The Obstacle Problem for the Total Variation

Thomas Schmidt
Department Mathematik, FAU Erlangen-Nürnberg

(joint work with Christoph Scheven, Universität Duisburg-Essen)

Benasque, August 31, 2015
The TV obstacle problem

Consider

- a bounded open set Ω in \( \mathbb{R}^n \), \( n \) positive integer,
- an obstacle \( \psi : \overline{\Omega} \to \mathbb{R} \) with \( \psi \leq 0 \) on \( \partial \Omega \).

**Obstacle problem:** Minimize the total variation (TV)

\[
\int_{\Omega} |\nabla u| \, dx
\]

among functions \( u : \overline{\Omega} \to \mathbb{R} \) with

\[
u \equiv 0 \text{ on } \partial \Omega \quad \text{and} \quad u \geq \psi \text{ on } \Omega.\]
The generalized TV obstacle problem

Natural space for existence results (thanks to weak* compactness):

\[ \text{BV}_0(\Omega) := \left\{ u \in L^1(\mathbb{R}^n) : \text{gradient } D u \text{ is finite measure on } \mathbb{R}^n \right. \]
\[ \quad \left. \text{and } u \equiv 0 \text{ a.e. on } \mathbb{R}^n \setminus \Omega \right\} \]

(contains \( W^{1,1}_{(0)}(\Omega) \), but also \( u \) with jumps along hypersurfaces in \( \overline{\Omega} \)).
The generalized TV obstacle problem

Natural space for existence results (thanks to weak* compactness):

$$BV_0(\Omega) := \left\{ u \in L^1(\mathbb{R}^n) : \text{gradient } Du \text{ is finite measure on } \mathbb{R}^n \right\}$$

and $$u \equiv 0 \text{ a.e. on } \mathbb{R}^n \setminus \Omega$$

(contains $$W_{1,1}^{1,1}(\Omega)$$, but also $$u$$ with jumps along hypersurfaces in $$\Omega$$).

**Generalized obstacle problem:** Minimize the total mass of $$Du$$

$$|Du|(\Omega) = |Du|(\Omega) + |Du|(\partial \Omega) \approx ||\text{int trace}(u)||_{L^1(\partial \Omega)}$$

among

$$u \in BV_0(\Omega) \quad \text{with } u \geq \psi \text{ a.e. on } \Omega.$$
Existence and duality for $W^{1,1}$ obstacles

Basic results:

- The generalized TV obstacle problem has a minimizer.
Existence and duality for $W^{1,1}$ obstacles

Basic results:

- The generalized TV obstacle problem has a minimizer.

- For $\partial \Omega$ Lipschitz, $\psi \in W^{1,1}_0(\Omega)$, one has the duality formula

\[
\min \{ |Du|(\Omega) : u \in BV_0(\Omega), u \geq \psi \text{ a.e. on } \Omega \}
= \max \left\{ \int_\Omega \sigma \cdot \nabla \psi \, dx : \sigma \in S^\infty(\Omega, \mathbb{R}^n), \text{div} \sigma \leq 0 \text{ in } \mathcal{D}'(\Omega) \right\}.
\]

\text{sub-unit vector fields}
Existence and duality for $W^{1,1}$ obstacles

Basic results:

- The generalized TV obstacle problem has a minimizer.
- For $\partial \Omega$ Lipschitz, $\psi \in W^{1,1}_{0}(\Omega)$, one has the duality formula

$$
\min\{|D\!u|(\bar{\Omega}) : u \in BV_{0}(\bar{\Omega}), u \geq \psi \text{ a.e. on } \Omega\} = \max \left\{ \int_{\Omega} \sigma \cdot \nabla \psi \, dx : \sigma \in S^{\infty}(\Omega, \mathbb{R}^{n}), \text{div} \sigma \leq 0 \text{ in } D'(\Omega) \right\}.
$$

$\leadsto$ to say more, need products $\sigma \cdot D\!\psi$ and $\sigma \cdot D\!u$ if merely $\psi, u \in BV$ (e.g. if $\psi$ is a characteristic function).
The Anzellotti pairing

Consider:

- \( u \in BV_{loc}(\Omega) \),
- a vector field \( \sigma \in L^\infty_{loc}(\Omega, \mathbb{R}^n) \) (w.r.t. Lebesgue measure \( dx \)).

