Killing–Yano symmetries of Black Holes

Superstring Solutions, Supersymmetry and Geometry, Benasque

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Introduction

Symmetry: key tool throughout physics

Killing vectors k_a: most basic symmetries of classical solutions in gravity, give conserved quantities e.g. energy,sangular momentum
$\longrightarrow$
Killing *tensors* $k_{a_1 \ldots a_p}$: "hidden" symmetries of phase space, rather than configuration space, give further constants of motion

Key examples: rotating black holes, e.g. Kerr

Killing tensors are basic geometric properties, helpful for:

- Finding exact solutions:
  - For asymptotically AdS black holes, few algorithmic methods
  - Known exact AdS black holes have these symmetries
- Studying properties:
  - Integrability of geodesics
  - Separability of Klein–Gordon equation, Dirac equation, ...

Summary

Overview of basics

Motivation from black hole examples in supergravity

Lifting lower-dimensional Killing tensors into higher-dimensions, e.g. 10d string theory — general geometrical results

"Lift condition" constraint and its appearance elsewhere

Partly based on 1511.09310, and also earlier works

Conformal Killing vectors

2 equivalent definitions of Killing vectors:

$$\boxed{\nabla_a K_b = \nabla_{[a} K_{b]} \Longleftrightarrow \nabla_{(a} K_{b)} = 0}$$

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$$\boxed{\nabla_a k_b = \nabla_{[a} k_{b]} + g_{ab} \widetilde{k} \Longleftrightarrow \nabla_{(a} k_{b)} = \widetilde{k} g_{ab}}$$

$$\widetilde{k} = \frac{1}{D} \nabla_a k^a, \qquad \text{spacetime dimension } D$$

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Each definition generalizes simply to higher ranks:

1. Antisymmetric conformal Killing–Yano (CKY) p-forms
   $k_{a_1 \ldots a_p} = k_{[a_1 \ldots a_p]}$
2. Symmetric rank-p conformal Killing–Stäckel (CKS) tensors
   $k_{a_1 \ldots a_p} = k_{(a_1 \ldots a_p)}$

Antisymmetric generalization: CKY p-forms

Conformal Killing vector k_a:

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$$\widetilde{k} = \frac{1}{D} \nabla_a k^a$$

Antisymmetric generalization: CKY p-forms

Conformal Killing–Yano (CKY) p-forms $k_{a_1 \dots a_p} = k_{[a_1 \dots a_p]}$ (Tachibana 69, Kashiwada 68):

$$\boxed{\nabla_a k_{b_1 b_2 \dots b_p} = \nabla_{[a} k_{b_1 b_2 \dots b_p]} + p g_{a[b_1} \widetilde{k}_{b_2 \dots b_p]}}$$

$$\widetilde{k}_{b_2 \dots b_p} = \frac{1}{D - p + 1} \nabla_a k^a{}_{b_2 \dots b_p}$$

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$$\widetilde{k}_{b_2 \dots b_p} = \frac{1}{D-p+1} \nabla_a k^a{}_{b_2 \dots b_p}$$

In differential form notation:

$$\nabla_X k = \frac{1}{p+1} X \lrcorner \, \mathrm{d}k - \frac{1}{D-p+1} X^\flat \wedge \delta k$$

exterior derivative   divergence

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exterior derivative divergence

From general decomposition of $\nabla_a k_{b_1 b_2 \dots b_p}$:

$$\nabla_a k_{b_1 b_2 \dots b_p} \equiv \nabla_{[a} k_{b_1 b_2 \dots b_p]} + p g_{a[b_1} \widetilde{k}_{b_2 \dots b_p]} + \cancel{(k_{a\perp})_{b_1 \dots b_p}}$$

1-form $\otimes$ p-form $(p+1)$-form $(p-1)$-form   "projection"/ twistor operator

Antisymmetric generalization: KY and CCKY p-forms

2 special cases of conformal Killing–Yano (CKY) p-forms:

