Gravitational Wave Observatories III: Ground Based Interferometers

Neil J. Cornish
Outline

• Interferometer design
• Sources of noise
• LIGO data analysis, theory and search results
• Astrophysical rates
Gravitational Wave Telescopes
\[ \Phi(t) = 2\pi \nu_0 \Delta T(t) \]

**Interferometer Design**

**Consideration 1**

\[ \Delta T \propto hL \quad \text{for} \quad f_{gw} < \frac{c}{L} \]

Want L as large as possible up to

\[ L \sim \frac{c}{10^3 \text{ Hz}} = 300 \text{ km} \]

Expensive!

Solution: folded arms

**Consideration 2**

\[ \Phi \propto \nu_0 \Delta T \]

Fluctuations in Laser frequency can masquerade as GW signal

Partial Solution: Highly stable lasers

Solution: Michelson topology - cancel laser frequency noise
Interferometer Design

The 4 km Fabry-Perot cavities effectively fold the arms.

Can calculate the response using basic E&M, electric field transmission and reflection coefficients at each mirror. (See Maggiore’s text)

In the long wavelength limit, phase shift is

$$|\Delta \Phi_{FP}| = \left(\frac{\nu_0 h L}{c}\right) \left[ \frac{8F}{\sqrt{1 + \left(f_{gw}/f_p\right)^2}} \right]$$

$$F = \frac{\pi \sqrt{r_1 r_2}}{1 - r_1 r_2}$$

Cavity Finesse

$$f_p = \frac{1}{4\pi \tau_s} \approx \frac{c}{4FL}$$

Pole frequency
Single Fabry-Perot Cavity

Basic design

\[ |\Delta \Phi_{FP}| = \left( \frac{\nu_0 h L}{c} \right) \left[ \frac{8F}{\sqrt{1 + (\frac{f_{gw}}{f_p})^2}} \right] \]

\[ F = \frac{\pi \sqrt{r_1 r_2}}{1 - r_1 r_2} \quad \text{Cavity Finesse} \]

\[ f_p = \frac{1}{4\pi \tau_s} \approx \frac{c}{4FL} \quad \text{Pole frequency} \]

Reflectivities are very close to unity

e.g. advanced LIGO \( F = 450 \quad f_p = 42 \text{ Hz} \)

Note that increasing the Finesse improves the low frequency sensitivity (good), but lowers the pole frequency (bad)

Advanced LIGO gets around this problem by using signal recycling
Michelson Interferometer with coupled Fabry-Perot Cavities

The actual LIGO design is much more complicated. FP cavities are coupled by a power recycling and signal recycling mirrors.

There is a common mode and a differential mode

\[ L_+ = \frac{L_1 + L_2}{2} \quad L_- = \frac{L_1 - L_2}{2} \]

Differential mode contains the GW signal

\[ f_- = \frac{2}{4\pi L_+} \ln \left( \frac{1 - r_i r_s}{r_e r_i - r_e r_s (r_i^2 + r_s^2)} \right) \]

\( i = \) input mirror
\( e = \) end mirror
\( s = \) signal recycling mirror

Advanced LIGO \( f_- = 350 \text{ Hz} \)

Transfer function

\[ \frac{1}{\sqrt{1 + (f_{gw}/f_-)^2}} \]

The gain from folding and recycling is now a complicated combination of terms. Comes out at a factor of \( \sim 1,100 \)
Signal recycling and response shaping

The cavity transfer function for the Michelson-Fabry-Perot topology with signal recycling allows us to shape the response and target particular signals by changing the distance to the signal recycling mirror

\[
C(f) = \frac{t_s e^{-i(2\pi f \ell_s / c + \phi_s)}}{1 - r_s \left( \frac{r_i e^{-4\pi i f L / c}}{1 - r_i e^{-4\pi i f L / c}} \right) e^{-2i(2\pi f \ell_s / c + \phi_s)}}
\]

\[
\phi_s = 2\pi \nu_0 \ell_s / c
\]

The pole frequency quoted on the last slide was for the zero detuned configuration
Sources of Noise

