Observation from measurable sets for analytic parabolic equations

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A joint work with
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Interior null-controllability of the Heat equation

Theorem (Lebeau-Robbiano, Imanuvilov)

Let $\Omega \subseteq \mathbb{R}^n$ an open bounded set with sufficiently smooth boundary, $T > 0$ and $\omega \subseteq \Omega$ be an open set, then there is a constant $N = N(\omega, \Omega, T)$ s.t. for each $u_0 \in L^2(\Omega)$ exists $f \in L^2(\Omega \times (0, T))$ s.t.

$$\|f\|_{L^2(\Omega\times(0,T))} \leq N\|u_0\|_{L^2(\Omega)}$$

and the solution to

$$\begin{cases} \partial_t u - \Delta u = \chi_\omega f, & \text{in } \Omega \times (0, T], \\ u = 0, & \text{on } \partial\Omega \times (0, T], \\ u(0) = u_0. & \text{in } \Omega, \end{cases}$$

satisfies $u(T) \equiv 0$.

Interior observability inequality over open sets

The interior null-controllability property for the Heat equation is equivalent to the *interior observability*, i.e., there exists a constant $N = N(\omega, \Omega, T)$ s.t. the solution to

$$\begin{cases} \partial_t v - \Delta v = 0, & \text{in } \Omega \times (0, T], \\ v = 0, & \text{on } \partial\Omega \times (0, T], \\ v(0) = v_0. & \text{in } \Omega, \end{cases}$$

satisfies the *observability inequality*

$$\|v(T)\|_{L^2(\Omega)} \leq N \|v\|_{L^2(\omega \times (0,T))}.$$

An interior observability inequality over measurable sets

Theorem (J. Apraiz, L. Escauriaza, G. Wang, C. Zhang, 2014)

Let $0 < T < 1$, $\mathcal{D} \subset \Omega \times (0, T)$ ($\partial\Omega$ Lipschitz) be a measurable set, $|\mathcal{D}| > 0$. Then $\exists\ N = N(\mathcal{D}, \Omega, T)$ s.t.

$$\|u(T)\|_{L^2(\Omega)} \leq N \int_{\mathcal{D}} |u(x,t)|\, dxdt$$

holds for all solutions to

$$\begin{cases} \partial_t u - \Delta u = 0, & \text{in } \Omega \times (0, T], \\ u = 0 & \text{on } \partial\Omega \times (0, T], \\ u(0) = u_0, & u_0 \in L^2(\Omega). \end{cases}$$

Null-controllability of a parabolic equations from measurable sets

Corollary (J. Apraiz, L. Escauriaza, G. Wang, C. Zhang, 2014)

Let $0 < T < 1$ and $\mathcal{D} \subseteq \Omega \times (0, T)$ ($\partial\Omega$ Lipschitz) be a measurable set, $|\mathcal{D}| > 0$. Then for each $u_0 \in L^2(\Omega)$ exists $f \in L^\infty(\Omega \times (0, T))$ s.t.

$$\|f\|_{L^\infty(\mathcal{D})} \leq N(\mathcal{D}, \Omega, T)\|u_0\|_{L^2(\Omega)}$$

and the solution to

$$\begin{cases} \partial_t u - \Delta u = \chi_{\mathcal{D}} f, & \text{in } \Omega \times (0, T], \\ u = 0, & \text{on } \partial\Omega \times (0, T], \\ u(0) = u_0. & \text{in } \Omega, \end{cases}$$

satisfies $u(T) \equiv 0$.