Can one define a product \( [\sigma, Du] \)?
The Anzellotti pairing

Consider:
- \( u \in \text{BV}_{\text{loc}}(\Omega) \),
- a vector field \( \sigma \in L^\infty_{\text{loc}}(\Omega, \mathbb{R}^n) \) (w.r.t. Lebesgue measure \( dx \)).

Can one define a product \( \llbracket \sigma, Du \rrbracket \)? If \( \text{div} \sigma \) is suitably good, yes:

**Definition (Kohn & Temam '82/’83, Anzellotti ‘83, . . .)**

*For \( u, \sigma \) as above, the distribution*

\[
\llbracket \sigma, Du \rrbracket := \text{div}(\sigma u) - u \text{div} \sigma \in \mathcal{D}'(\Omega).
\]

*makes sense (and behaves reasonably) if . . .*
- . . . either \( u \in L^\infty_{\text{loc}}(\Omega) \), \( \text{div} \sigma \in L^1_{\text{loc}}(\Omega) \)
- . . . or \( \text{div} \sigma \in L^n_{\text{loc}}(\Omega) \) (*then uses Sobolev’s embedding*).
A pairing for divergence-measure fields

But even if $\text{div} \, \sigma \notin L^1_{\text{loc}}(\Omega)$, we still have:

**Definition (a new Anzellotti type pairing, Scheven & S.)**

For $u \in BV_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ and $\sigma \in L^\infty_{\text{loc}}(\Omega, \mathbb{R}^n)$ such that $\text{div} \, \sigma$ is Radon measure (in particular if $\text{div} \, \sigma \leq 0$ in $\mathcal{D}'(\Omega)$), we define

$$[\sigma, Du^+] := \text{div}(\sigma u) - u^+ \text{div} \, \sigma \in \mathcal{D}'(\Omega).$$
A pairing for divergence-measure fields

But even if $\text{div} \, \sigma \notin L^1_{\text{loc}}(\Omega)$, we still have:

**Definition (a new Anzellotti type pairing, Scheven & S.)**

For $u \in \text{BV}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ and $\sigma \in L^\infty_{\text{loc}}(\Omega, \mathbb{R}^n)$ such that $\text{div} \, \sigma$ is Radon measure (in particular if $\text{div} \, \sigma \leq 0$ in $\mathcal{D}'(\Omega)$), we define

$$[\sigma, D u^+] := \text{div}(\sigma u) - u^+ \text{div} \, \sigma \in \mathcal{D}'(\Omega).$$

- makes sense because
  - $\text{div} \, \sigma$ vanishes on $\mathcal{H}^{n-1}$-negligible sets (Chen & Frid ’99),
  - $u$ has $\mathcal{H}^{n-1}$-a.e. defined representatives $u^\pm$ s.t., for $\mathcal{H}^{n-1}$-a.e. $x$,
    - either $u^+(x) = u^-(x)$ is the Lebesgue value of $u$ at $x$
    - or $u^-(x) < u^+(x)$ are the approximate jump values of $u$ at $x$.
- pairing $[\sigma, D u^*]$ with representative $u^* := \frac{u^+ + u^-}{2}$ already used by Mercaldo & Segura de León & Trombetti ‘09.
Properties of the pairing

Theorem (properties of $[\sigma, Du^+]$, Scheven & S.)

For $u \in BV_{\text{loc}}(\Omega)$ and $\sigma \in L^\infty(\Omega, \mathbb{R}^n)$ with $\text{div} \, \sigma \leq 0$ in $\mathcal{D}'(\Omega)$,

- $[\sigma, Du^+]$ is a Radon measure with product estimate
  \[ ||[\sigma, Du^+]| \leq ||\sigma||_{L^\infty(\Omega, \mathbb{R}^n)}|Du| \quad \text{on } \Omega, \]

- and its absolutely continuous part is the pointwise product, i.e.
  \[ [\sigma, Du^+]^a = (\sigma \cdot \nabla^a u)dx \quad \text{on } \Omega. \]

- In particular, $[\sigma, Du^+] = (\sigma \cdot \nabla u)dx$ trivializes for $u \in W^{1,1}_{\text{loc}}(\Omega)$. 
Properties of the pairing

**Theorem (properties of $[\sigma, Du^+]$, Scheven & S.)**

For $u \in BV_{loc}(\Omega)$ and $\sigma \in L^\infty(\Omega, \mathbb{R}^n)$ with $\text{div } \sigma \leq 0$ in $\mathcal{D}'(\Omega)$,