- Fundamental noise
- Facility noise
Fundamental (quantum) noise

\[ S_{\text{shot}}^{1/2} \propto \frac{1}{\sqrt{I_0}} \]

\[ S_{\text{rp}}^{1/2} \propto \sqrt{I_0} \]

\[ \text{SQL} \Delta x \Delta p \geq \hbar \]
Shot noise: Digital camera, photons per pixel

\[ S_{\text{shot}}^{1/2} \propto \frac{1}{\sqrt{I_0}} \]
Radiation pressure noise

The test masses are essentially free (inertial) in the horizontal direction for frequencies above the pendulum frequency of the suspension

\[ \ddot{x} = \frac{F_{\text{rad}}}{M} \]

\[ \Rightarrow \quad \ddot{x} = -\frac{\tilde{F}_{\text{rad}}}{4\pi^2 f^2 M} \]

\[ \mathbb{E}[\tilde{x}] = 0 \quad \mathbb{E}[\tilde{x}\tilde{x}^*] \propto \frac{I_0}{f^4 M^2} \]

\[ \Rightarrow \quad S_{\text{rp}}^{1/2} \propto \frac{\sqrt{I_0}}{f^2 M} \]
Fundamental (quantum) noise

The intensity of the laser light is frequency dependent due to the cavity response:

\[ S_Q = \frac{\hbar}{\pi^2 f^2 M L^2} \left( \frac{I(f)}{\pi^2 f^2 M c} + \frac{\pi^2 f^2 M c}{I(f)} \right) \]

\[ I(f) \sim \frac{I_0}{1 + (f/f_-)^2} \]

SQL is reached when the Shot and RP contributions are equal (minimum of \( S_Q \)):

\[ S_{SQL} = \frac{2\hbar}{\pi^2 f_{opt}^2 M L^2} \]
Standard Quantum Limit

\[
\text{SQL} \quad x \Delta p \geq \hbar
\]

\[
S_{\text{SQL}} = \frac{2\hbar}{\pi^2 f_{\text{opt}}^2 M L^2}
\]

The SQL is not a fundamental limit to GW detector sensitivity

1) Measure momentum change rather than position change (speedmeter)

2) The RP and Shot noise can be made to be correlated. Allows us to reshape uncertainty ellipse using squeezed light (Quantum no-demolition measurement)
Facility Noise

![Graph showing various types of noise](image_url)

- Facility noise
- Quantum noise
- Seismic noise
- Gravity Gradients
- Suspension thermal noise
- Coating Brownian noise
- Coating Thermo-optic noise
- Substrate Brownian noise
- Excess Gas
- Total noise

**Vertical Axis:** Strain [1/√Hz]

**Horizontal Axis:** Frequency [Hz]
Seismic Noise

$S_{\text{seis}}^{1/2} \sim 10^{-12} \left( \frac{10 \text{ Hz}}{f} \right)^2 \text{Hz}^{-1/2}$

10$^{10}$ too large

Need one of these
Seismic Noise

\[ S_{\text{seis}}^{1/2} \sim 10^{-12} \left( \frac{10 \text{ Hz}}{f} \right)^2 \text{Hz}^{-1/2} \]

Single pendulum suspension

\[ S_{\text{seis,filt}}^{1/2} = \frac{1}{|1 - (f / f_{\text{pend}})^2|} S_{\text{seis}}^{1/2} \]

\[ f_{\text{pend}} \sim 1 \text{ Hz} \]

\[ S_{\text{seis,filt}}^{1/2} \approx 10^{-12} \left( \frac{f_{\text{pend}}}{f} \right)^2 \left( \frac{10 \text{ Hz}}{f} \right)^2 \text{Hz}^{-1/2} \]

aLIGO 5-stage pendulum suspension

\[ S_{\text{seis,filt}}^{1/2} \approx 10^{-12} \left( \frac{f_{\text{pend}}}{f} \right)^{10} \left( \frac{10 \text{ Hz}}{f} \right)^2 \text{Hz}^{-1/2} \]
Thermal Noise