1. Killing–Yano (KY) p-form $(\delta k = 0)$ (Yano 52):

$$\boxed{\nabla_a k_{b_1 \dots b_p} = \nabla_{[a} k_{b_1 \dots b_p]}}$$

2. Closed conformal Killing–Yano (CCKY) p-form $(\mathrm{d}k = 0)$:

$$\boxed{\nabla_a k_{b_1 \dots b_p} = p g_{a[b_1} \widetilde{k}_{b_2 \dots b_p]}}$$

Hodge dual:

KY $(D-p)$-form $\stackrel{\star}{\longleftrightarrow}$ CCKY p-form

Symmetric generalization: CKS tensors

Rank-p Killing–Stäckel (KS) tensor $K_{a_1 \dots a_p} = K_{(a_1 \dots a_p)}$ (Stäckel 1893):

$$\boxed{\nabla_{(a} K_{b_1 \dots b_p)} = 0}$$

Constant of motion: $K^{a_1 \dots a_p} P_{a_1} \dots P_{a_p}$

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Rank-p conformal Killing–Stäckel (CKS) tensor $Q_{a_1 \dots a_p} = Q_{(a_1 \dots a_p)}$:

$$\boxed{\nabla_{(a} Q_{b_1 \dots b_p)} = g_{(ab_1} q_{b_2 \dots b_p)}}$$

Constant of motion along *null* geodesics: $Q^{a_1 \dots a_p} P_{a_1} \dots P_{a_p}$

Symmetric generalization: rank-2 KS tensors

Trivial rank-2 KS tensors:

- metric g_{ab}
- symmetrized Killing vectors $k_{(a}l_{b)}$

Irreducible KS tensor: not decomposable into trivial/lower-rank KS tensors

KY p-form "squared" $=$ rank-2 KS tensor:

$$(Y \bullet Y)_{ab} := Y^{c_1 \dots c_{p-1}}{}_a Y_{c_1 \dots c_{p-1} b} = K_{ab}$$

$\implies$ KY p-forms more fundamental

Converse not true: rank-2 KS tensor $\not\Longrightarrow$ KY "square root"

Starting point: Kerr black hole

Kerr metric (Kerr 63):

$$\mathrm{d}s^2 = \frac{-R(r)(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi)^2 + \sin^2\theta\,[a\,\mathrm{d}t - (r^2+a^2)\,\mathrm{d}\phi]^2}{r^2 + a^2\,\cos^2\theta} + (r^2 + a^2\cos^2\theta)\left(\frac{\mathrm{d}r^2}{R(r)} + \mathrm{d}\theta^2\right), \quad R(r) = r^2 + a^2 - 2mr$$

Spacetime admits:

- Time translation Killing vector $\partial/\partial t$
- Rotational Killing vector $\partial/\partial\phi$
- Killing–Yano 2-form Y_{ab} (Floyd 74, Penrose 73)

Each gives constant of geodesic motion, E, J_ϕ, Carter constant ("$\mathbf{J}^2$")

Generalizations in Einstein gravity:

- $D > 4$, $\Lambda \neq 0$, NUT charges
- Killing tensor structure generalizes (e.g. (Kubizňák 08))

Charged, rotating black holes

Further exact, charged, rotating generalizations of the Kerr black hole known in string theory

Theories and solutions involve 3-form field strength H

Examples admit Killing–Yano p-forms with modified connection:

$$\Gamma^{H\,a}{}_{bc} = \Gamma^{a}{}_{bc} + \tfrac{1}{2} H^{a}{}_{bc}$$

Levi-Civita torsion

Define Killing–Yano p-forms with torsion (KYT p-forms) by replacing $\Gamma \to \Gamma^{H}$ in previous definitions

- Not necessary to generalize symmetric KS tensor definition
- Not necessary to couple test particles to H, can regard H as just a matter field

Conformal frames

Under change of conformal frame (note indices raised):
conformal Killing tensor $k^{a_1 \cdots a_p} \longrightarrow$ conformal Killing tensor $k^{a_1 \cdots a_p}$

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In string theory, 2 conformal frames (Callan Friedan Martinec Perry 85):

1. Einstein frame:

$$\mathcal{L} = R \star 1 - \tfrac{1}{2} \star \mathrm{d}\varphi \wedge \mathrm{d}\varphi - \tfrac{1}{2} \mathrm{e}^{\sqrt{8/(D-2)}\varphi} \star H \wedge H + \dots$$

