Chiral Haldane-SPT phases of SU(N) quantum spin chains in the adjoint representation

Thomas Quella
University of Cologne

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In collaboration with:
Kasper Duivenvoorden (arXiv:1206.2462 & 1208.0697)
Abhishekk Roy (arXiv:1512.05229)

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– Introduction –

The Haldane phase of SU(2) quantum spin chains
Haldane’s Conjecture

Setup and large-spin continuum limit

Spin-$s$ Heisenberg spin chain

\[ H = \sum_{\langle kl \rangle} \vec{S}_k \cdot \vec{S}_l \]

$O(3)$ non-linear $\sigma$-model (field $\vec{n} : S^2 \rightarrow S^2$)

\[ S[\vec{n}] = S_{\text{kin}}[\vec{n}] + \theta \cdot \text{Winding}[\vec{n}] \]

$\theta$-term: $\theta = 2\pi s$

Conjecture

The physics depends crucially on the “parity” of the spin $s$

Half-integer: Gapless spin liquid $\rightarrow$ SU(2) WZW model

Integer: Gapped spin liquid $\rightarrow$ “Haldane phase”
Definition of a Haldane-SPT phase

What do I mean by a Haldane-SPT phase?

A 1D quantum spin system with continuous symmetry whose ground state
- is unique (no spontaneous symmetry breaking) → spin liquid
- is gapped
- exhibits symmetry fractionalization (realizes a non-trivial SPT phase)

Warning

The definition of a Haldane phase may vary, in particular with regard to
- the nature of the symmetry (continuous vs. discrete)
- the exclusion of spontaneous symmetry breaking
- Historically, the term simply seems to be associated with the integer spin phase of the SU(2) Heisenberg model
Phases of SU(2) spin models

The bilinear-biquadratic SU(2) spin-1 chain

\[ H = \sum_{\langle kl \rangle} \left[ \cos \theta \vec{S}_k \vec{S}_l + \sin \theta (\vec{S}_k \vec{S}_l)^2 \right] \]

Phase diagram

[Läuchli, Schmid, Trebst]
The Haldane phase of SU(2) spin models

The Haldane phase of spin-1 chains

- Unique ground state $\rightarrow$ SU(2) singlet
- Diluted anti-ferromagnetic order
- Symmetry protected topological phase

\[ H = J \sum_{\langle kl \rangle} \left[ \vec{S}_k \cdot \vec{S}_l + \xi (\vec{S}_k \cdot \vec{S}_l)^2 \right] \]

AKLT ($\xi = \frac{1}{3}$)

Heisenberg

Peculiar property: Emergent massless boundary modes

Open BC

\[ \text{Spin 1/2 (fractionalization)} \]

Periodic BC

Virtual boundary modes

(through bipartite entanglement)

[Affleck, Kennedy, Lieb, Tasaki] [Den Nijs, Rommelse] [Gu, Wen] [Pollmann, Berg, Turner, Oshikawa]
Outline of this talk

**Goal:** Discuss exotic SU(N) spin liquid phases

Why SU(N) systems?
- Open fundamental theoretical issues, e.g. Haldane’s Conjecture
- Large number of exotic phases $\sim N$ (even in 1D)
- Experimental realization, numerical challenge, large-$N$ considerations, ...

More specifically: SU(N) Haldane-SPT phases
- Classification (via AKLT states)
- Construction (of parent Hamiltonians)
- If time permits: Topological phase transitions
– Classification –

The Haldane-SPT phases of SU(N)
Results on anti-ferromagnetic gapped SU(N) spin chains

Anti-ferromagnetic SU(N) spin model in 1D

Spin operators: $\vec{S}_k \in su(N)$

Classification of gapped symmetry protected topological phases

Open BC

The symmetry can fractionalize in up to $N$ topologically distinct ways.*

* Note: Similar results have been derived for all simple Lie groups G
Sketch of the physical situation

Space of gapped PSU(N) spin chains

SSB

Haldane phases

Phase transition (closure of gap)

PSU(N) spin chain

Fractional spins

Phase characterized by topological invariant

\[ [B] \in \mathbb{Z}_N \]

\(* PSU(N) = \text{"variant of SU(N)"} \)

Haldane phases

\[ [0] \]

\[ [1] \]

\[ [2] \]

\[ [3] \]

\[ ... \]

\[ [N-1] \]

\[ ... \]
– Construction –

AKLT states and the design of entanglement properties
The AKLT construction

Basic idea: Realize physical spin $\mathcal{P}$ in terms of auxiliary spins $B^*$ and $B$...

