Homogeneous Boltzmann equation for bosons particles. Isotropic solutions

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Boltzmann equation

Distribution function at time $t \geq 0$, position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$: $f(t,x,v)$.

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f).$$

First simplification: Spatially homogeneous: $f(t,x,v) = f(t,v)$.

$$\frac{\partial f}{\partial t} = Q(f).$$

Number of particles (N), Energy (E) and Momentum (P):

$$N = \int_{\mathbb{R}^3} f(t,v)\, dv, \quad E = \int_{\mathbb{R}^3} f(t,v)|v|^2\, dv, \quad P = \int_{\mathbb{R}^3} f(t,v) v\, dv$$

The collision operator

$$Q(f)(t,v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v*, \sigma)(f' f'_* - f f_*) \, d\sigma \, dv_*.$$

$f' = f(v'), \quad f'_* = f(v'_*), \quad f = f(v), \quad f_* = f(v_*).$

$B(v - v_*, \sigma)$: information about how particles collide. There are many models.

Collision conserve momentum and kinetic energy:

$$v' + v'_* = v + v_*, \quad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2$$

Classical framework: $f' f'_* - f f_*$

Quantum framework:

$$f' f'_* (1 + \tau f)(1 + \tau f_*) - f f_* (1 + \tau f')(1 + \tau f'_*).$$

$\tau = 0$: classical particles. $\tau = -1$: fermions particles. $\tau = 1$: bosons particles

Second simplification: Radially symmetric: $f(t, v) = f(t, |v|^2)$. Dependence on the energy $|v|^2$ instead on the velocity v.

[1] **X. Lu**, *On Isotropic Distributional Solutions to the Boltzmann Equation for Bose-Einstein Particles* , J. Statist. Phys., vol. **116**, **(2004)**, 1597-1649.

Existence of solution is probed in the sense of positive Borel measures.

[2] **D.V. Semikov, I.I. Tkachev**, *Condensation of Bosons in the kinetic regime*, Phys. Rev. D 55, 2, **(1997)** 489-502.

The distribution function is splited in two parts:

$$f(t, |v|^2) = g(t, |v|^2) + c(t)\delta_0(v).$$

g is the gas, and $c(t)$ is the condensate. A coupled system of two equations is considered.

Existence of solution for a simplified version of the coupled system obtained in [2]

Distribution function at time $t \geq 0$ and energy $x \geq 0$: $g(t, x)$.

The condensate at time $t \geq 0$:

$$c(t) = c_0 e^{-\int_0^\infty g(t,x)\sqrt{x}\,dx}, \qquad c_0 > 0.$$

The gas $g(t, x)$:

$$\begin{aligned}\frac{\partial g}{\partial t}(t,x) = c(t)\Bigg[&\int_0^x \frac{g(t,x-y)}{\sqrt{x-y}}\frac{g(t,y)}{\sqrt{y}}\,dy \\ &+ 2\int_x^\infty \frac{g(t,y)}{\sqrt{y}}\frac{g(t,y-x)}{\sqrt{y-x}}\,dy \\ &+ 2\int_x^\infty \frac{g(t,y)}{\sqrt{y}}\,dy \\ &- g(t,x)\left(\sqrt{x} + \frac{4}{\sqrt{x}}\int_0^x \frac{g(t,y)}{\sqrt{y}}\,dy\right)\Bigg].\end{aligned}$$

Differential form:

$$\frac{\partial g}{\partial t} = c(t)\Big[Q(g,g) + L(g) - g(t,x)A(g)\Big]. \qquad (1)$$

Integral form:

$$g(t,x) = g_0(x)e^{-\int_0^t A(g)\,d\sigma} + \int_0^t c(s)e^{-\int_s^t A(g)\,d\sigma}\Big[Q(g,g) + L(g)\Big]\,ds.$$

Number of particles and Energy:

$$N(t) = \int_0^\infty g(t,x)\,dx, \qquad E(t) = \int_0^\infty g(t,x)x\,dx.$$

Considering smooth approximations of $\frac{1}{\sqrt{x}}$ we have the following:

Lemma 1

For every $T > 0$ and every positive initial datum $g_0 \in L^\infty(0,\infty) \cap L^1([0,\infty), x dx)$, equation (1) has a unique positive solution $g \in C([0,T], L^\infty(0,\infty) \cap L^1([0,\infty), x dx))$.

Moreover, the number of particles is nondecreasing and the energy is conserved: $\forall t \geq 0$

$$\int_0^\infty g(t,x)\, dx \leq \int_0^\infty g_0(x)\, dx + c_0,$$

$$\int_0^\infty g(t,x) x\, dx = \int_0^\infty g_0(x) x\, dx.$$

Ingredients of the proof: Banach contraction principle (local solution)+ a priori estimates (using the weak formulation)

Weak formulation

Multiply the equation by a test function $\varphi(x)$ and integrate:

$$\int_0^\infty (g(t,x) - g(s,x))\varphi(x)\,dx$$
$$= 2\int_s^t c(\sigma)\int_0^\infty g(\sigma,x)\phi_n(x)\int_0^x g(\sigma,y)\widetilde{\varphi}(x,y)\,dy\,dx\,d\sigma$$
$$+ \int_s^t c(\sigma)\int_0^\infty g(\sigma,x)\phi_n(x)\overline{\varphi}(x)\,dx\,d\sigma.$$

where

$$\widetilde{\varphi}(x,y) = \varphi(x+y) + \varphi(x-y) - 2\varphi(x),$$
$$\overline{\varphi}(x) = 2\int_0^x \varphi(y)\,dy - \varphi(x)x.$$

$\varphi(x) = 1$ and $\varphi(x) = x \quad \Rightarrow$ conservations.

Moment's estimates

The k-moment of g at time $t \geq 0$ is defined as

$$M_k(g)(t) = \int_0^\infty g(t,x)x^k\,dx.$$

Lemma 2

Let $k \geq 0$. If the initial data g_0 satisfies

$$\int_0^\infty g_0(x)x^k\,dx < +\infty,$$

then the solution satisfies

$$\int_0^\infty g(t,x)x^k\,dx \leq C \quad \forall t \geq 0.$$

Ingredients of the proof: bounded cut offs of x^k + weak formulation + limit process.

On going work

Limit process: To prove an existence result for

$$\frac{\partial g}{\partial t} = c(t)\Big[Q(g,g) + L(g) - g(t,x)A(g)\Big] \quad \text{whithout cut off of } \frac{1}{\sqrt{x}}$$

using the solutions of the simplified equation obtained by lemma 1, and the uniformly bounds of lemma 2.

Thank you for your attention