2\pi^2$.

$$\text{Minimal time:} \qquad T_{min} = \limsup_{k \to +\infty} \frac{-\ln|\varphi_k(x_0)|}{\lambda_k} = T_0$$

With a zero-order coupling term supported outside the control domain

F. Ammar Khodja, A. Benabdallah, M. González Burgos & L. de Teresa (2016).

$$\begin{cases} \partial_t y + \begin{pmatrix} -\partial_{xx} & 0 \\ 0 & -\partial_{xx} \end{pmatrix} y + \begin{pmatrix} 0 & q(x) \\ 0 & 0 \end{pmatrix} y = \begin{pmatrix} 0 \\ \mathbf{1}_\omega u(t,x) \end{pmatrix}, & t \in (0,T), x \in (0,1), \\ y(t,0) = y(t,1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{cases}$$

Notations: $\varphi_k(x) = \sqrt{2}\sin(k\pi x)$, $\lambda_k = k^2\pi^2$.

$$\omega = (a,b), \qquad \mathbf{Supp}(\boldsymbol{q}) \cap \boldsymbol{\omega} = \boldsymbol{\emptyset}$$

$$I_k(q) = \int_0^1 q(x)\varphi_k(x)^2 \mathrm{d}x, \qquad I_{1,k}(q) = \int_0^a q(x)\varphi_k(x)^2 \mathrm{d}x.$$

where $|I_k(q)| + |I_{1,k}(q)| \neq 0$ for any $k \in \mathbb{N}^*$.

Minimal time: $$T_{min} = \limsup_{k \to +\infty} \frac{\min\left(-\ln|I_k(q)|, -\ln|I_{1,k}(q)|\right)}{\lambda_k} = T_0.$$

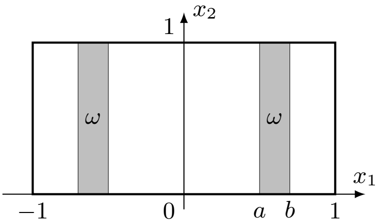
K. Beauchard, P. Cannarsa & R. Guglielmi (2014) and K. Beauchard, L. Miller & M. M. (2015).

$$\begin{cases} \partial_t y - \partial_{x_1 x_1} y - x_1^2 \partial_{x_2 x_2} y = \mathbf{1}_\omega u(t, x_1, x_2), \quad t \in (0,T), (x_1, x_2) \in \Omega \\ y(t, x_1, x_2) = 0, \quad (x_1, x_2) \in \partial\Omega. \end{cases}$$

Notations:

$$\Omega = (-1, 1) \times (0, 1),$$

$$\omega = \Big[(-b, -a) \cup (a, b)\Big] \times (0, 1).$$



Minimal time: $\quad T_{min} = \dfrac{a^2}{2} = T_0.$

A system with different diffusion coefficients

F. Ammar Khodja, A. Benabdallah, M. González Burgos & L. de Teresa (2014).

$$\begin{cases} \partial_t y + \begin{pmatrix} -\partial_{xx} & 0 \\ 0 & -d\partial_{xx} \end{pmatrix} y + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} y = 0, \quad t \in (0,T), x \in (0,1), \\ y(t,0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \quad y(t,1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{cases}$$

Notations : $\Lambda = \left\{ k^2\pi^2,\ dk^2\pi^2;\ k \in \mathbb{N}^* \right\}$.

Minimal time: $\quad T_{min} = c(\Lambda) = T_0.$

where $c(\Lambda)$ is the condensation index defined by

$$c(\Lambda) = \limsup_{k \to +\infty} \frac{-\ln |E'(\lambda_k)|}{\lambda_k} \quad \text{with} \quad E(z) = \prod_{j=1}^{+\infty} \left(1 - \frac{z^2}{z_k^2} \right).$$

Setting

- Assume that the operator $-A^*$ admits a sequence of eigenvalues $\Lambda = (\lambda_k)_{k\in\mathbb{N}^*}$ such that

$$\exists \delta > 0,\ \mathrm{Re}(\lambda_k) \geq \delta|\lambda_k|,\ \forall k \in \mathbb{N}^* \quad \text{and} \quad \sum_{k=1}^{+\infty} \frac{1}{|\lambda_k|} < +\infty. \qquad \text{(Spectrum)}$$

- For any $k \in \mathbb{N}^*$, we denote by $r_k = \dim(\mathrm{Ker}(A^* + \lambda_k))$ the geometric multiplicity of the eigenvalue λ_k and assume that $\sup_{k\in\mathbb{N}^*} r_k < +\infty$.