2. String frame:

$$\mathcal{L} = \mathrm{e}^{\sqrt{(D-2)/2}\varphi} (R \star 1 + \tfrac{D-2}{2} \star \mathrm{d}\varphi \wedge \mathrm{d}\varphi - \tfrac{1}{2} \star H \wedge H) + \dots$$

For known black holes, ~~conformal~~ Killing tensors in string frame (DC 08)
$\implies$ conformal Killing tensors in other (e.g. Einstein) frames

Example 1: arbitrary dimensional bosonic string

D-dimensional theory in Einstein/string frame ($\mathrm{d}s_{\mathsf{s}}^2 = X^2\,\mathrm{d}s_{\mathsf{E}}^2$):

$$\mathcal{L}_{\mathsf{E}} = R\star 1 - \tfrac{1}{2}\star\mathrm{d}\varphi\wedge\mathrm{d}\varphi - X^{-2}\star F\wedge F - \tfrac{1}{2}X^{-4}\star H\wedge H$$

$$\mathcal{L}_{\mathsf{s}} = X^{-(D-2)}(R\star 1 + \tfrac{D-2}{2}\star\mathrm{d}\varphi\wedge\mathrm{d}\varphi - \star F\wedge F - \tfrac{1}{2}\star H\wedge H)$$

$$H = \mathrm{d}B - A\wedge F, \qquad X = \mathrm{e}^{-\varphi/\sqrt{2(D-2)}}$$

Lifts to $(D+1)$ dimensional theory

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Charged Kerr–NUT in $D = 2n + \varepsilon$ dimensions, $\varepsilon = 0, 1$ (DC 08)

- Parameters: Mass, $n - 1 - \varepsilon$ NUT charges, $n - 1 + \varepsilon$ rotations, 1 electric charge
- Killing coordinates: ψ_k, $k = 0, \ldots, n - 1 + \varepsilon$
- Non-Killing coordinates: x_μ, $\mu = 1, \ldots, n$

Solution admits CCKYT 2-form (Houri Kubizňák Warnick Yasui 10):

Lifting KYT p-forms

Are symmetries of lower-dimensional solutions also symmetries of higher(e.g. 10)-dimensional solution?

Motivations:

- Understanding from string theory
- Higher-dimensional solution may have more symmetry

(Kaluza–Klein) lift, coordinates $\{x^a\} = \{x^i, z\}$, $F = \mathrm{d}A$:

$$\mathrm{d}\bar{s}^2 = \mathrm{d}s^2 + (\mathrm{d}z + A)^2, \qquad \overline{H} = H + F \wedge (\mathrm{d}z + A)$$

Ansatz motivated by string theory black holes with KYT p-forms:

- No scalar in metric ansatz, enough for string frame
- Only 1 gauge field (from more general 2 gauge fields set equal)

Related recent works:

- Basic examples (Houri Yamamoto 12)
- String dualities (Chervonyi Lunin 15)
- Different lift (Krtouš Kubizňák Kolář 15)

Lifting KYT p-forms

Lemma: The non-vanishing components of the difference of connections including torsion $(\Delta\Gamma^H)^a{}_{bc} = \overline{\Gamma}^{\overline{H}\,a}{}_{bc} - \Gamma^{H\,a}{}_{bc}$ are

$$(\Delta\Gamma^H)^i{}_{jk} = A_j F_k{}^i.$$

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Proposition: Suppose the metric $\mathrm{d}s^2$ admits a KYT p-form $Y_{i_1 \dots i_p}$ with 3-form torsion H_{ijk}. Suppose also that the "lift condition'

$$\boxed{F^j{}_{[i_1} Y_{i_2 \dots i_p]j} = 0}$$

holds. Then $Y_{i_1 \dots i_p}$ is a KYT p-form for the metric $\mathrm{d}\overline{s}^2$, with torsion $\overline{H}_{abc}$.