In *Observation from measurable sets for parabolic analytic evolutions and applications* (Escauriaza, Montaner, Zhang (2015)), these results are extended to some equations and systems with real-analytic coefficients not depending on time such as:

- higher-order parabolic evolutions,
- strongly coupled second-order systems with a possibly non-symmetric structure,
- one-component control of a weakly coupled system of two equations,

In this work, the real-analyticity of coefficients is quantified as:

$$|\partial_x^\gamma a_\alpha(x)| \leq {\rho_0}^{-1-|\gamma|}|\gamma|! \ \text{ in } \overline{\Omega} \times [0, T].$$

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- New quantitative estimates of space-time analyticity of the form

$$|\partial_x^\gamma \partial_t^p u(x,t)| \leq e^{1/\rho t^{1/(2m-1)}} \rho^{-|\gamma|-p} \, |\gamma|! \, p! \, t^{-p} \|u_0\|_{L^2(\Omega)},$$

  $0 < t \leq 1$, $\gamma \in \mathbb{N}^n$, $p \geq 0$ and $2m$ is the order of the parabolic problem solved by u. These estimates are obtained quantifying each step of a reasoning developed by Landis and Oleinik (1974) which reduces the strong UCP within characteristic hyperplanes of parabolic equations to its elliptic counterpart and is based on a spectral representation of solutions.

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- The so-called *telescoping series method* (L. Miller; K. D. Phung, G. Wang).

S. Vessella. *A continuous dependence result in the analytic continuation problem.* Forum Math. **11**, 6 (1999), 695–703.

Lemma. (Propagation of smallness from measurable sets)

Let $\omega \subset B_R$ be a measurable set $|\omega| > 0$. Let f be a real-analytic function in B_{2R} s.t. there exist numbers M and ρ for which

$$|\partial_x^\gamma f(x)| \leq M(\rho R)^{-|\gamma|}|\gamma|!$$

holds when $x \in B_{2R}$ and $\gamma \in \mathbb{N}^n$. Then, there are $N = N(B_R, \rho, |\omega|)$ and $\theta = \theta(B_R, \rho, |\omega|)$, $0 < \theta < 1$, such that

$$\|f\|_{L^\infty(B_R)} \leq NM^{1-\theta}\left(\frac{1}{|\omega|}\int_\omega |f|dx\right)^\theta .$$

Some remarks on the quantitative estimates

The quantitative estimate of space-time real-analyticity

$$|\partial_x^\gamma \partial_t^p u(x,t)| \le e^{t^{-\frac{1}{2m-1}}} \rho^{-1-|\gamma|-p} \, |\gamma|! \, p! \, t^{-p} \|u_0\|_{L^2(\Omega)}$$

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These features of the quantitative estimates of analyticity are essential in the proof of the interior observability estimate over measurable sets.

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$$L = \sum_{|\alpha|\le 2m} a_\alpha(x,t)\partial_x^\alpha = \sum_{|\alpha|,|\beta|\le m} \partial_x^\alpha(A_{\alpha\beta}(x,t)\partial_x^\beta) + \sum_{|\gamma|\le m} A_\gamma(x,t)\partial_x^\gamma,$$

assume that for some ρ_0, $0<\rho_0<1$

$$\sum_{|\alpha|=|\beta|=m} A_{\alpha,\beta}(x,t)\xi^{\alpha+\beta} \ge \rho_0|\xi|^{2m} \quad \forall \xi\in\mathbb{R}^n, \text{ in } \overline{\Omega}\times[0,T],$$

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$$|\partial_x^\gamma\partial_t^p a_\alpha(x,t)| \le \rho_0{}^{-1-|\gamma|-p}|\gamma|!p! \quad \text{in } \overline{\Omega}\times[0,T].$$

As far as we know, the best estimate that follows from the works of S. D. Eidelman, A. Friedman, D. Kinderlehrer, L. Nirenberg, G. Komatsu and H. Tanabe is:

Theorem

There is $0 < \rho \leq 1$, $\rho = \rho(\rho_0, n, \partial\Omega)$ *such that* $\forall \alpha \in \mathbb{N}^n, p \in \mathbb{N}$

$$|\partial_x^\gamma \partial_t^p u(x,t)| \leq \rho^{-1-\frac{|\gamma|}{2m}-p} |\gamma|!\ p!\ t^{-\frac{|\gamma|}{2m}-p-\frac{n}{4m}} \|u_0\|_{L^2(\Omega)},$$

in $\overline{\Omega} \times (0, T]$ when u solves

$$\begin{cases} \partial_t u + (-1)^m L u = 0, & \text{in } \Omega \times (0, T], \\ u = Du = \ldots = D^{m-1} u = 0, & \text{in } \partial\Omega \times (0, T], \\ u(\cdot, 0) = u_0, & u_0 \in L^2(\Omega). \end{cases}$$

and $\partial\Omega$ is a real-analytic hypersurface.