- We denote by $(\varphi_{k,j})_{k\in\mathbb{N}^*, 1\leq j\leq r_k}$ the associated sequence of normalised eigenvectors and we assume that it forms a complete sequence in H i.e.

$$\Big(\langle \Phi, \varphi_{k,j}\rangle = 0, \quad \forall k \in \mathbb{N}^*, \forall j \in \{1, \dots, r_k\}\Big) \implies \Phi = 0.$$

Equivalence between quantitative Fattorini-Hautus test and null controllability

F. Ammar Khodja, A. Benabdallah, M. González Burgos & M. M. (submitted)

Theorem 1

Assume that the condensation index of the sequence $\Lambda = (\lambda_k)_{k\in\mathbb{N}^*}$ satisfies $\mathrm{c}(\Lambda) = 0$. Let T_0 be defined by $(*)$. Then:

- If $T_0 > 0$ and $T < T_0$, system (S) is not null controllable in time T;
- If $T_0 < +\infty$ and $T > T_0$, system (S) is null controllable in time T.

- Condition (Spectrum) in a parabolic setting: "*restriction to space dimension* 1".
- Generalization to "*suitable growth*" of the geometric multiplicity.
- Qualitative result ($T_{min} \in [T_0, 2T_0]$) for algebraic multiplicity 2 (i.e. Jordan chain of length 2)
- Assumption $c(\Lambda) = 0$ is not so strong.

Sketch of proof: simple eigenvalues I

- Definition of solutions

$$\langle y(T), \varphi_j \rangle - \langle y_0, e^{-\lambda_j T} \varphi_j \rangle = \int_0^T e^{-\lambda_j (T-t)} \langle u(t), B^* \varphi_j \rangle_U \mathrm{d}t.$$

- Complete family of eigenvectors

$$y(T) = 0 \quad \Longleftrightarrow \quad \int_0^T e^{-\lambda_j (T-t)} \langle u(t), B^* \varphi_j \rangle_U \mathrm{d}t = -e^{-\lambda_j T} \langle y_0, \varphi_j \rangle, \ \forall j \in \mathbb{N}^*.$$

- Biorthogonal family

Let $T > 0$ and let $\sigma = (\sigma_k)_{k \in \mathbb{N}}$ be a normally ordered complex sequence satisfying (Spectrum). Then, there exists a biorthogonal family $(q_k)_{k \in \mathbb{N}}$ to the exponentials associated with σ i.e.

$$\int_0^T e^{-\sigma_j t} q_k(t) \mathrm{d}t = \delta_{k,j}, \quad \forall k, j \in \mathbb{N},$$

such that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\|q_k\|_{L^2(0,T;\mathbb{C})} \leq C_\varepsilon e^{\mathrm{Re}(\sigma_k)(\mathrm{c}(\sigma)+\varepsilon)}.$$

Sketch of proof: simple eigenvalues II

- "*New*" form of controls for the moment problem

$$u(t) = \sum_{k\in\mathbb{N}^*} \alpha_k \, q_k(T-t) \, B^*\varphi_k,$$

where the scalar coefficient α_k has to be determined.

$$\int_0^T e^{-\lambda_j(T-t)} \langle u(t), B^*\varphi_j\rangle_U \mathrm{d}t = -e^{-\lambda_j T}\langle y_0, \varphi_j\rangle, \ \forall j \in \mathbb{N}^*$$

$$\iff \quad \sum_{k\in\mathbb{N}^*} \alpha_k \langle B^*\varphi_k, B^*\varphi_j\rangle \int_0^T e^{-\lambda_j(T-t)} q_k(T-t)\mathrm{d}t = -e^{-\lambda_j T}\langle y_0, \varphi_j\rangle, \ \forall j \in \mathbb{N}^*$$

$$\iff \quad \alpha_k = -\frac{e^{-\lambda_k T}\langle y_0, \varphi_k\rangle}{\|B^*\varphi_k\|_U^2}, \ \forall k \in \mathbb{N}^*.$$

- Convergence of the series thanks to quantitative Fattorini-Hautus test

$$u(t) = -\sum_{k\in\mathbb{N}^*} e^{-\lambda_k T}\langle y_0, \varphi_k\rangle \, q_k(T-t) \, \frac{B^*\varphi_k}{\|B^*\varphi_k\|_U^2}.$$

→ Multiple eigenvalues: construct and estimate a biorthogonal family to $(B^*\varphi_{k,1}, \dots, B^*\varphi_{k,r_k})$.

