Gravitational waves from phase transitions

3. Generation of gravitational waves

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Outline

Yesterday ...

- Bubble nucleation rate/volume

$$\Gamma(T) = \Gamma_0(T) e^{-S(T)}$$

- Transition rate parameter β

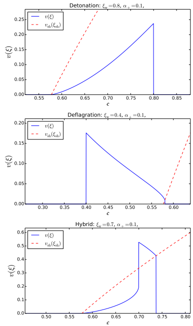
$$\Gamma(t) = \Gamma_f e^{\beta(t-t_f)}$$

with $\frac{4\pi}{3} v_{\rm w}^3 \left(3!/\beta^4\right) \Gamma_f = 1$

- Wall speed $v_{\rm w}$
- Transition strength $\alpha_+ = \frac{4\epsilon}{3w_+}$
    - Junction conditions ($r = w_+/w_-$)

$$v_+ v_- = \frac{1-(1-3\alpha_+)r}{3-3(1+\alpha_+)r}, \qquad \frac{v_+}{v_-} = \frac{3+(1-3\alpha_+)r}{1+3(1+\alpha_+)r}$$

    - Similarity solution for bubble growth: detonation, deflagration, hybrid
    - Conversion efficiency κ



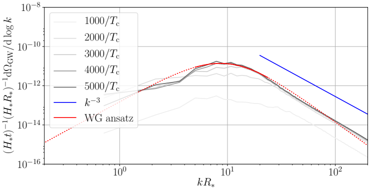
Gravitational waves from shear stresses

Metric perturbations h_{ij} from transverse-traceless part of EM tensor Π_{ij}:

$$\ddot{h}_{ij} - \nabla^2 h_{ij} = 16\pi G \Pi_{ij}$$

$$\Pi^{f}_{ij} = \left[(e+p)\gamma^2 v_i v_j + p\delta_{ij}\right]^{TT}$$

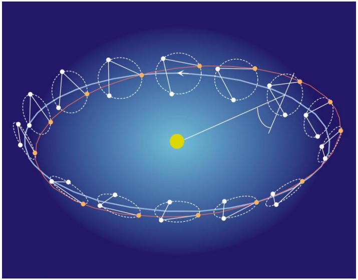
$$\Pi^{\phi}_{ij} = \left[\partial_i\phi\partial_j\phi - \tfrac{1}{2}(\partial\phi)^2\delta_{ij}\right]^{TT}$$



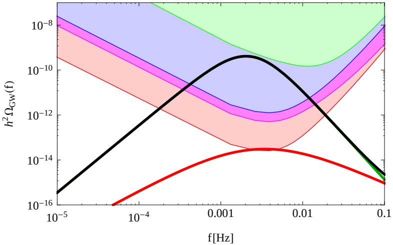
GW power spectrum: $\dfrac{d\rho_{GW}(k)}{d\ln k} = \dfrac{k^3}{32\pi G}\displaystyle\int \dfrac{d\Omega}{(2\pi)^3}\,\dot{h}_{ij}(t,\mathbf{k})\dot{h}^*_{ij}(t,\mathbf{k})$

Detection prospects

Space-based GW detector LISA (launch by 2034):



- e.g. strong EW transition with eLISA configurations: best (r), worst (g) (Caprini et al. 2015)
- $\alpha = 0.5$, $\beta/H = 100$, $v_w = 0.95$
- Total (k) sound (g) turbulence (r)



Gravitational wave equation (and an auxiliary equation)

- Assume processes happen much faster than Hubble rate ($\beta/H \gg 1$)
- Assume metric perturbations are small
- Linearised GR, neglect expansion[1]

$$ds^2 = -dt^2 + (\delta_{ij} + h_{ij})dx^i dx^j$$

- Linearised Einstein eqn for transverse-traceless part:

$$\ddot{h}_{ij} - \nabla^2 h_{ij} = 16\pi G \Pi_{ij}$$

- Convenient to avoid TT for evolution[2]

$$\ddot{u}_{ij} - \nabla^2 u_{ij} = 16\pi G \tau_{ij}$$

  where $h_{ij} = u_{ij}^{TT}$ and $\Pi_{ij} = \tau_{ij}^{TT}$.
- Can take

$$\tau_{ij}^{\mathrm{f}} = (e+p)\gamma^2 v_i v_j, \qquad \tau_{ij}^{\phi} = \partial_i\phi\partial_j\phi$$