Note:

- Geometrical result, independent of field equations
- "Lift condition" will appear in other contexts

Lift condition application 2: charged particle motion

Associated symmetric Killing–Stäckel tensor

$$K_{ab} = Y^{c_1 \dots c_{p-1}}{}_a Y_{c_1 \dots c_{p-1} b} =: (Y \bullet Y)_{ab}$$

Lift condition $F^b{}_{[a_1} Y_{a_2 \dots a_p] b} = 0 \implies$

$$F^c{}_{(a} K_{b)c} = 0$$

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Charged particle motion:

$$P^a \nabla_a P^b = e F^b{}_a P^a$$

Constant of motion $K_{bc} P^b P^c$:

$$P^a \nabla_a (K_{bc} P^b P^c) = P^a P^b P^c \nabla_{(a} K_{bc)} + 2e K_{bc} F^b{}_a P^a P^c$$

Shown for KY 2-forms (Hughston Penrose Sommers Walker 72), p-forms (Açık Ertem Önder Verçin 08)

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KS equation

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KS equation   lift condition

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Lift condition application 3: Dirac equation

KY 2-form Y_{ab} satisfying

$$F^c{}_{[a}Y_{b]c} = 0$$

$\implies$ operator K $(= \mathrm{i}\gamma_5(\gamma^a Y_a{}^b D_b - \frac{1}{6}\gamma^{abc}\nabla_a Y_{bc}))$ commuting with Dirac operator $\mathcal{D} = \gamma^a D_a = \gamma^a(\nabla_a - \mathrm{i}eA_a)$ (Carter McLenaghan 79):

$$[\mathcal{D}, K] = 0$$

$\implies$ good quantum number

- Underlies separability of Dirac equation in Kerr
- Flat spacetime: $K^2 \sim \mathbf{J}^2$

Lift condition application 3: Dirac equation

Proposition: Let $Y_{a_1 \dots a_p}$ be a KY p-form, and define the operator

$$K_Y = \gamma^{b_1 \dots b_{p-1}} Y^a{}_{b_1 \dots b_{p-1}} (\nabla_a - \mathrm{i} e A_a) + \frac{\gamma^{b_1 \dots b_{p+1}}}{2(p+1)} \nabla_{b_1} Y_{b_2 \dots b_{p+1}}.$$

Suppose also that

$$F^b{}_{[a_1} Y_{a_2 \dots a_p]b} = 0.$$

Then K_Y graded anti-commutes with the Dirac operator $\mathcal{D}$:

$$\mathcal{D} K_Y + (-1)^p K_Y \mathcal{D} = 0.$$

Note: multiplying K_Y by γ^5 etc. switches commutation/
anti-commutation with $\mathcal{D}$

Lift condition application 4: worldline supersymmetry

Pseudo-classical charged spinning particle (e.g. Brink Di Vecchia Howe 77):

$$L = \frac{m}{2} g_{ab} \dot{x}^a \dot{x}^b + e A_a \dot{x}^a + \frac{\mathrm{i}}{2} \left(\eta_{\mu\nu} \psi^\mu \frac{D\psi^\nu}{D\tau} - \frac{e}{m} F_{\mu\nu} \xi^\mu \xi^\nu \right)$$

ψ^μ Grassmann-valued, $D\psi^\mu / D\tau = \dot{\xi}^\mu + \omega^\mu{}_{\nu a} \xi^\nu \dot{x}^a$

Constraints:

- $H := (m/2) g_{ab} \dot{x}^a \dot{x}^b + (\mathrm{i}e/2m) F_{ab} \xi^a \xi^b = -m/2$ (time reparameterization)
- $Q := \dot{x}^a \psi_a = 0$ (spacelike spin)

Supersymmetry algebra (generic, i.e. g_{ab}-independent):

$$\{Q, H\} = 0, \qquad \{Q, Q\} = -\frac{2\mathrm{i}}{m} H$$

Poisson–Dirac bracket, $a_F = 0, 1$ Grassmann parity of F:

$$\{F, G\} = \frac{\partial F}{\partial x^a} \frac{\partial G}{\partial p_a} - \frac{\partial F}{\partial p_a} \frac{\partial G}{\partial x^a} + \mathrm{i}(-1)^{a_F} \frac{\partial F}{\partial \psi^\mu} \frac{\partial G}{\partial \psi_\mu}$$