Theorem 2

Assume that the eigenvectors of A^* form a Riesz basis of H. Assume that for any $k \neq j \in \mathbb{N}^*$,

$$\mathrm{Ker}(B^*) \cap \mathrm{Span}(\varphi_k, \varphi_j) \neq \{0\}$$

and for any $\varepsilon > 0$

$$\|B^* \varphi_k\|_U e^{\varepsilon \,\mathrm{Re}(\lambda_k)} \underset{k \to +\infty}{\longrightarrow} +\infty.$$

Finally, assume that the condensation index of the sequence $\Lambda = (\lambda_k)_{k \in \mathbb{N}^*}$ satisfies $\mathrm{c}(\Lambda) = \mathrm{Bohr}(\Lambda)$.
Let T_0 be defined by $(*)$. Then:

- If $T_0 > 0$ and $T < T_0$, system (S) is not null controllable in time T;
- If $T_0 < +\infty$ and $T > T_0$, system (S) is null controllable in time T.

- Structural assumption on B^* $\rightarrow$ generalizes $\dim U = 1 \implies$ simple eigenvalues.
- Technical (?) assumption on the condensation index: "*allow eigenvalues to get exponentially close but only two by two*".
- Remove the sufficient observation of eigenvectors hypothesis: qualitative result $T_{min} \in [T_0, 2T_0]$.

The general setting

Theorem 3

Let T_0 be defined by $(*)$. Then, there exists $\widetilde{T} \in [T_0, T_0 + \mathrm{c}(\Lambda)]$ such that

- if $\widetilde{T} > 0$ and $T < \widetilde{T}$, system (S) is not null controllable in time T;
- if $\widetilde{T} < +\infty$ and $\mathrm{c}(\Lambda) < +\infty$, for $T > \widetilde{T}$, system (S) is null controllable in time T.

- Quite general setting for (systems) of one dimensional parabolic equations.

On the spectral assumption $\sum_{k=1}^{+\infty} \frac{1}{|\lambda_k|} < +\infty$

- Used for the moment method: needed for the biorthogonal family with respect to the time exponentials $\longrightarrow$ biorthogonal family to $e^{-\lambda_k \cdot} B^* \varphi_k$ in $L^2(0,T;U)$?

- Crucial for the characterization of controllability through the quantitative Fattorini-Hautus test.
  Quantum harmonic oscillator, T. Duyckaerts & L. Miller (2012).
  Let $H = L^2(\mathbb{R})$, $U = L^2(\mathbb{R})$, $B = \mathbf{1}_{(-\infty,x_0)}$ with $x_0 \in \mathbb{R}$ and

$$A = -\partial_{xx} + x^2, \qquad D(A) = \{y \in H^2(\mathbb{R}); x \mapsto x^2 y(x) \in L^2(\mathbb{R})\}.$$

  A is self-adjoint and generates a C^0-semigroup. Its eigenvalues are $\{\lambda_k = 2k-1; k \in \mathbb{N}^*\}$ and its eigenvectors are the Hermite polynomials (Hilbert basis).
  For this operator $\boldsymbol{T_{min} = +\infty}$ and $\exists M, m \in \mathbb{R}$ such that

$$\|y\|^2 \leq M\|(A^* + \lambda)y\|^2 + m\|B^* y\|_U^2, \qquad \forall y \in D(A^*),\ \forall \lambda \in \mathbb{R} \quad \Longrightarrow \quad \boldsymbol{T_0 = 0}.$$

  Notice that this example satisfies every assumption of Theorem 3 except that $\sum_{k\in\mathbb{N}^*} \frac{1}{\lambda_k} = +\infty$.

Open problems

- A "better" quantitative Fattorini-Hautus test: capture the minimal time coming from both the condensation of eigenvalues and the "localization" of eigenvectors.

- Minimal time for null controllability of parabolic systems in higher dimensions ? Lack of methods...