[1] Can put expansion in, not much changes
[2] Garcia-Bellido, Figueroa, Sartre 2008

Gravitational wave spectral density

- GW energy density (average over many wavelengths, periods)

$$\rho_{\mathrm{gw}} = \frac{1}{32\pi G}\overline{\dot{h}_{ij}(x)\dot{h}_{ij}(x)}$$

- Fourier transform: $\dot{h}_{ij}(\mathbf{k},t) = \int d^3x \dot{h}_{ij}(\mathbf{x},t)e^{-i\mathbf{k}\cdot\mathbf{x}}$
- Define spectral density $P_{\dot{h}}$ through

$$\langle \dot{h}_{ij}(\mathbf{k},t)\dot{h}^*_{ij}(\mathbf{k}',t)\rangle = P_{\dot{h}}(\mathbf{k})\,\bar{\delta}^3(\mathbf{k}-\mathbf{k}')$$

- Assume gravitational waves are generated by a process which is
  - isotropic: $P_{\dot{h}}(\mathbf{k}) \to P_{\dot{h}}(k)$
  - homogeneous (hence $\bar{\delta}^3(\mathbf{k}-\mathbf{k}')$)
  - Random, Gaussian
- GW energy density becomes

$$\rho_{\mathrm{gw}} = \frac{1}{32\pi G}\int \bar{d}^3k P_{\dot{h}}(k) = \frac{1}{32\pi G}\frac{4\pi}{(2\pi)^3}\int dk k^2 P_{\dot{h}}(k)$$

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$$\rho_{\mathrm{gw}} = \frac{1}{32\pi G}\int \tilde{d}^3k P_{\dot{h}}(k) = \frac{1}{32\pi G}\frac{1}{2\pi^2}\int dk k^2 P_{\dot{h}}(k)$$

Gravitational wave power spectrum

- Recall GW energy density from spectral density

$$\rho_{\rm gw} = \frac{1}{32\pi G}\frac{1}{2\pi^2}\int dk k^2 P_{\dot{h}}(k)$$

- Convenient to introduce power spectrum $\mathcal{P}_{\dot{h}} = \frac{k^3}{2\pi^2}P_{\dot{h}}(k)$

$$\rho_{\rm gw} = \frac{1}{32\pi G}\int \frac{dk}{k}\mathcal{P}_{\dot{h}}(k)$$

- Cosmology: characterisation better in terms of $\Omega_{\rm gw} = \rho_{\rm gw}/\bar{e}$.
- Define gravitational wave power spectrum

$$\mathcal{P}_{\rm gw}(k) \equiv \frac{d\Omega_{\rm gw}}{d\ln(k)} = \frac{1}{\bar{e}}\frac{1}{32\pi G}\mathcal{P}_{\dot{h}}(k) = \frac{1}{12H^2}\mathcal{P}_{\dot{h}}(k)$$

GW from stochastic sources

- Equation for auxiliary tensor $u_{ij}(\mathbf{x}, t)$

$$(\partial_t^2 - \nabla^2) u_{ij}(\mathbf{x}, t) = (16\pi G)\tau_{ij}(\mathbf{x}, t)$$

- Solution with oscillator Green's function[3]

$$u_{ij}(\mathbf{k}, t) = (16\pi G) \int_0^t dt' \frac{\sin[k(t - t')]}{k} \tau_{ij}(\mathbf{k}, t')$$