Thermal noise contributions
Thermal Noise

Fluctuation-dissipation theorem: PSD of fluctuations of a system in equilibrium at temperature $T$ is determined by the dissipative terms that return the system to equilibrium

$$S_T(f) = \frac{4k_B T}{(2\pi f)^2} \Re[Y(f)]$$

For an anelastic spring with loss angle $\phi$ Hooke's law becomes $F = -k x (1 + i\phi)$

$$S_T(f) = \frac{k_B T}{2\pi^3 M f} \frac{f_{res}^2 \phi(f)}{(f_{res}^2 - f^2)^2 + f_{res}^4 \phi^2(f)}$$
Mirror Coating Thermal Noise

\[ S_T(f) = \frac{k_B T}{2\pi^3 M f} \frac{f_{\text{res}}^2 \phi(f)}{(f_{\text{res}}^2 - f^2)^2 + f_{\text{res}}^4 \phi^2(f)} \]

The resonant frequencies for the coatings are very high (tens of kHz), and the loss angle small (millionths)

\[ S_{MC} = \left( \frac{2k_B T \phi}{\pi^3 M L^2 f_{MC}^2} \right) \frac{1}{f} \]

For advanced LIGO

\[ S_{MC}^{1/2} = 2.5 \times 10^{-24} \left( \frac{100 \text{ Hz}}{f} \right)^{1/2} \text{ Hz}^{-1/2} \]
Suspension Thermal Noise

\[ S_T(f) = \frac{k_B T}{2\pi^3 M f} \left( \frac{f_{\text{res}}^2 \phi(f)}{(f_{\text{res}}^2 - f^2)^2 + f_{\text{res}}^4 \phi^2(f)} \right) \]

Pendulum mode
\[ f_{\text{pen}} = 9 \, \text{Hz} \]

Violin modes
\[ f_v = 500, 1000, \ldots \, \text{Hz} \]
Pendulum Suspension Thermal Noise

\[ S_T(f) = \frac{k_B T}{2\pi^3 M f} \frac{f_{\text{res}}^2 \phi(f)}{(f_{\text{res}}^2 - f^2)^2 + f_{\text{res}}^4 \phi^2(f)} \]

\[ f_{\text{res}} = f_{\text{pen}} \ll f \]

For advanced LIGO

\[ S_{\text{pen}}^{1/2} = 3.5 \times 10^{-25} \left( \frac{100 \text{ Hz}}{f} \right)^{5/2} \text{ Hz}^{-1/2} \]
Violin mode Thermal Noise

\[ S_T(f) = \frac{k_B T}{2\pi^3 M f} \frac{f_{\text{res}}^2 \phi(f)}{(f_{\text{res}} - f^2)^2 + f_{\text{res}}^4 \phi^2(f)} \]

\[ f_{\text{res}} = f_v = 500 \text{ Hz, 1000 Hz, ...} \]

For advanced LIGO, first harmonic

\[ S_v^{1/2} = \frac{3 \times 10^{-24}}{1 + (f_v^2 - f^2)^2/\delta f^4} \text{ Hz}^{-1/2} \]

\[ \delta f = f_v \phi^{1/2} \approx 2 \text{ Hz} \]
Gravity Gradient Noise

We can’t escape Newton

\[ \ddot{x} = G \int \frac{\delta \rho(x', t)}{|x - x'|^3} (x - x') \, d^3x \]

This is why we need LISA!
Real aLIGO noise spectra from O1

Suspension Thermal noise
Violin modes

Strain noise, $1/\sqrt{Hz}$ vs Frequency, Hz

Advanced LIGO, L1 (2015)
Enhanced LIGO (2010)
Advanced LIGO, H1 (2015)
Advanced LIGO design
Real aLIGO noise spectra from O1

Electronic noise
60 Hz power line and harmonics
Real aLIGO noise spectra from O1/S6

Calibration lines

Calibration and the control system is a whole other topic...
LIGO searches for binary inspired signals

Recall: Likelihood for Stationary Gaussian Noise

\[ p(d|\tilde{\lambda}) = \text{const.} \ e^{-\frac{\chi^2(\tilde{\lambda})}{2}} \]

\[ \chi^2(\tilde{\lambda}) = (d - h(\tilde{\lambda})|d - h(\tilde{\lambda})) \]

\[ (a|b) = 2 \int_0^\infty \frac{\tilde{a}(f)\tilde{b}^*(f) + \tilde{a}^*(f)\tilde{b}(f)}{S_n(f)} df \]