Lift condition application 4: worldline supersymmetry

KY 2-form $\implies$ enhanced (non-generic, i.e. g_{ab}-dependent) worldline supersymmetry ($e = 0$):

- KY 2-form $\iff$ superinvariant $\mathcal{Y}$, $\{Q, \mathcal{Y}\} = 0$ (Gibbons Rietdijk van Holten 93)
- Jacobi identity $0 = \{\mathcal{Y}, \{Q, Q\}\} + \{Q, \{\mathcal{Y}, Q\}\} + \{Q, \{Q, \mathcal{Y}\}\}$ $\implies$ $\{H, \mathcal{Y}\} = 0$, constants of motion

$$F^b{}_{[a_1} Y_{a_2 \ldots a_p]b} = 0$$

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$$F^b{}_{[a_1} Y_{a_2 \ldots a_p]b} = 0$$

$\implies$ $e \neq 0$ generalizations:

- Arbitrary p-forms, no torsion, $e \neq 0$ (Tanimoto 95)
- 2-forms, with torsion, $e \neq 0$ (Rietdijk van Holten 95)
- (Arbitrary p-forms, torsion, $e = 0$ (Kubizňák Kunduri Yasui 09))

Lift condition $\implies$ invariance of general $N = 1$ supersymmetric particle action (Papadopoulos 11)

Lift condition summary

"Lift condition" $F^b{}_{[a_1} Y_{a_2 \ldots a_p]b} = 0$ appears in several different contexts:

1. Kaluza–Klein lift of KYT p-form
2. Constant of motion for charged particles
3. Existence of operator commuting with Dirac operator
4. Enhanced worldline supersymmetry of pseudo-classical charged spinning particle

All are general, geometric results, independent of

- Field equations
- Precise form of spacetime

Lifting CCKYT p-forms

Recall Hodge duality:

$$\text{KYT } (D-p)\text{-form} \overset{\star}{\longleftrightarrow} \text{CCKYT } p\text{-form}$$

Corollary: Suppose that the metric $\mathrm{d}s^2$ admits a CCKYT p-form $\widetilde{Y}$

with 3-form torsion H. Suppose also that

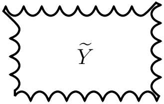
$$\epsilon_{kj_1\ldots j_{D-p}i_1\ldots i_{p-1}} F^k{}_{i_p} \widetilde{Y}^{i_1\ldots i_p} = 0.$$

Then $\widetilde{Y} \wedge (\mathrm{d}z + A)$ is a CCKYT $(p+1)$-form for the metric $\mathrm{d}\bar{s}^2$, with torsion $\overline{H}$.

Tower of Killing tensors for Kerr generalizations

Spacetime dimension $D = 2n + \varepsilon$, $\varepsilon = 0, 1$

CCKYT 2-form



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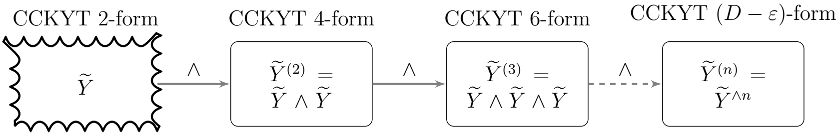

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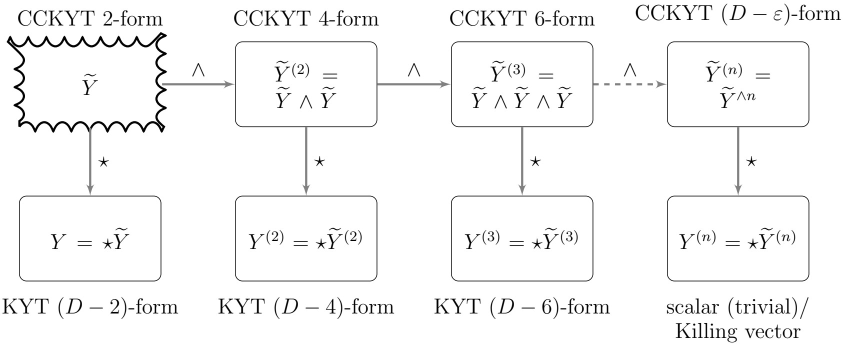