- Gravitational wave from TT projector $\dot{h}_{ij}(\mathbf{k}, t) = \lambda_{ij,kl}(\mathbf{k}) \dot{u}_{kl}(\mathbf{k}, t)$
- Projector: $\lambda_{ij,kl}(\mathbf{k}) = P_{ik}(\mathbf{k}) P_{jl}(\mathbf{k}) - \frac{1}{2} P_{ij}(\mathbf{k}) P_{kl}(\mathbf{k})$ with $P_{ij}(\mathbf{k}) = \delta_{ij} - \hat{k}_i \hat{k}_j$.
- GW power spectrum obtained from

$$\langle \dot{h}^{ij}_{\mathbf{k}}(t) \dot{h}^{ij}_{\mathbf{k}'}(t) \rangle =$$

$$(16\pi G)^2 \int_0^t dt_1 dt_2 \cos[k(t - t_1)] \cos[k(t - t_2)] \lambda_{ij,kl}(\mathbf{k}) \langle \tau_{\mathrm{f}}^{ij}(\mathbf{k}, t_1) \tau_{\mathrm{f}}^{kl}(\mathbf{k}', t_2) \rangle.$$

[3] Boundary condition: $u_{ij}(\mathbf{k}, t) \to 0$ as $t \to 0$

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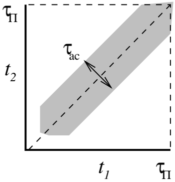
Unequal time correlator (UETC) for shear stress

- Define shear stress UETC Π^2 by

$$\lambda_{ij,kl}(\mathbf{k})\langle\tau^{ij}(\mathbf{k},t_1)\tau^{kl}(\mathbf{k}',t_2)\rangle = \Pi^2(k,t_1,t_2)\,\delta^3(\mathbf{k}+\mathbf{k}')$$

- Form of UETC
  - source is "on" for a time τ_Π
  - auto-correlated for a time τ_{ac}
  - peak at wavenumber $k \sim L_\Pi$



GW spectral density from shear stress UETC

- Averaging over a many periods of the wave[4]

$$P_{\dot{h}}(k,t) = (16\pi G)^2 \tfrac{1}{2} \int_0^t dt_1 dt_2 \cos[k(t_1 - t_2)] \Pi^2(k, t_1, t_2).$$

- Assumed form of UETC (write $t_+ = (t_1 + t_2)/2$, $t_- = t_1 - t_2$)
  $\Pi^2(k, t_-/\tau_{\mathrm{ac}})\theta(\tau_\Pi - t_+)$
- Results in spectral density for $\dot{h}$

$$P_{\dot{h}}(k,t) = (16\pi G)^2 \tau_\Pi \tau_{\mathrm{ac}} \mathcal{C}_\Pi(k)$$

where $\mathcal{C}_\Pi(k) = \frac{1}{2} \int \frac{dt_-}{\tau_{\mathrm{ac}}} \cos(kt_-) \Pi^2(k, t_-/\tau_{\mathrm{ac}})$

- Available scales for auto-correlation time τ_{ac}:
  - $\tau_{\mathrm{ac}} \sim k^{-1}$
  - $\tau_{\mathrm{ac}} \sim L_\Pi$
  - $\tau_{\mathrm{ac}} \sim \tau_\Pi$

[4] So that $\cos[k(t - t_1)] \cos[k(t - t_2)] \to \frac{1}{2} \cos[k(t_1 - t_2)]$

Sources of shear stress: fluid vs. scalar field

- Estimate size of shear stress correlator: $\Pi^2 \sim \langle \tau\tau \rangle$
  - Fluid source tensor $\tau_{\mathrm{f}}^{ij} = w\gamma^2 v^i v^j$
  - Field source tensor $\tau_{\phi}^{ij} = \partial^i \phi \partial^j \phi$
- Kinetic energies $K_{\mathrm{f}} = \int d^3x \tau_{ii}^{\mathrm{f}}$, $K_\phi = \int d^3x \tau_{ii}^{\phi}$
  - Fluid: $K_{\mathrm{f}} = \int d^3x\, w\gamma^2 v^2 = \frac{4\pi}{3} R^3 \bar{w} \frac{3}{4} \alpha\kappa$
  - Field: $K_\phi = \int d^3x\, (\nabla\phi)^2 = 4\pi R^2 \sigma$
- Ratio $K_{\mathrm{f}}/K_\phi \sim R\bar{w}/\sigma \sim R/\ell \gg 1$
  - Bubble size R grows to Hubble length, ℓ is a microscopic scale (wall width).
- Scalar coupled to fluid, similarity solution: fluid shear stress dominant
  - Fluid shear stresses come from compression/rarefaction: sound waves
- Runaway: field shear stress also grows as R^3 (not considered here)