Suppose that we have two hypotheses:

\[ H_1 : \text{A signal with parameters } \tilde{\lambda} \text{ is present} \]
\[ H_0 : \text{No signal is present} \]

Likelihood ratio:

\[ \Lambda(\tilde{\lambda}) = \frac{p(d|h(\tilde{\lambda}), H_1)}{p(d, H_0)} \]

For Gaussian noise:

\[ \Lambda(\tilde{\lambda}) = e^{-(d|h) + \frac{1}{2}(h|h)} \]
Frequentist Hypothesis Testing

$\Lambda$ – Detection Statistic

$H_0$ – Noise Hypothesis

$H_1$ – Noise + Signal Hypothesis

Set threshold $\Lambda_*$ such that $\Lambda > \Lambda_*$ favors hypothesis $H_1$

Type I error - False Alarm

Type II error - False Dismissal
Neyman-Pearson Theorem

For a fixed false alarm rate, the false dismissal rate is minimized by the likelihood ratio statistic

\[ \Lambda(\vec{\lambda}) = \frac{p(d|h(\vec{\lambda}), H_1)}{p(d, H_0)} \]

The likelihood ratio is maximized over the signal parameters.

The rho statistic is often used in place of the likelihood ratio

Writing \( h = \rho \hat{h} \) where \((\hat{h}|\hat{h}) = 1 \)

\[ \Lambda(\vec{\lambda}) = e^{\rho(d|\hat{h}) - \frac{1}{2} \rho^2} \]

Maximizing wrt rho

\[ \frac{\partial \Lambda(\vec{\lambda})}{\partial \rho} = 0 \quad \Rightarrow \quad \rho(\vec{\lambda}) = (d|\hat{h}(\vec{\lambda})) \]

\[ \log \Lambda(\vec{\lambda}) = \frac{1}{2} \rho^2(\vec{\lambda}) \]
The rho statistic and SNR

The signal-to-noise ratio (SNR) is defined:

\[ \text{SNR} = \frac{\text{Expected value when signal present}}{\text{RMS value when signal absent}} = \frac{E[\rho]}{\sqrt{E[\rho_0^2] - E[\rho_0]^2}} = (h|\hat{h}) = \sqrt{(h|h)} \]

In practice, the detector noise is not perfectly Gaussian, and variants of the rho statistic are now used, notably the “new SNR” statistic, introduced by B. Allen Phys.Rev. D71 (2005) 062001

\[ \text{SNR}^2 = 4 \int_0^\infty \frac{|\tilde{h}(f)|^2}{S_n(f)} \, df \]
Matched Filtering = Maximum Likelihood

Convolve the data with a filter (template) $K$:

$$Y = \int dt \int du \, K(t - u) d(u)$$

SNR

$$= \frac{E[Y]}{\sqrt{E[Y^2] - E[Y]^2}}$$

$$= \frac{2 \int_0^\infty df (\tilde{h}^*(f) \tilde{K}(f) + \tilde{h}(f) \tilde{K}^*(f))}{\sqrt{4 \int_0^\infty df |\tilde{K}(f)|^2 S(f)}}$$

Maximizing the SNR yields

$$\tilde{K}(f) = \frac{\tilde{h}(f)}{S(f)} \quad \Rightarrow \quad Y = (d|h) = \rho \sqrt{(h|h)}$$
Frequentist Detection Threshold

For stationary, Gaussian noise the detection statistic $\rho$ is Gaussian distributed.

For the null hypothesis we have

\[ p_0(\rho) = \frac{1}{\sqrt{2\pi}} e^{-\rho^2/2} \]

For the detection hypothesis we have

\[ p_1(\rho) = \frac{1}{\sqrt{2\pi}} e^{-(\rho^2 - \text{SNR}^2)/2} \]

Setting a threshold of $\rho_*$ gives the false alarm and false dismissal probabilities

\[ P_{\text{FA}} = \frac{1}{2} \text{erfc}(\rho_*/\sqrt{2}) \]
\[ P_{\text{FD}} = \frac{1}{2} \text{erfc}((\rho_* - \text{SNR})/\sqrt{2}) \]

LIGO/Virgo analyses do not use SNR thresholds, but rather use False Alarm Rate thresholds

\[ \text{FAR} = \frac{P_{\text{FA}}}{T_{\text{obs}}} \]

E.g. FAR = One in million years and an observation time of one year

\[ P_{\text{FA}} = 10^{-6} \text{ aka } 4.9 \sigma \]
\[ \rho_* = 4.8 \]
Grid Based Searches

Goal is to lay out a grid in parameter space that is fine enough to catch any signal with some good fraction of the maximum matched filter SNR.