Auxiliary layer: $B^* \otimes B = 0 \oplus \mathcal{P} \oplus \cdots$

Physical layer: Projection to $\mathcal{P}$

Fractionalized boundary spins from valence bonds

[Affleck, Kennedy, Lieb, Tasaki]
The AKLT states

Fractionalized boundary spins from valence bonds

Boundary spin $B^*$

Physical spin $P$

Boundary spin $B$

$\dim(B)^2$ states $|\psi_{\alpha\beta}\rangle$

Question

How to construct a Hamiltonian with $|\psi_{\alpha\beta}\rangle$ as exact ground states?
The AKLT Hamiltonian

Task

- Start with the AKLT states $|\psi_{\alpha\beta}\rangle$
- Construct a Hamiltonian which has $|\psi_{\alpha\beta}\rangle$ as its (unique) ground states

The two-site Hamiltonian as a projector

Auxiliary layer: $\mathcal{B}^* \otimes \mathcal{B} \otimes \mathcal{B}^* \otimes \mathcal{B}$
The AKLT Hamiltonian

**Task**
- Start with the AKLT states $|\psi_{\alpha\beta}\rangle$
- Construct a Hamiltonian which has $|\psi_{\alpha\beta}\rangle$ as its (unique) ground states

**The two-site Hamiltonian as a projector**

- Auxiliary layer: $B^* \otimes B \otimes B^* \otimes B$
- Physical layer: $P \otimes P = B^* \otimes B \oplus \text{others}$

[Affleck, Kennedy, Lieb, Tasaki]
The AKLT Hamiltonian

**Task**

- Start with the AKLT states $|\psi_{\alpha\beta}\rangle$
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**The two-site Hamiltonian as a projector**

- Auxiliary layer: $B^* \otimes B \otimes B^* \otimes B$
- Physical layer: $\mathcal{P} \otimes \mathcal{P} = B^* \otimes B \oplus \text{others}$
- Singlet bond: $B^* \otimes B$
The AKLT Hamiltonian

**Task**
- Start with the AKLT states $|\psi_{\alpha\beta}\rangle$
- Construct a Hamiltonian which has $|\psi_{\alpha\beta}\rangle$ as its (unique) ground states

**The two-site Hamiltonian as a projector**

**Auxiliary layer:**
- $B^* \otimes B \otimes B^* \otimes B$

**Physical layer:**
- $P \otimes P = B^* \otimes B \oplus \text{others}$

**Singlet bond:**
- $H_{2\text{-site}} = \text{“Projector onto } (B^* \otimes B)^\perp\text{”} = \text{Function of } \vec{S}_1 \text{ and } \vec{S}_2$

[Affleck, Kennedy, Lieb, Tasaki]
Ground states of the AKLT Hamiltonian

Periodic or infinite chain: Unique ground state (usually but not guaranteed)

Open chain: \((\text{dim } B)^2\)-fold ground state degeneracy (usually)

Protection by symmetry

The existence of boundary spins is a robust feature as long as

1) they are fractionalized
2) the system remains gapped

[Pollmann,Berg,Turner,Oshikawa] [Chen,Gu,Wen] [Schuch,Perez-Garcia,Cirac]
Symmetry
Fractionalization
Symmetry fractionalization

What is symmetry fractionalization?

Symmetry of emerging boundary spins
different from
Symmetry of physical spins

The case of SO(3)

Two types of su(2) reps

Trivial phase
same rep type

Haldane phase
different rep type

[Chen, Gu, Wen] [Schuch, Perez-Garcia, Cirac]
Symmetry fractionalization in the SU(2) AKLT model

Basic idea

Physical and boundary spin behave differently under the action of the center $\mathbb{Z}_2 = \{\pm I\} \subset SU(2)$

Visualization of the action

Action of $-I$: $-1 -1 -1 -1$
**Symmetry fractionalization in the SU(2) AKLT model**

### Basic idea

Physical and boundary spin behave differently under the action of the center $\mathbb{Z}_2 = \{\pm I\} \subset SU(2)$.