Sound waves

Consider EM tensor for perturbations with z dependence only

$$T^{tt} = w\gamma^2 - p, \qquad T^{tz} = w\gamma^2 v^z, \qquad T^{zz} = w\gamma^2 (v^z)^2 + p$$

Perturbations: $\delta e = e - \bar{e}$, $\delta p = p - \bar{p}$, v^z all $\ll 1$

$$\partial_t T^{tt} + \partial_z T^{zt} = 0 \implies \partial_t(\delta e) + \bar{w}\partial_z v^z = 0 \tag{1}$$

$$\partial_t T^{tz} + \partial_z T^{zz} = 0 \implies \bar{w}\partial_t v^z + \partial_z(\delta p) = 0 \tag{2}$$

Note that δp and δe both depends temperature T: $\delta p = \left(\frac{\partial p}{\partial T} / \frac{\partial e}{\partial T}\right) \delta e = c_s^2 \delta e$
Hence equations (1) and (2) can be combined

$$\left(\partial_t^2 - c_s^2 \partial_z^2\right) v^z = 0, \qquad \left(\partial_t^2 - c_s^2 \partial_z^2\right) \delta T = 0$$

Sound wave is a collective mode of fluid velocity v^i and temperature T.
It is longitudinal: v^i is in direction of travel of wave.

Shear stress UETC from sound waves 1

- Recall shear stress UETC Π^2:
$$\lambda_{ij,kl}(\mathbf{k}_1)\langle\tau^{ij}(\mathbf{k}_1,t_1)\tau^{kl}(\mathbf{k}_2,t_2)\rangle = \Pi^2(k_1,t_1,t_2)\,\bar{\delta}^3(\mathbf{k}_1+\mathbf{k}_2)$$
- Source tensor dominated by fluid: $\tau^{ij} = \tau^{ij}_{\mathrm{f}} = w\gamma^2 v^i v^j$
- Non-relativistic fluid velocities: $\tau^{ij}_{\mathrm{f}} \simeq \bar{w} v^i v^j$
- Fourier transform of velocity field $\tilde{v}^i(\mathbf{q},t) = \int d^3x\, v^i(\mathbf{x},t)e^{-i\mathbf{q}\cdot\mathbf{x}}$
- Hence
$$\tau^{ij}_{\mathrm{f}}(\mathbf{k},t) = \bar{w}\int \bar{d}^3q\, \tilde{v}^i(\mathbf{q},t)\tilde{v}^j(\tilde{\mathbf{q}},t), \qquad \tilde{\mathbf{q}} = \mathbf{q}-\mathbf{k}$$
- Assume velocity field is Gaussian: $\langle\tau\tau\rangle \sim \langle vvvv\rangle = \sum\langle vv\rangle\langle vv\rangle$
- Velocity unequal time correlator:
$$\langle\tilde{v}^i_{\mathbf{q}_1}(t_1)\tilde{v}^{*j}_{\mathbf{q}_2}(t_2)\rangle = \Big[P_{ij}(q)F(q,t_1,t_2) + \hat{q}^i\hat{q}^j G(q,t_1,t_2)\Big]\,\bar{\delta}^3(\mathbf{q}_1-\mathbf{q}_2).$$
  - Tranverse projector $P_{ij}(q) = \delta_{ij} - \hat{q}^i\hat{q}^j$
  - Sound waves contribute only to longitudinal part G

