Evolution of spoon-shaped networks

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Spoon-shaped network

**Definition**

\[ \Gamma_0 = \gamma^1([0, 1]) \cup \gamma^2([0, 1]) , \]

\[ \gamma^1 : [0, 1] \to \Omega , \]

\[ \gamma^2 : [0, 1] \to \overline{\Omega} . \]

\[ \gamma^1(0) = \gamma^1(1) = \gamma^2(0) = O . \]

\[ \gamma^2(1) = P \in \partial \Omega . \]
The evolution by curvature of a spoon-shaped network is the geometric gradient flow of the Length functional, that is, the sum of the lengths of all the curves of the network:

\[ L(Γ) = L_1 + L_2 = \sum_{i=1}^{2} \int_0^1 |γ^i_x(ξ)| \, dξ. \]

The equation that describes the motion by curvature is

\[ γ^i_t(x, t) = \frac{γ^i_{xx}(x, t)}{|γ^i_x(x, t)|^2}. \]
Closed curve and tree-like network

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Evolution of spoon-shaped networks
Theorem (Gage-Hamilton-Grayson)

A simple closed curve evolving by curvature becomes eventually convex and then shrinks to a point in a finite time.
Comparison between closed curve and spoon-shaped network

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Evolution of spoon-shaped networks
Existence

Theorem

For any initial smooth, spoon-shaped network $\Gamma_0$ in a smooth, convex, open set $\Omega \subset \mathbb{R}^2$ there exists a unique smooth solution of the problem in a maximal time interval $[0, T)$.

Theorem

If $[0, T)$, with $T < \infty$, is the maximal time interval of existence of a smooth solution $\Gamma_t$ of the problem, then at least one of the following possibilities holds:

\begin{itemize}
  \item $\liminf_{t \to T} L_2(t) = 0$,
  \item $\limsup_{t \to T} \int_{\Gamma_t} k^2 ds = +\infty$.
\end{itemize}
We rescale the flow in its maximal time interval \([0, T)\).
Fixed \(x_0 \in \mathbb{R}^2\), let \(\tilde{F}_{x_0} : \Gamma \times [-1/2 \log T, +\infty) \rightarrow \mathbb{R}^2\) be the map

\[
\tilde{F}_{x_0}(p, t) = \frac{F(p, t) - x_0}{\sqrt{2(T - t)}} \quad t(t) = -\frac{1}{2} \log (T - t) \tag{1}
\]

then, the rescaled networks are given by

\[
\tilde{\Gamma}_{x_0, t} = \frac{\Gamma_t - x_0}{\sqrt{2(T - t)}} \tag{2}
\]

and they evolve according to the equation

\[
\frac{\partial}{\partial t} \tilde{F}_{x_0}(p, t) = \tilde{v}(p, t) + \tilde{F}_{x_0}(p, t) \tag{3}
\]

where

\[
\tilde{v}(p, t) = \frac{v(p, t(t))}{\sqrt{2(T - t(t))}} = \tilde{k} + \tilde{\lambda} = \tilde{k}\nu + \tilde{\lambda}\tau \quad \text{and} \quad t(t) = T - e^{-2t}. \tag{4}
\]

We obtained a smooth flow of spoon-shaped networks \(\tilde{\Gamma}_t\) defined for \(t \in [-1/2 \log T, +\infty)\).
Long time behavior

Assume that the length of the curve $\gamma^2(x, t)$ is uniformly bounded away from zero for $t \in [0, T)$. Then, for every $x_0 \in \mathbb{R}^2$ a sequence of rescaled networks $\tilde{\Gamma}_{x_0, t}$ converges in the $C^1_{loc}$ topology to a limit set which is one of the following:

- a halfline from the origin,
- a straight line through the origin,
- an infinite flat triod centered at the origin,
- a Brakke spoon.