The match measures the fractional loss in SNR in recovering a signal with a template and defines a natural metric on parameter space:

\[ M(\bar{x}, \bar{y}) = \frac{(h(\bar{x})|h(\bar{y}))}{\sqrt{(h(\bar{x})|h(\bar{x}))(h(\bar{y})|h(\bar{y}))}} \]

Taylor expanding

\[ M(\bar{x}, \bar{x} + \Delta \bar{x}) = 1 - g_{ij} \Delta x^i \Delta x^j + \ldots \]

where \( g_{ij} = \frac{(h_i|h_j)}{(h|h)} - \frac{(h|h_i)(h|h_j)}{(h|h)^2} \) (Owen Metric)

Number of templates (for a hypercube lattice in D dimensions)

\[ N = \frac{V}{\Delta V} = \frac{\int d^D x \sqrt{g}}{(2\sqrt{(1 - M_{\text{min}})/D})^D} \]

Cost grows geometrically with D for any lattice.
LIGO Style Grid Searches

Typically 2-3 dimensional, 1000’s points
Reducing the cost of a search

In most cases it is possible to analytically maximize over 3 or more parameters

Distance:

The unit normalized template $\hat{h}$ defines a reference distance $D$

Scaling this template to distance $D$ gives

$$ h = \frac{D}{\bar{D}} \hat{h} $$

The distance is then estimated from the data as

$$ D = \frac{\bar{D}}{(d|\hat{h})} $$
Reducing the cost of a search

Phase Offset:

Generate two templates $h(\phi = 0)$ and $h(\phi = \pi/2)$

Then $$(d|h)_{\max \phi} = \sqrt{(d|h(0))^2 + (d|h(\pi/2))^2}$$

Easy to see this in the Fourier domain.

Suppose $\tilde{d} = \tilde{h}_0 e^{i\phi}$, then

$$(d|h(0)) = (h_0|h_0) \cos \phi$$
$$(d|h(\pi/2)) = (h_0|h_0) \sin \phi$$
Reducing the cost of a search

Time Offset:

Fourier transform treats time as periodic - use this to our advantage

Compute the inverse Fourier transform of the product of the Fourier transforms:

$$(d|h)(\Delta t) = 4 \int \frac{\tilde{d}^*(f)\tilde{h}(f)}{S(f)} e^{2\pi if\Delta t} df$$

Then if the template and data differ by a time shift: \(d(t) = h(t - t_0)\)

$$(d|h)_{\text{max}} = (d|h)(\Delta t = t_0)$$
Workflow for pyCBC search

- Template bank constructed
- Matched filtering is done per-detector (not coherent)
- Detection statistic computed ("new SNR")
- Coincidence in time/mass enforced
- Data quality vetoes applied
- Monte Carlo background to compute FAR vs new SNR
Things that go bump in the night

(Bandpass filtered, whitened, time domain)

Samples from the Syracuse Audio Study of Glitches
Contending with non-stationary, non-Gaussian noise

Non-stationary:

- Adiabatic drifts in the PSD
  - work with short data segments

- Glitches
  - vetoes and time-slides

Non-Gaussian:
Analysis of 16 days of data from September 2015

Divide by 16 days to get FAR

“new SNR”
Early Morning, September 14 2015
BayesWave reconstruction of the signal
Fig. 1. The gravitational-wave event GW150914 observed by the LIGO Hanford (H1, left column panels) and Livingston (L1, right column panels) detectors. Times are shown relative to September 14, 2015 at 09:50:45 UTC. For visualization, all time series
President Obama @POTUS

Einstein was right! Congrats to @NSF and @LIGO on detecting gravitational waves - a huge breakthrough in how we understand the universe.