1. $[\sigma, Du^+]$ is a Radon measure with product estimate
   \[ |[\sigma, Du^+]| \leq ||\sigma||_{L^\infty(\Omega, \mathbb{R}^n)} |Du| \quad \text{on } \Omega, \]
2. and its absolutely continuous part is the pointwise product, i.e.
   \[ [\sigma, Du^+]^a = (\sigma \cdot \nabla^a u)dx \quad \text{on } \Omega. \]

- In particular, $[\sigma, Du^+] = (\sigma \cdot \nabla u)dx$ trivializes for $u \in W^{1,1}_{loc}(\Omega)$.
- Proofs based on fine (semi)continuity and capacity methods (e.g., since $u^+$ is not the limit of standard mollifications, need one-sided approximations of Carriero-Dal Maso-Leaci-Pascali ‘88).
Properties of the pairing

Theorem (properties of $[\sigma, Du^+]$, Scheven & S.)

For $u \in BV_{\text{loc}}(\Omega)$ and $\sigma \in L^\infty(\Omega, \mathbb{R}^n)$ with $\text{div} \sigma \leq 0$ in $\mathcal{D}'(\Omega)$,

- $[\sigma, Du^+]$ is a Radon measure with product estimate
  $$|[[\sigma, Du^+]]| \leq \|\sigma\|_{L^\infty(\Omega, \mathbb{R}^n)}|Du| \quad \text{on } \Omega,$$

- and its absolutely continuous part is the pointwise product, i.e.
  $$[\sigma, Du^+]^a = (\sigma \cdot \nabla u^a)dx \quad \text{on } \Omega.$$

- In particular, $[\sigma, Du^+] = (\sigma \cdot \nabla u)dx$ trivializes for $u \in W^{1,1}_{\text{loc}}(\Omega)$.

- Proofs based on fine (semi)continuity and capacity methods (e.g., since $u^+$ is not the limit of standard mollifications, need one-sided approximations of Carrero-Dal Maso-Leaci-Pascali ’88).

- Up-to-the-boundary pairing $[\sigma, Du^+]_0$ on $\overline{\Omega}$ accounts for zero Dirichlet datum (on mildly regular $\partial \Omega$; cf. S. ‘15, Beck & S. ‘15).
Duality for BV obstacles

Theorem (duality for the TV obstacle problem, Scheven & S.)

For mildly regular $\partial \Omega$, $\psi \in BV_0(\overline{\Omega}) \cap L^\infty(\Omega)$ with $|D\psi|(\partial \Omega) = 0$:

$$\min \{ |Du|(\overline{\Omega}) : u \in BV_0(\overline{\Omega}), u \geq \psi \text{ a.e. on } \Omega \}$$

$$= \max \{ [\sigma, D\psi^+](\Omega) : \sigma \in S^\infty(\Omega, \mathbb{R}^n), \text{div } \sigma \leq 0 \text{ in } \mathcal{D}'(\Omega) \}.$$
Duality for BV obstacles

**Theorem (duality for the TV obstacle problem, Scheven & S.)**

For mildly regular $\partial \Omega$, $\psi \in BV_0(\Omega) \cap L^\infty(\Omega)$ with $|D\psi|(\partial \Omega) = 0$:

$$\min \{ |D u|(\Omega) : u \in BV_0(\Omega), u \geq \psi \text{ a.e. on } \Omega \}$$

$$= \max \{ [\sigma, D\psi^+](\Omega) : \sigma \in S^\infty(\Omega, \mathbb{R}^n), \text{div } \sigma \leq 0 \text{ in } D'(\Omega) \}.$$

Two methods of proof (both rely on the properties of the pairing):

- Either look at obstacle problems for the $p$-Laplace in $W^{1,p}_0$ and pass $p \downarrow 1$ (this way, if $\psi \in W^{1,1+\varepsilon}_0$, also get a convergence result for minimizers when $p \downarrow 1$),
Theorem (duality for the TV obstacle problem, Scheven & S.)