Shear stress UETC from sound waves 2

- Recall shear stress UETC Π^2:
  $$\lambda_{ij,kl}(\mathbf{k}_1)\langle\tau^{ij}(\mathbf{k}_1,t_1)\tau^{kl}(\mathbf{k}_2,t_2)\rangle = \Pi^2(k_1,t_1,t_2)\,\bar{\delta}^3(\mathbf{k}_1+\mathbf{k}_2)$$
- With $\tau_{\mathrm{f}}^{ij}(\mathbf{k},t) = \bar{w}\int \bar{d}^3q\,\tilde{v}^i(\mathbf{q},t)\tilde{v}^j(\tilde{\mathbf{q}},t), \qquad \tilde{\mathbf{q}} = \mathbf{q}-\mathbf{k}$
- And the velocity unequal time correlator from sound waves: $\langle\tilde{v}^i_{\mathbf{q}_1}(t_1)\tilde{v}^{*j}_{\mathbf{q}_2}(t_2)\rangle = \hat{q}^i\hat{q}^j G(q,t_1,t_2)\,\bar{\delta}^3(\mathbf{q}_1-\mathbf{q}_2)$.
- A long calculation gives:[(5)]

$$\Pi^2(k,t_1,t_2) = \bar{w}^2\int \bar{d}^3q\,\frac{q^2}{\tilde{q}^2}(1-\mu^2)^2 G(q,t_1,t_2)G(\tilde{q},t_1,t_2)$$

where $\mu = \hat{\mathbf{k}}\cdot\hat{\mathbf{q}}$, $\tilde{q}^2 = q^2 - 2qk\mu + k^2$.

(5) Caprini, Durrer, Servant 2007, 2009

Velocity UETC from sound waves

- Following completion of transition at t_i, sound waves propagate freely
- General sound wave solution:

$$v^i(\mathbf{x}, t) = \int \bar{d}^3 q \left(v^i_{\mathbf{q}} e^{-i\omega t + i\mathbf{q}\cdot\mathbf{x}} + v^{*i}_{\mathbf{q}} e^{i\omega t - i\mathbf{q}\cdot\mathbf{x}} \right), \qquad \omega = c_s q$$

- Spectral density of velocity plane wave amplitudes

$$\langle v^i_{\mathbf{q}} v^j_{\mathbf{q}'} \rangle = \hat{q}^i \hat{q}^j P_v(q)\, \bar{\delta}^3(\mathbf{q} + \mathbf{q}')$$

- Recall velocity UETC $\langle \tilde{v}^i_{\mathbf{q}}(t_1) \tilde{v}^{*j}_{\mathbf{q}'}(t_2) \rangle = \hat{q}^i \hat{q}^j G(q, t_1, t_2)\, \bar{\delta}^3(\mathbf{q} - \mathbf{q}')$.
- Plane wave amplitudes related to Fourier transform $\tilde{v}^i_{\mathbf{q}}(t) = 2 v^i_{\mathbf{q}} \cos(\omega t)$
- Hence (recall $t_+ = (t_1 + t_2)/2$, $t_- = t_1 - t_2$)

$$G(q, t_1, t_2) = 4 P_v(q) \cos(\omega t_1) \cos(\omega t_2) = 2 P_v(q) \left[\cos(\omega t_-) + \cos(2\omega t_+) \right]$$

Shear stress UETC from sound waves 3

- Recall expressions for shear stress UETC and velocity UETC

$$\Pi^2(k,t_1,t_2) = \bar{w}^2 \int \bar{d}^3q \, \frac{q^2}{\tilde{q}^2}(1-\mu^2)^2 G(q,t_1,t_2)G(\tilde{q},t_1,t_2)$$

$$G(q,t_1,t_2) = 2P_v(q)\left[\cos(\omega t_-) + \cos(2\omega t_+)\right]$$

- Argument:
  - Convolution to get Π^2 has integral over $q = \omega/c_s$
  - For large times $t_+ \gg \omega^{-1}$ and $\cos(2\omega t_+)$ is highly oscillatory
  - $\implies$ Π^2 dominated by $\cos^2(\omega t_-)$ terms

$$\Pi^2(k,t_1,t_2) = \bar{w}^2 \int \bar{d}^3q \, \frac{q^2}{\tilde{q}^2}(1-\mu^2)^2 P_v(q)P_v(\tilde{q}) \cos(\omega t_-)\cos(\tilde{\omega} t_-)$$