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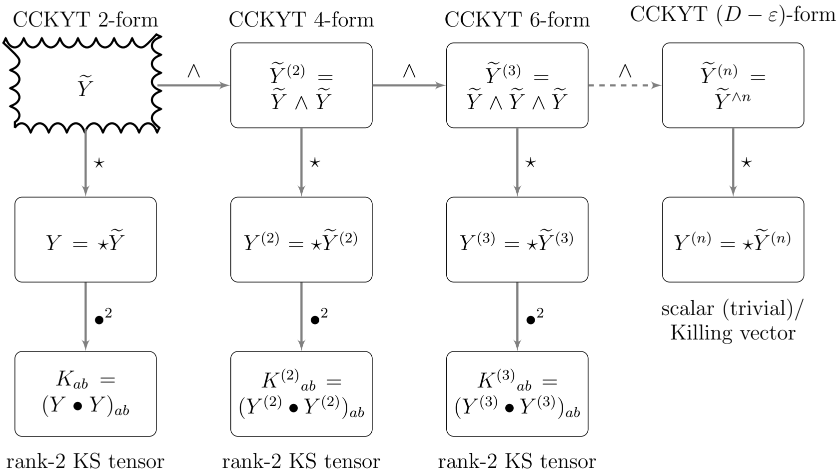

$g_{ab} + (n+\varepsilon)$ Killing vectors $+ (n-1)$ KS tensors $\Longrightarrow$ D constants

Lifting the tower of Killing tensors

Corollary: Suppose that the metric $\mathrm{d}s^2$ admits a CCKYT p-form $\widetilde{Y}_1$ and a CCKYT q-form $\widetilde{Y}_2$ with 3-form torsion H, and that they satisfy the conditions to lift to CCKYT forms $\widetilde{Y}_1 \wedge (\mathrm{d}z + A)$ and $\widetilde{Y}_2 \wedge (\mathrm{d}z + A)$ on $\mathrm{d}\bar{s}^2$. Then $\widetilde{Y}_1 \wedge \widetilde{Y}_2$ also satisfies the condition to lift to a $(p + q + 1)$-form $\widetilde{Y}_1 \wedge \widetilde{Y}_2 \wedge (\mathrm{d}z + A)$ on $\mathrm{d}\bar{s}^2$.

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Lift condition for CCKYT 2-form $\widetilde{Y}$
$\implies$ lift condition for CCKYT $(2j)$-forms $\widetilde{Y}_{(j)} = \widetilde{Y}^{\wedge j}$
$\implies$ lift condition for KYT $(D-2j)$-forms $Y_{(j)} = \star \widetilde{Y}_{(j)}$
$\implies$ whole tower of Killing tensors lifts

Lifting KS tensors

Proposition: Suppose that the metric $\mathrm{d}s^2$ admits a rank-p KS tensor $K_{i_1 \dots i_p}$. Suppose also that the KS lift condition

$$\boxed{F^j{}_{(i_1} K_{i_2 \dots i_p) j} = 0}$$

holds. Then $K_{i_1 \dots i_p}$ is a rank-p KS tensor for the metric $\mathrm{d}\bar{s}^2$.

Note: KYT lift condition $\implies$ rank-2 KS lift condition

Example 1: arbitrary dimensional bosonic string

D-dimensional theory in Einstein/string frame ($\mathrm{d}s_{\mathsf{s}}^2 = X^2\,\mathrm{d}s_{\mathsf{E}}^2$):

$$\mathcal{L}_{\mathsf{E}} = R \star 1 - \tfrac{1}{2} \star \mathrm{d}\varphi \wedge \mathrm{d}\varphi - X^{-2} \star F \wedge F - \tfrac{1}{2} X^{-4} \star H \wedge H$$

$$\mathcal{L}_{\mathsf{s}} = X^{-(D-2)}\big(R \star 1 + \tfrac{D-2}{2} \star \mathrm{d}\varphi \wedge \mathrm{d}\varphi - \star F \wedge F - \tfrac{1}{2} \star H \wedge H\big)$$

$$H = \mathrm{d}B - A \wedge F, \qquad X = \mathrm{e}^{-\varphi/\sqrt{2(D-2)}}$$

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String frame Kaluza–Klein ansatz:

$$\mathrm{d}s_{D+1}^2 = \mathrm{d}s^2 + (\mathrm{d}z + A)^2, \qquad H_{D+1} = H + F \wedge (\mathrm{d}z + A)$$

