Optimization problems involving L^∞-functionals:relaxation and convexity issues

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PDEs, optimal design and numerics, Benasque, August 2017

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Note that u_Ω is a minimizer of

$$\int_\Omega \left(\frac{1}{2}\langle a(x)\nabla, \nabla u\rangle - f(x)u(x)\right) dx$$

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Ω non-homogeneous body obtained as mixture of different materials giving on the deformations u the differential constraint

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L^∞ functionals involving the gradient

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where $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \bar{\mathbb{R}}$ is a Borel function

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Note that the "right" notion is that of absolute minimizer!

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Main problem: compute a suitable formula for the relaxation of a general supremal functional F

Theorem Let Ω be bounded and connected with Lipschitz

boundary. Let F be the supremal functional represented by a Borel function $f : \Omega \times \mathbb{R}^N \to \bar{\mathbb{R}}$. Assume that F satisfies:
(H_Ω) there exists $(u_n)_{n\in\mathbb{N}}$ such that, set $u_\xi(x) := \xi \cdot x$, it holds

$$\lim_{n\to\infty} F(u_n) = \inf_{W^{1,\infty}(\Omega)} F (\in \bar{\mathbb{R}})$$

$$\limsup_{\xi\to 0} F(u_n + u_\xi) = F(u_n) \quad \forall\, n \in \mathbb{N}.$$

Then the relaxed functional $\Gamma_\tau(F)$ is a level convex functional when τ is one of the topologies $\tau_\infty, w^*, w^*_{seq}$.

Characterization

Let $f : (x, \cdot)$ be l.s.c for any $x \in \Omega$ and assume F satisfies (H_Ω). Then the following facts are equivalent:

(i) F is w^*-lower semicontinuous in $W^{1,\infty}(\Omega)$;

(ii) F is w^*_{seq}-lower semicontinuous in $W^{1,\infty}(\Omega)$;

(iii) F is a level convex supremal functional;

(iv) there exists a level convex normal supremand φ such that

$$F(u) = \operatorname*{ess\,sup}_{x \in \Omega} \varphi(x, Du(x)).$$

Moreover φ satisfies the following property: there esists a negligible set $N \subset \Omega$ such that for every $x \in \Omega \setminus N$ and for every $\xi \in \mathbb{R}^N$ $\varphi(x, \xi) \geq f(x, \xi)$.