- Conclusions:
  - $\Pi^2(k,t_1,t_2)$ depends mostly on $t_- = t_1 - t_2$ ("stationary")
  - Autocorrelation time of mode with wavenumber k is $\tau_{\rm ac} \sim k^{-1}$

GW from UETC

- Recall in spectral density for $\dot{h}$

$$P_{\dot{h}}(k,t) = (16\pi G)^2 \tau_\Pi \tau_{\rm ac} \mathcal{C}_\Pi(k)$$

where $\mathcal{C}_\Pi(k) = \frac{1}{2}\int \frac{dt_-}{\tau_{\rm ac}} \cos(kt_-)\Pi^2(k, t_-/\tau_{\rm ac})$

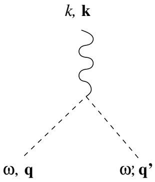
- We justified dropping the dependence of Π^2 on t_+
- We argued that $\tau_{\rm ac} = k^{-1}$, so
  $\mathcal{C}_\Pi(k) = \frac{1}{2}k\int dt_- \cos(kt_-)\Pi^2(k, kt_-)$
- Hence

$$\mathcal{C}_\Pi(k) = 4\bar{w}^2 \int \bar{d}^3 q \, \frac{q^2}{\tilde{q}^2}(1-\mu^2)^2 P_v(q) P_v(\tilde{q}) \Delta(k,\omega,\tilde{\omega})$$

where $\Delta(k,\omega,\tilde{\omega}) = \frac{1}{2}\int dt_- \cos(kt_-)\cos(\omega t_-)\cos(\tilde{\omega}t_-)$

Kinematics of GW production from sound waves

- Spectral density of metric perturbation $P_{\dot{h}}(k,t) = (16\pi G)^2 \tau_\Pi k^{-1}\, \mathcal{C}_\Pi(k)$
  $\mathcal{C}_\Pi(k) = 4\bar{w}^2 \int \bar{d}^3q\, \frac{q^2}{\tilde{q}^2}(1-\mu^2)^2 P_v(q) P_v(\tilde{q}) \Delta(k,\omega,\tilde{\omega})$
  $\Delta(k,\omega,\tilde{\omega}) = \frac{1}{2}\int dt_-\, \cos(kt_-)\cos(\omega t_-)\cos(\tilde{\omega} t_-)$
- For large time differences:
  $\Delta(k,\omega,\tilde{\omega}) \to \pi\delta(k \pm \omega \pm \tilde{\omega})$
  - Recall $\omega = c_s q$,
    $\tilde{\omega} = c_s \tilde{q} = c_s(q^2 + k^2 - 2kq\mu)$
  - Kinematics: only $k - \omega - \tilde{\omega}$ can vanish
  - Conservation of energy for production of GWs
- Hence



$$\mathcal{C}_\Pi(k) = 4\pi\bar{w}^2 \int \bar{d}^3q\, \frac{q^2}{\tilde{q}^2}(1-\mu^2)^2 P_v(q) P_v(\tilde{q}) \delta(k - \omega - \tilde{\omega})$$

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- Use δ function to perform integral over $\mu = \hat{\mathbf{k}}\cdot\hat{\mathbf{q}}$
  - Solution: $\mu = \mu_* = \frac{1}{c_s}\left(1 - \frac{1}{2}(1-c_s^2)\frac{k}{q}\right)$
  - Giving $\tilde{q} = \tilde{q}_* = \frac{1}{c_s}k - q$
- Only $q_- < q < q_+$ can produce GW with frequency k: with $q_\pm = k\frac{1\pm c_s}{2c_s}$

$$\mathcal{C}_\Pi(k) = \frac{\bar{w}^2}{2\pi c_s}\int_{q_-}^{q_+} dq\, q^2 \frac{q}{\tilde{q}}(1-\mu_*^2)^2 P_v(q) P_v(\tilde{q}_*)$$