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$(D+1)$-dimensional theory in Einstein/string frame:

$$\mathcal{L}_{\mathsf{E}} = R\star 1 - \tfrac{D-2}{2(D-1)}\star\mathrm{d}\varphi\wedge\mathrm{d}\varphi - \tfrac{1}{2}X^{-4(D-2)/(D-1)}\star H\wedge H$$

$$\mathcal{L}_{\mathsf{s}} = X^{-(D-2)}(R\star 1 + \tfrac{D-2}{2}\star\mathrm{d}\varphi\wedge\mathrm{d}\varphi - \tfrac{1}{2}\star H\wedge H)$$

$$H = \mathrm{d}B, \qquad \mathrm{d}s_{\mathsf{s}}^2 = X^{2(D-2)/(D-1)}\mathrm{d}s_{\mathsf{E}}^2$$

Example 1: arbitrary dimensional bosonic string theory

Charged Kerr–NUT in $D = 2n + \varepsilon$ dimensions, $\varepsilon = 0, 1$ (DC 08)

- Parameters: Mass, $n - 1 - \varepsilon$ NUT charges, $n - 1 + \varepsilon$ rotations, 1 electric charge
- Killing coordinates: ψ_k, $k = 0, \dots, n - 1 + \varepsilon$
- Non-Killing coordinates: x_μ, $\mu = 1, \dots, n$

D-dimensional metric and gauge field strength:

$$\mathrm{d}s^2 = \sum_{\mu=1}^{n} (e^\mu e^\mu + e^{\hat{\mu}} e^{\hat{\mu}}) + \varepsilon e^0 e^0, \quad F = \frac{c}{s} \sum_{\mu=1}^{n} H_\mu(x_\mu) e^\mu \wedge e^{\hat{\mu}}$$

$$e^\mu = \frac{\mathrm{d}x_\mu}{\sqrt{Q_\mu(x_\mu)}}, \quad e^{\hat{\mu}} = \sqrt{Q_\mu(x_\mu)} \Big(\mathcal{A}_\mu - \sum_{\nu=1}^{n} \frac{2N_\nu s^2}{HU_\nu} \mathcal{A}_\nu \Big), \quad e^0 = \sqrt{S} \Big(\mathcal{A} - \sum_{\nu=1}^{n} \frac{2N_\nu s^2}{HU_\nu} \mathcal{A}_\nu \Big)$$

CCKYT 2-form (Houri Kubizňák Warnick Yasui 10):

$$\widetilde{Y} = \sum_{\mu=1}^{n} x_\mu e^\mu \wedge e^{\hat{\mu}}$$

Lift condition: ✓ (also for $\mathrm{AdS}_{4,5,6,7}$ generalizations in gauged sugra)

Example 2: 4-dimensional supergravity

$\mathcal{N} = 2$ supergravity coupled to 1 vector multiplet, Einstein frame:

$$\mathcal{L}_4 = R \star 1 - \tfrac{1}{2} \star \mathrm{d}\varphi \wedge \mathrm{d}\varphi - \tfrac{1}{2}\mathrm{e}^{2\varphi} \star \mathrm{d}\chi \wedge \mathrm{d}\chi$$
$$- \mathrm{e}^{-\varphi}(\star F^1 \wedge F^1 + \star \widetilde{F}_2 \wedge \widetilde{F}_2) + \chi(F^1 \wedge F^1 + \widetilde{F}_2 \wedge \widetilde{F}_2)$$

- Special case: Einstein–Maxwell theory, $F^1 = F^2$, $\varphi = \chi = 0$
- With V: abelian truncation of $\mathcal{N} = 4$, $\mathrm{SO}(4)$-gauged sugra

Charged rotating black holes (Cvetič Youm 96, Lozano-Tellechea Ortín 99):