If u satisfies

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This estimate is useless for applications to observability inequalities from measurable sets.

Main result

Theorem (L. Escauriaza, S. Montaner, C. Zhang, in preparation, 2015)

Let $T \in (0,1]$ and $\partial\Omega$ be a real-analytic hypersurface. There are constants ρ and N s.t. for any $\alpha \in \mathbb{N}^n$ and $p \in \mathbb{N}$

$$|\partial_x^\alpha \partial_t^p u(x,t)| \leq N e^{N t^{-\frac{1}{2m-1}}} \rho^{-|\alpha|-p} t^{-p} |\alpha|! p! \|u\|_{L^2(\Omega\times(0,T))} \text{ in } \overline{\Omega} \times (0,T],$$

if u solves

$$\begin{cases} \partial_t u + (-1)^m L u = 0, & \text{in } \Omega \times (0,T], \\ u = Du = \ldots = D^{m-1} u = 0 & \text{in } \partial\Omega \times (0,T], \\ u(0) = u_0, & u_0 \in L^2(\Omega). \end{cases}$$

This estimate is adequate to prove the interior observabililty estimate over measurable sets when the coefficients of L are space-time real-analytic.

Idea of the proof of the quantitative estimates of analyticity

We prove a L^2 estimate by induction on $|\gamma|$ and p, let $B_r \subseteq B_1 \subseteq$ s.t. $B_r \cap \overline{\Omega} \neq \emptyset$:

$$
\begin{aligned}
&(1-r)^{2m} \| t^{p+1} e^{-\theta t^{-1/2m-1}} \partial_t^{p+1} \partial_x^\gamma u \|_{L^2(\Omega \cap B_r \times (0,T)} \\
&\qquad + \sum_{k=0}^{2m} (1-r)^k \, \| t^{p+\frac{k}{2m}} e^{-\theta t^{-1/2m-1}} D^k \partial_t^p \partial_x^\gamma u \|_{L^2(\Omega \cap B_r \times (0,T))} \\
&\qquad\qquad \leq \rho^{-1-|\gamma|-p} \theta^{-\frac{|\gamma|}{2}} (1-r)^{-|\gamma|} |\gamma|! p! \|u\|_{L^2(\Omega \times (0,T))}. \qquad (1)
\end{aligned}
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$$\leq \rho^{-1-|\gamma|-p} \theta^{-\frac{|\gamma|}{2}} (1-r)^{-|\gamma|} |\gamma|! p! \|u\|_{L^2(\Omega \times (0,T))}. \quad (1)$$

The precise form of the weights $t^{p+1} e^{-\theta t^{-1/2m-1}}$ is crucial to obtain:

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$$\leq \rho^{-1-|\gamma|-p} \theta^{-\frac{|\gamma|}{2}} (1-r)^{-|\gamma|} |\gamma|! p! \|u\|_{L^2(\Omega \times (0,T))}. \quad (1)$$

The precise form of the weights $t^{p+1} e^{-\theta t^{-1/2m-1}}$ is crucial to obtain:

- the lower bound $\rho \theta^{\frac{1}{2}} (1-r)$, (not depending on t) for the spatial radius of convergence of the Taylor series of u.