Kinematics of GW production from sound waves

- Assume that sound waves have length scale L_f
- Scale out mean square velocity $\overline{U}_\mathrm{f}^2$
- Hence $P_v(q) = \overline{U}_\mathrm{f}^2 L_\mathrm{f}^3 \tilde{P}_v(qL_\mathrm{f})$, where $\tilde{P}_v$ is dimensionless.
- Define $z = qL_\mathrm{f}$ and $z_\pm = (kL_\mathrm{f})\frac{1\pm c_\mathrm{s}}{2c_\mathrm{s}}$

$$\mathcal{C}_\Pi(k) = \frac{\bar{w}^2\overline{U}_\mathrm{f}^4}{2\pi c_\mathrm{s}} L_\mathrm{f}^3 \int_{z_-}^{z_+} \frac{dz}{z} \frac{(z-z_+)^2(z-z_-)^2}{z_+ + z_- - z} \tilde{P}_v(z)\tilde{P}_v(z_+ + z_- - z)$$

- Define dimensionless function

$$\tilde{\mathcal{C}}_\Pi(kL_\mathrm{f}) = \int_{z_-}^{z_+} \frac{dz}{z} \frac{(z-z_+)^2(z-z_-)^2}{z_+ + z_- - z} \tilde{P}_v(z)\tilde{P}_v(z_+ + z_- - z)$$

  - is dimensionless and a function of $y = kL_\mathrm{f}$
  - peaks at $y \sim 1$ (definition of L_f)
  - Magnitude O(1)

Power spectrum of GWs from sound waves

- The final pieces are

$$\mathcal{P}_{\rm gw}(k) \equiv \frac{d\Omega_{\rm gw}}{d\ln(k)} = \frac{1}{12H^2}\frac{k^3}{2\pi^2}P_{\dot{h}}(k)$$

$$P_{\dot{h}}(k,t) = (16\pi G)^2 \tau_\Pi k^{-1}\,\mathcal{C}_\Pi(k)$$

$$\mathcal{C}_\Pi(k) = \frac{\bar{w}^2\overline{U}_{\rm f}^4}{2\pi c_{\rm s}} L_{\rm f}^3 \tilde{\mathcal{C}}_\Pi(kL_{\rm f})$$

$$\tilde{\mathcal{C}}_\Pi(kL_{\rm f}) = \int_{z_-}^{z_+}\frac{dz}{z}\,\frac{(z-z_+)^2(z-z_-)^2}{z_+ + z_- - z}\tilde{P}_v(z)\tilde{P}_v(z_+ + z_- - z)$$

- The gravitational wave power spectrum is

$$\boxed{\mathcal{P}_{\rm gw}(k) = 3\Gamma^2\overline{U}_{\rm f}^4(H\tau_\Pi)(HL_{\rm f})\frac{(kL_{\rm f})^3}{2\pi^2}\frac{\tilde{\mathcal{C}}_\Pi(kL_{\rm f})}{2\pi c_{\rm s} kL_{\rm f}}.}$$

where $\Gamma = \bar{w}/\bar{e}$ (adiabatic index, $\Gamma \simeq 4/3$)

Power spectrum of GWs from sound waves: amplitude

- From last slide

$$\mathcal{P}_{\rm gw}(k) = 3\Gamma^2\overline{U}_{\rm f}^4 (H\tau_\Pi)(HL_{\rm f})\tilde{\mathcal{P}}_{\rm gw}(kL_{\rm f})$$

$$\tilde{\mathcal{P}}_{\rm gw}(kL_{\rm f}) = \frac{(kL_{\rm f})^3}{2\pi^2}\frac{\tilde{\mathcal{C}}_\Pi(kL_{\rm f})}{2\pi c_{\rm s} kL_{\rm f}}$$

$$\tilde{\mathcal{C}}_\Pi(kL_{\rm f}) = \int_{z_-}^{z_+} \frac{dz}{z}\,\frac{(z-z_+)^2(z-z_-)^2}{z_+ + z_- - z}\tilde{P}_v(z)\tilde{P}_v(z_+ + z_- - z)$$

where $z_\pm = (kL_{\rm f})\frac{1\pm c_{\rm s}}{2c_{\rm s}}$

- Power proportional to square of kinetic energy: $\Gamma\overline{U}_{\rm f}^2 \simeq \kappa\alpha$
  - Phase transition strength parameter α
  - Vacuum to kinetic energy conversion efficiency κ
- Power proportional to length scale $L_{\rm f}$
  - Length scale set by bubble separation: $L_{\rm f} \sim R_* = (8\pi)^{\frac{1}{3}} v_{\rm w}/\beta$
- Power proportional to time sound waves last τ_Π, which is the shorter of
  - Hubble time $\tau_H = H^{-1}$
  - Eddy turn-over time $\tau_{\rm e} \sim L_{\rm f}/\overline{U}_{\rm f}$ (time for turbulence to develop)