### Visualization of the action

[Diagram showing the action of $-I$ on boundary $S=\frac{1}{2}$ spin and physical $S=1$ spins.]
Symmetry fractionalization in the SU(N) AKLT model

Basic idea

Physical and boundary spin behave differently under the action of the center $\mathbb{Z}_N = \{\omega^I | \omega^N = 1\} \subset SU(N)$

Visualization of the action

Action of $\omega^I$: $\tilde{\omega}^B \omega^B + 1 = +1$
Symmetry fractionalization in the SU(N) AKLT model

Basic idea
Physical and boundary spin behave differently under the action of the center
\( \mathbb{Z}_N = \{ \omega \mathbb{I} | \omega^N = 1 \} \subset SU(N) \)

Visualization of the action

Action of \( \omega \mathbb{I} \):
**Representation types of SU(N) spins**

### Representations of SU(N)

**Young tableau**

\[ \lambda = \begin{array}{c}
\cline{1-3}
& & \\
\cline{1-1}
& & \\
\cline{1-1}
\end{array} \]

\[ \Rightarrow \quad [\lambda] = \text{Boxes}(\lambda) \mod N \]

**Remark:** The center acts by \( \rho_\lambda(\omega I) = \omega^{[\lambda]} \)

---

**Symmetry fractionalization for SU(2) in terms of Young tableaux**

\[ \square \square \cong \text{Spin} 1 \]

\[ \square \quad \bullet \quad \bullet \quad \square \]

\[ \text{Spin 1/2} \quad (\text{fractionalization}) \]
Haldane phases of PSU(N) spin chains

Symmetry fractionalization for PSU(N) = SU(N)/\mathbb{Z}_N

Physical spins $\mathcal{P}$ (with $[\mathcal{P}] = 0$)

Fractional spins, class $[\mathcal{B}]$ $\mathcal{B}^*$ $\mathcal{B}$

Classification

PSU(N) chains admit $N–1$ distinct types of Haldane-SPT phases: $[\mathcal{B}] \in \mathbb{Z}_N \setminus \{0\}$

Topological invariant

The representation type $[\mathcal{B}]$ of the boundary spin $\mathcal{B}$ is a topological invariant. It can be measured using a non-local string order parameter.
A Haldane-SPT phase is called chiral if $B \neq B^*$

**SU(N) examples**

- A necessary condition for $B = B^*$ is $[B] \in \{[0], [\frac{N}{2}]\}$
- The Haldane-SPT phases of SU(2) chains are all non-chiral

<table>
<thead>
<tr>
<th>SU(N)</th>
<th>$N$ even</th>
<th>$N$ odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chiral</td>
<td>$[B] \notin {[0], [\frac{N}{2}]}$</td>
<td>$[B] \neq [0]$</td>
</tr>
<tr>
<td>Non-chiral</td>
<td>$[B] = [\frac{N}{2}]$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>
Combination with spontaneous symmetry breaking

Realizations of chiral Haldane-SPT phases require the use of chiral Hamiltonians

Previous considerations in the literature

“Naive” (non-chiral) AKLT Hamiltonians lead to a two-fold degenerate ground state

[Affleck, Arovas, Marston, Rabson] [Greiter, Rachel] [Rachel, Schuricht, Scharfenberger, Thomale, Greiter] [Morimoto, Ueda, Momi, Furusaki]
Examples
Illustration for PSU(3) (adjoint representation)

Sketch of possible edge modes

**Trivial phase**

- Class [0]

**Chiral Haldane-SPT phases**

- Class [1]
- Class [2]
Illustration for PSU(4) (adjoint representation)

Sketch of possible edge modes

- **Trivial phase**
  - Class [0]
  - Class [2]

- **Chiral Haldane-SPT phases**
  - Class [1]
  - Class [3]

[Roy,TQ]
Illustration for PSU(4) (self-dual representation)

Sketch of possible edge modes

[Nonne, Molinet, Capponi, Lecheminant, Totsuka]

Trivial phase

Class [0]

Non-chiral Haldane-SPT phase

Class [2]
Technical details
Realization of non-trivial topological phases (arbitrary $N$)

Potential symmetry fractionalizations

(I) \[ \left\{ \begin{array}{c} \square \\ \bullet \end{array} \right\} ^{N-1} \]

Fractional spins

Dim of physical spins

Dim of boundary spins (I)

Dim of boundary spins (II)

Symmetry group | SU(2) | SU(3) | SU(4) | SU(6) | SU(8) | SU(N) |
---|---|---|---|---|---|---|
Dim of physical spins | 3 | 8 | 15 | 35 | 63 | $N^2 - 1$ |
Dim of boundary spins (I) | 2 | 3 | 4 | 6 | 8 | $N$ |
Dim of boundary spins (II) | $\emptyset$ | 3 | 6 | 15 | 28 | $\frac{1}{2}N(N - 1)$ |
Sketch of the AKLT construction

For SU(4):

\[
\begin{array}{c}
\begin{array}{c}
\text{Adjoint rep}
\end{array}
\end{array}
\]

(I)

For SU(4):

\[
\begin{array}{c}
\begin{array}{c}
\text{Adjoint rep}
\end{array}
\end{array}
\]

(II)

Two-site Hilbert space and Hamiltonian

\[
\begin{array}{c}
\begin{array}{c}
\text{unwanted}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{project onto this}
\end{array}
\end{array}
\]

Auxiliary layer

<table>
<thead>
<tr>
<th>Case (I)</th>
<th>Case (II)</th>
</tr>
</thead>
</table>
| \[
\begin{array}{c}
\begin{array}{c}
\text{Adjoint rep}
\end{array}
\end{array}
\]
| \[
\begin{array}{c}
\begin{array}{c}
\text{Adjoint rep}
\end{array}
\end{array}
\] |
How to construct the projectors?