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$$\begin{aligned}(1-r)^{2m}\|t^{p+1}e^{-\theta t^{-1/2m-1}}\partial_t^{p+1}\partial_x^\gamma u\|_{L^2(\Omega\cap B_r\times(0,T)}\\ +\sum_{k=0}^{2m}(1-r)^k\,\|t^{p+\frac{k}{2m}}e^{-\theta t^{-1/2m-1}}D^k\partial_t^p\partial_x^\gamma u\|_{L^2(\Omega\cap B_r\times(0,T))}\\ \leq \rho^{-1-|\gamma|-p}\theta^{-\frac{|\gamma|}{2}}(1-r)^{-|\gamma|}|\gamma|!p!\|u\|_{L^2(\Omega\times(0,T))}.\end{aligned} \quad (1)$$

The precise form of the weights $t^{p+1}e^{-\theta t^{-1/2m-1}}$ is crucial to obtain:

- the lower bound $\rho\theta^{\frac{1}{2}}(1-r)$, (not depending on t) for the spatial radius of convergence of the Taylor series of u.
- the adequate factors $|\gamma|!p!$ in the right hand side of (1).

A consequence on null-controllability

Theorem

Let Ω be an open bounded set with real-analytic boundary, $\mathcal{D} \subseteq \Omega \times (0, T)$ be a Lebesgue measurable, $|\mathcal{D}| > 0$ and assume the abovementioned real-analyticity regularity on the coefficients of L, then: $\forall u_0 \in L^2(\Omega)$, $\exists f \in L^\infty(\mathcal{D})$ with

$$\|f\|_{L^\infty(\mathcal{D})} \leq N \|u_0\|_{L^2(\Omega)},$$

such that the solution to

$$\begin{cases} \partial_t u + (-1)^m L u = f \chi_{\mathcal{D}}, & \text{in } \Omega \times (0, T], \\ u = Du = \ldots = D^{m-1} u = 0, & \text{in } \partial\Omega \times (0, T], \\ u(0) = u_0, & \text{in } \Omega, \end{cases}$$

satisfies $u(T) \equiv 0$. Also, the control f with minimal $L^\infty(\mathcal{D})$-norm is unique and has the bang-bang property; i.e., $|f(x,t)| = const.$ for a.e. (x,t) in $\mathcal{D}$.

Thank you!

If $t \in (0, T)$, we set

$$\mathcal{D}_t = \{x \in \Omega : (x, t) \in \mathcal{D}\} \quad , \quad E = \{t \in (0, T) : |\mathcal{D}_t| \geq |\mathcal{D}|/(2T)\}.$$

By Vessella's result on *propagation of smallness* and the obtained analyticity estimates:
$\exists\ N = N(\Omega, |\mathcal{D}|/T, \rho)$ and $\theta = \theta(\Omega, |\mathcal{D}|/T, \rho)$ in $(0, 1)$ such that

$$\|u(L)\|_{L^2(\Omega)} \leq N\|u(L)\|^{\theta}_{L^1(\mathcal{D}_L)} M^{1-\theta}, \quad \text{with} \;\; M = Ne^{NL^{-1/(2m-1)}}\|u(0)\|_{L^2(\Omega)}.$$

Finally, we arrive to the telescoping series

$$
e^{-\frac{N}{(l_k - l_{k+1})^{1/(2m-1)}}} \|u(l_k)\|_{L^2(\Omega)} - e^{-\frac{N}{(l_{k+1} - l_{k+2})^{1/(2m-1)}}} \|u(l_{k+1})\|_{L^2(\Omega)}
\leq N \int_{l_{k+1}}^{l_k} \chi_E \|u(t)\|_{L^1(\mathcal{D}_t)} dt,
$$

where $\{l_k\}_{k\geq 1}$ is a monotone decreasing sequence satisfying $\lim_{k\to\infty} l_k = l$, $l < l_1 \leq T$, where $l \in (0, T)$ is a Lebesgue point of E. Summing from $k = 1$ to $+\infty$ and using energy estimate we obtain

$$
\|u(T)\|_{L^2(\Omega)} \leq N \|u\|_{L^1(\mathcal{D})}.
$$