Power spectrum of GWs from sound waves: shape

- From last-but-one slide

$$\mathcal{P}_{\rm gw}(k) = 3\Gamma^2\overline{U}_{\rm f}^4(H\tau_\Pi)(HL_{\rm f})\tilde{\mathcal{P}}_{\rm gw}(kL_{\rm f})$$

$$\tilde{\mathcal{P}}_{\rm gw}(kL_{\rm f}) = \frac{(kL_{\rm f})^3}{2\pi^2}\frac{\tilde{\mathcal{C}}_\Pi(kL_{\rm f})}{2\pi c_{\rm s} kL_{\rm f}}$$

$$\tilde{\mathcal{C}}_\Pi(kL_{\rm f}) = \int_{z_-}^{z_+}\frac{dz}{z}\frac{(z-z_+)^2(z-z_-)^2}{z_+ + z_- - z}\tilde{P}_v(z)\tilde{P}_v(z_+ + z_- - z)$$

where $z_\pm = (kL_{\rm f})\frac{1\pm c_{\rm s}}{2c_{\rm s}}$

- If velocity power spectrum $\mathcal{P}_v \sim q^n$, then $\tilde{P}_v(z) \sim z^{n-3}$
- $\implies \tilde{\mathcal{C}}_\Pi(y) \sim y^3(y^{n-3})^2 \sim y^{2n-3}$
- $\implies \tilde{\mathcal{P}}_{\rm gw}(y) \sim y^3 y^{2n-4} \sim y^{2n-1} \propto k^{2n-1}$
- Prediction: $\mathcal{P}_{\rm gw}(k) \sim k^{2n-1}$ from $\mathcal{P}_v(q) \sim q^n$

Modelling GWs from sound waves

- Phenomenological model for GW power spectrum from linear sound waves (v_w not near sound speed c_s)
  - Bubble nucleation temperature T_n
  - Hubble rate H_n
  - Mean bubble separation R_*, peak of power spectrum at $z_p = k_p R_* \simeq 10$
- $$\frac{d\Omega_{gw,0}}{d\ln(f)} = 0.68 F_{gw,0}\Gamma^2 \overline{U}_f^4 (H_n R_*)\tilde{\Omega}_{gw} C\left(\frac{f}{f_{p,0}}\right).$$

  where

  $$C(s) = s^3\left(\frac{7}{4+3s^2}\right)^{7/2}.$$

  dilution of GWs since matter-domination

  $$F_{gw,0} = (3.57 \pm 0.05) \times 10^{-5}\left(\frac{100}{h_*}\right)^{\frac{1}{3}}.$$

  Peak frequency

  $$f_{p,0} \simeq 26\left(\frac{1}{H_n R_*}\right)\left(\frac{z_p}{10}\right)\left(\frac{T_n}{10^2\,\mathrm{GeV}}\right)\left(\frac{h_*}{100}\right)^{\frac{1}{6}} \mu\mathrm{Hz},$$

Summary and outlook

- Gravitational wave production from 1st order phase transition
  - Promising sensitivity to EW transition from LISA (Taiji ...)
- Areas for further work:
  - GWs from turbulent velocity field
  - GWs from runaway bubble (scalar field dominant)
  - Parameter extraction from GW power spectrum $\alpha, \beta, v_{\rm w}, T_{\rm c}$
  - Improve accuracy of calculations of $\alpha, \beta, v_{\rm w}, T_{\rm c}$ from fundamental theory
  - Collider signals and phase transition parameters
- Future:
  - information about fundamental physics from the GW background at LISA