Introducing birdtracks

**Fundamental rep**

\[(T^a)_{\alpha\beta} = \begin{array}{c}
\alpha \\
\beta
\end{array} \]

\[\begin{array}{c}
\beta \\
\alpha
\end{array} = \begin{array}{c}
\alpha \\
\beta
\end{array} = - S^a = \begin{array}{c}
\alpha \\
\alpha
\end{array} = - \begin{array}{c}
\alpha \\
\alpha
\end{array}
\]

**Adjoint rep**

Graphical representation of structure constants

\[f^{ab}_c T^c = \begin{array}{c}
\bullet \\
\bullet
\end{array} = \begin{array}{c}
\bullet \\
\bullet
\end{array} = - \begin{array}{c}
\bullet \\
\bullet
\end{array}
\]

\[d^{ab}_c T^c = \begin{array}{c}
\bullet \\
\bullet
\end{array} = \begin{array}{c}
\bullet \\
\bullet
\end{array} + \begin{array}{c}
\bullet \\
\bullet
\end{array}
\]
How to construct the projectors?

Graphical representation of some properties

**Tracelessness**

\[ = 0 \]

**Normalization**

\[ = \]

**Decomposition of unity**

\[ = \frac{1}{N} \]

**Commutation relations**

\[- = ]
How to construct the projectors?

Graphical representation of 2-site spin interactions

\[-S_1 \cdot S_2 = \]

\[d_{abc} S_1^a S_1^b S_2^c = \]

\[d_{abc} S_1^a S_2^b S_2^c = \]

\[d_{abc} d_{efg} S_1^a S_1^f S_2^g S_2^e S_2^b S_2^c = \]

Basis of invariant operators
Examples of projectors

\[ P_\bullet = \frac{1}{N^2 - 1} \]

\[ P_A = \frac{1}{2N} \]

\[ P_S = \frac{N}{2(N^2 - 4)} \]

\[ P_{A_1} = \frac{1}{2} \left\{ \begin{array}{c} \begin{array}{c} \text{Diagram 1} \end{array} \\
+ \frac{1}{2N} \\
+ \begin{array}{c} \text{Diagram 2} \end{array} \end{array} \right\} \]

\[ P_{S_1} = \frac{1}{2} \left\{ \begin{array}{c} \begin{array}{c} \text{Diagram 3} \end{array} \\
+ \frac{1}{2(N-2)} \\
- \begin{array}{c} \text{Diagram 4} \end{array} \\
- \frac{1}{N(N-1)} \end{array} \right\} \]
A universal AKLT Hamiltonian

**Hamiltonian on the auxiliary layer**

\[ h_{aux} = I - \frac{1}{\dim(B)} \]

**Projection to the physical level**

\[ \tilde{H}_{2\text{-site}} = I - \frac{1}{\dim(B)} \]

\[ \Rightarrow \quad \mathbb{P}\text{phys} h_{aux} \mathbb{P}\text{phys} = I_{\text{phys}} - \]

\[ \Rightarrow \quad \text{generally not equal weight superposition of projectors (but can be adjusted)} \]
The AKLT Hamiltonians for PSU(4)

For SU(4): 

Case (I): 
\[ H_{\text{2-site}} = \mathbb{I} - \frac{1}{56} C_A - \frac{1}{896} K + \frac{13}{210} \vec{S}_1 \cdot \vec{S}_2 - \frac{17}{840} (\vec{S}_1 \cdot \vec{S}_2)^2 \]
\[ + \frac{1}{420} (\vec{S}_1 \cdot \vec{S}_2)^3 + \frac{1}{1680} (\vec{S}_1 \cdot \vec{S}_2)^4 \]

Case (II): 
\[ H_{\text{2-site}} = \mathbb{I} - \frac{1}{128} K + \frac{31}{20} \vec{S}_1 \cdot \vec{S}_2 - \frac{7}{40} (\vec{S}_1 \cdot \vec{S}_2)^2 \]
\[ - \frac{1}{5} (\vec{S}_1 \cdot \vec{S}_2)^3 - \frac{3}{160} (\vec{S}_1 \cdot \vec{S}_2)^4 \]