- Parameters: mass, rotation, NUT, 2 electric + 2 magnetic charges
- Special case: dyonic Kerr–Newman
- AdS generalizations (DC Compère 13, Gnecchi Hristov Klemm Toldo Vaughan 13)

T^2 lift to 6d (more generally for STU supergravity):

$$\mathcal{L}_6 = R \star 1 - \tfrac{1}{2} \star \mathrm{d}\phi \wedge \mathrm{d}\phi - \tfrac{1}{2}\mathrm{e}^{-\sqrt{2}\phi} \star H \wedge H, \qquad H = \mathrm{d}B$$

Example 2: 4-dimensional supergravity

Black hole metric and gauge field strengths:

$$\mathrm{d}s^2 = -e^0 e^0 + \sum_{\mu=1}^{3} e^\mu e^\mu, \quad F^I = \frac{1}{W^2}(F_r^I e^0 \wedge e^1 + F_u^I e^2 \wedge e^3)$$

$$e^0 = \sqrt{\frac{(r_2^2 + u_2^2)R}{W}}(\mathrm{d}\tau + u_1 u_2 \mathrm{d}\psi), \quad e^1 = \sqrt{\frac{r_2^2 + u_2^2}{R}}\mathrm{d}r,$$
$$e^2 = \sqrt{\frac{(r_2^2 + u_2^2)U}{W}}(\mathrm{d}\tau - r_1 r_2 \mathrm{d}\psi_1), \quad e^3 = \sqrt{\frac{r_2^2 + u_2^2}{U}}\mathrm{d}u$$

KYT 2-form, $\star H = \mathrm{e}^{2\varphi}\mathrm{d}\chi$:

$$Y = u_2 e^0 \wedge e^1 + r_2 e^2 \wedge e^3$$

Lift condition: ✓

- Works for AdS generalization in gauged sugra
- KYT 2-form preserved under S^2 lift (trivial direct product)

Example 3: 5-dimensional minimal supergravity

Minimal $\mathcal{N} = 2$ supergravity:

$$\mathcal{L}_5 = R \star 1 - \tfrac{3}{2} \star F \wedge F + F \wedge F \wedge A$$

Charged rotating black holes of STU supergravity (Cvetič Youm 96):

- Parameters: mass, 2 rotations, 3 electric charges
- All 3 charges equal $\longrightarrow$ minimal supergravity

Metric and gauge field strength:

$$\mathrm{d}s^2 = -e^0 e^0 + \sum_{\mu=1}^{4} e^\mu e^\mu, \quad F = \frac{2q}{(r^2+y^2)^2}(re^0 \wedge e^1 + ye^2 \wedge e^3)$$

$$e^0 = \sqrt{\frac{R}{r^2+y^2}}\mathcal{A}, \quad e^1 = \sqrt{\frac{r^2+y^2}{R}}\mathrm{d}r, \quad e^2 = \sqrt{\frac{Y}{r^2+y^2}}(\mathrm{d}\tau - r^2\mathrm{d}\psi_1), \quad e^3 = \sqrt{\frac{r^2+y^2}{Y}}\mathrm{d}y$$

$$e^4 = \frac{ab}{ry}\left(\mathrm{d}\tau + (y^2-r^2)\mathrm{d}\psi_1 - r^2y^2\mathrm{d}\psi_2 + \frac{qy^2\mathcal{A}}{ab(r^2+y^2)}\right)$$

KYT 3-form, $H = \star F$ (Kubizňák Kunduri Yasui 09):

$$Y = (ye^0 \wedge e^1 + re^2 \wedge e^3) \wedge e^4$$

Lift condition: ✓ (also for AdS generalization in gauged sugra)

Some open issues

- Understand "lift condition" $F^b{}_{[a_1} Y_{a_2 \ldots a_p]b} = 0$ — why in different contexts?
- Situations without natural 3-form H:
  - 11d supergravity has 4-form F
  - 5d gauged supergravity lifts on S^5 to type IIB with 5-form F
- More general examples with multiple gauge fields have symmetric KS tensors but not antisymmetric KYT p-forms — symmetry enhancement in higher dimensions?
- Useful for construction of more general solutions, e.g. black holes in AdS$_4$?

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