RETTWEETS LIKES
7,787 16,365

3:43 PM - 11 Feb 2016

A worker installed a cable in 2010 to control light in the Laser Interferometer Gravitational-Wave Observatory in Hanford, Wash. The detector is 700 feet underground, wrapped in 50,000 tons of concrete and steel to shield it from earthquakes and ground vibrations.
LIGO detections to-date

GW150914
LVT151012
GW151226
GW170104
LVT151012
GW151226
GW150914
Triangulating the Source

Hanford
Triangulating the Source

Hanford + Livingston
Triangulating the Source

Hanford + Livingston + Virgo
Parameter estimation
Black Holes of Known Mass

Solar Masses

X-Ray Studies

GW150914

LIGO

GW170104

GW151226

LVT151012
Detection without templates

BayesWave  Cornish & Littenberg 2015

• Bayesian model selection
  • Three part model (signal, glitches, gaussian noise)
  • Trans-dimensional Markov Chain Monte Carlo

• Wavelet decomposition
  • Glitch & GW modeled by wavelets
  • Number, amplitude, quality and TF location of wavelets varies

Continuous Morlet/Gabor Wavelets
Lines and a drifting noise floor

![Graph showing power spectral density and amplitude spectral density with cubic spline and Lorentzian fits.](Image)

- **Power spectral density (Hz⁻¹)**
- **Frequency (Hz)**
- **Amplitude spectral density (1/√Hz)**

**Data**: Cubic spline fit, Lorentzian fit

**Model it**: Representative Spectra
Glitches

Model these too

\[ \sum \]
Gravitational Waves

and model these
"Caedite eos. Novit enim Dominus qui sunt eius" Arnaud Amalric
(Kill them all. For the Lord knoweth them that are His.)
Example from LIGO’s S5 science run
Reconstructing GW150914 with wavelets

LIGO Hanford Observatory: GW150914

LIGO Livingston Observatory: GW150914
Reconstructing GW170104 with wavelets
Match GR prediction

GW150914

GW170104
Would like to know merger rate to constrain population synthesis models. Even better, would like to know merger rate as a function of mass, spin, redshift etc.

Many cool techniques being developed to do this using things like Gaussian processes.

Only have time to discuss the total merger rate.
Astrophysical Rate Limits

Even without a detection we can produce interesting astrophysical results such as bounds on the binary merger rate for NS-NS...

Expect binary mergers to be a Poisson process. If the expected number of events is $\lambda$, then the probability of detecting $k$ events is

$$p(k|\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

If the event rate is $R$ [Mpc$^{-3}$ year$^{-1}$], and the observable 4-volume is $VT$ [Mpc$^3$ year]

$$\lambda = RVT$$

The probability of observing zero events ($k=0$) is then

$$p(R) = VT e^{-RVT}$$

Follows from $p(0|\lambda) = p(R) dR = e^{-RVT}$
Astrophysical Rate Limits

\[ p(R) = VT e^{-RV_T} \]

The probability distribution is peaked at a rate of zero. A 90% rate upper limit can be computed:

\[
\int_0^{R_*} p(R) dR = 1 - e^{-R_* VT} = 0.9
\]

\[
\Rightarrow R_* VT = \ln(0.1)
\]

\[
\Rightarrow R_* = \frac{2.3}{VT}
\]
e.g. NS-NS Merger Rate, aLIGO at design sensitivity

\[ V = \frac{4\pi}{3}(200 \text{ Mpc})^3 \]

Week 1

Truth 90% upper limit
Astrophysical Rate Limits

When a signal is detected the probability distribution for the rate is no longer peaked at zero. For example, with a single detection \((k=1)\) we have

\[
p(R) = R(VT)^2 e^{-RVT}
\]

This distribution is peaked at

\[
R = \frac{1}{VT}
\]

The 90% confidence interval now sets upper \textit{and} lower limits on the merger rate.
Simulated NS-NS Merger Rate Constraints

Week 2
Simulated NS-NS Merger Rate Constraints

Week 3
Simulated NS-NS Merger Rate Constraints

Week 4
Simulated NS-NS Merger Rate Constraints

Week 5
Simulated NS-NS Merger Rate Constraints