For mildly regular $\partial \Omega$, $\psi \in BV_0(\Omega) \cap L^\infty(\Omega)$ with $|D\psi|(\partial \Omega) = 0$:

$$\min \{|Du|(\Omega) : u \in BV_0(\Omega), u \geq \psi \text{ a.e. on } \Omega\} = \max \{\langle [\sigma, D\psi^+] \rangle(\Omega) : \sigma \in S^\infty(\Omega, \mathbb{R}^n), \operatorname{div} \sigma \leq 0 \text{ in } D'(\Omega)\}. $$

Two methods of proof (both rely on the properties of the pairing):

- Either look at obstacle problems for the $p$-Laplace in $W^{1,p}_0$ and pass $p \downarrow 1$ (this way, if $\psi \in W^{1,1+\varepsilon}_0$, also get a convergence result for minimizers when $p \downarrow 1$),
- or deduce it from (abstract) convex duality.
Heuristically, minimizers $u$ should satisfy
\[ \text{div} \frac{\nabla u}{|\nabla u|} \leq 0, \]
and we can now make this precise:

**Corollary (optimality conditions for the TV obstacle problem)**

*Every minimizer $u \in BV_0(\Omega)$ is super-1-harmonic on $\Omega$ in the sense that there exists some $\sigma \in S^\infty(\Omega, \mathbb{R}^n)$ with*

\[
[\sigma, Du^+]_0 = |Du| \text{ on } \overline{\Omega} \quad \text{and} \quad \text{div } \sigma \leq 0 \text{ in } \mathcal{D}'(\Omega).
\]

*BV-way of saying $\sigma = \frac{\nabla u}{|\nabla u|}$*
BV optimality conditions

Heuristically, minimizers \( u \) should satisfy

\[
\text{div} \frac{\nabla u}{|\nabla u|} \leq 0,
\]

and we can now make this precise:

**Corollary (optimality conditions for the TV obstacle problem)**

*Every minimizer \( u \in BV_0(\Omega) \) is super-1-harmonic on \( \Omega \) in the sense that there exists some \( \sigma \in S^\infty(\Omega, \mathbb{R}^n) \) with*

\[
\left\{ \begin{array}{c}
\left[ \sigma, Du^+ \right]_0 = \left| Du \right| \text{ on } \overline{\Omega} \\
\text{BV-way of saying } \sigma = \frac{\nabla u}{|\nabla u|}
\end{array} \right.
\]

*and \( \text{div } \sigma \leq 0 \text{ in } \mathcal{D}'(\Omega) \).*

*Moreover, \( u \) is 1-harmonic away from the obstacle in the sense of*

\[
\text{div } \sigma \equiv 0 \text{ on } \Omega \cap \{ u^+ > \psi^+ \}.
\]
Extensions

We can also treat . . .

- (much) more general obstacles:
  - thin and, most generally, quasi upper semicontinuous obstacles (then need additional tools: relaxation, De Giorgi’s measure, . . . ),
  - obstacles which are positive up to \( \partial \Omega \) (then need modified pairing),
Extensions

We can also treat . . .

- (much) more general obstacles:
  - thin and, most generally, quasi upper semicontinuous obstacles (then need additional tools: relaxation, De Giorgi’s measure, . . .),
  - obstacles which are positive up to \(\partial \Omega\) (then need modified pairing),

- the non-parametric area \(\int_{\Omega} \sqrt{1+|\nabla u|^2} \, dx\) and similar functionals,
Extensions

We can also treat ...

- (much) more general obstacles:
  - thin and, most generally, quasi upper semicontinuous obstacles (then need additional tools: relaxation, De Giorgi’s measure, ...),
  - obstacles which are positive up to $\partial \Omega$ (then need modified pairing),

- the non-parametric area $\int_\Omega \sqrt{1+|\nabla u|^2} \, dx$ and similar functionals,

- general boundary values.
Related topics

Related work in progress concerns . . .

- **BV supersolutions** to 1-Laplace and minimal surface equations, in particular:
  - compactness results,
  - the question if simultaneous super- and sub-solutions are solutions (for the 1-Laplace surprisingly non-trivial, since $\sigma$ is not unique $\rightsquigarrow$ duality argument of possible interest; cf. Yan ‘11),
Related topics

Related work in progress concerns . . .

- **BV supersolutions** to 1-Laplace and minimal surface equations, in particular:
  - compactness results,
  - the question if simultaneous super- and sub-solutions are solutions (for the 1-Laplace surprisingly non-trivial, since $\sigma$ is not unique $\leadsto$ duality argument of possible interest; cf. Yan ‘11),

- variational **existence results for measure data problems** to the 1-Laplace equation and the prescribed mean curvature equation (parametric or non-parametric; in the last case yields an alternative to the approach of Dai & Trudinger & Wang ‘12).