AKLT Hamiltonians for PSU(4)
For SU(4): \( \square \) (I) \( \square \) (II)

The AKLT Hamiltonians feature higher-order Casimir operators, e.g. terms like

\[
C_A = d_{abc} (S_1^a S_1^b S_2^c - S_1^a S_2^b S_2^c)
\]
or

\[
K = d_{abc} d_{def} S_1^a S_1^b S_1^d S_2^c S_2^e S_2^f
\]

where \( d_{abc} \) is the completely symmetric rank-3 tensor
Transfer matrix and correlation lengths

Step 1: Write the transfer matrix as a sum over projectors

\[
\begin{align*}
&= c_1 \frac{1}{N} - c_2 \\
&= c_1 \frac{2}{N(N-1)} + c_2 \frac{1}{N-2} + c_3 \alpha(N)
\end{align*}
\]

Step 2: The largest two eigenvalues determine the correlation length

\[
\xi_\square = \frac{1}{\ln(N^2 - 1)} \quad \xi_\square = \frac{1}{\ln\left[\frac{N^2 - 2N - 4}{2(N+1)(N-2)}\right]}
\]
### Open issue

Which phase is realized in the SU(N) Heisenberg model?

### Conjecture for the case SU(4)

The Heisenberg model realizes the class [2] Haldane-SPT phase

### Expectation for SU(odd)

The Heisenberg model realizes a superposition of two Haldane-SPT phases

### Conjecture on topological phase transitions

2nd order topological phase transitions are generically described by SU(N)\(_1\) for odd \(N\) and by SU(N)\(_2\) for even \(N\) (absence of a \(\mathbb{Z}_N\)-anomaly). Fine-tuned transitions may lead to larger values of the level
Other SU(N) setups
Realization of non-trivial topological phases (even $N$)

Potential symmetry fractionalization

Physical spin: $\frac{N}{2}$

Fractional spins

Topological class $\left[ \frac{N}{2} \right]$ 

Conjecture (so far only verified for $N = 4$)

These non-trivial topological phases are realized in the Heisenberg model

<table>
<thead>
<tr>
<th>Symmetry group</th>
<th>SU(2)</th>
<th>SU(4)</th>
<th>SU(6)</th>
<th>SU(8)</th>
<th>SU(10)</th>
<th>SU(N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dim of physical spins</td>
<td>3</td>
<td>20</td>
<td>175</td>
<td>1764</td>
<td>19404</td>
<td>$\frac{n!(n+1)!}{[(n/2)!(n/2+1)!]^2}$</td>
</tr>
<tr>
<td>Dim of boundary spins</td>
<td>2</td>
<td>6</td>
<td>20</td>
<td>70</td>
<td>252</td>
<td>$\frac{n!}{(n/2)!^2}$</td>
</tr>
</tbody>
</table>
Corresponding AKLT Hamiltonians

**SU(2) “AKLT” Hamiltonian**

$$H_{2\text{-site}} = \frac{2}{3} + \vec{S}_1 \cdot \vec{S}_2 + \frac{1}{3} (\vec{S}_1 \cdot \vec{S}_2)^2$$

**SU(4) “AKLT” Hamiltonian**

$$H_{2\text{-site}} = \frac{8}{3} + \vec{S}_1 \cdot \vec{S}_2 + \frac{13}{108} (\vec{S}_1 \cdot \vec{S}_2)^2 + \frac{1}{216} (\vec{S}_1 \cdot \vec{S}_2)^3$$

**SU(6) “AKLT” Hamiltonian**

$$H_{2\text{-site}} = \frac{504}{127} + \vec{S}_1 \cdot \vec{S}_2 + \frac{47}{508} (\vec{S}_1 \cdot \vec{S}_2)^2 + \frac{17}{4572} (\vec{S}_1 \cdot \vec{S}_2)^3 + \frac{1}{18288} (\vec{S}_1 \cdot \vec{S}_2)^4$$

**SU(8) and above**

Hamiltonians involve operators beyond powers of $\vec{S}_1 \cdot \vec{S}_2$
Conclusions
Conclusions

Summary

SU(N) spin chains exhibit various types of Haldane-SPT phases, most of them chiral. The construction of parent Hamiltonians for the adjoint representation is by no means straightforward but can be achieved for general $N$ using birdtracks.

Features

- They can exhibit different types of protected gapless edge modes
- For SU(4) we have a complete realization of Haldane-SPT phases

Related topics not covered in this talk

- Non-local string order, hidden symmetry breaking, etc.
- Multifractality of topological phase transitions
- Realization in alkaline-earth Fermi gases (Talk by Lecheminant)