Young's law for Nonlocal Fractional Perimeters

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Capillarity and wetting phenomena

Capillarity is the ability of a liquid to flow in narrow spaces, even in opposition to external forces like gravity. *Wetting* is the ability of a liquid to maintain contact with a solid surface.
-connected and essentially due to intermolecular forces between the liquid and surrounding solid surfaces.
Given Ω open set in $\mathbb{R}^n$ (*the container*), $E \subset \Omega$ (regione occupied by) *the liquid* the equilibrium state of E in Ω is determined by the *Gauss' free energy*

$$\int_{\partial E \cap \Omega} \varphi(x, \nu_E(x))\, d\mathcal{H}^{n-1} + \int_{\partial E \cap \partial \Omega} \sigma(x)\, d\mathcal{H}^{n-1} + \int_E g(x)\, dx$$

$+$ condition $|E| = m$ for $m \in (0, |\Omega|)$.

$\varphi(x, \nu(x))$ *(anisotropic) surface tension density* at point x of the surface with normal ν, $\sigma(x)$ *adhesion coefficient* at point x of $\partial\Omega$

Variational formulation

$\hookrightarrow$ weak formulation on sets of finite perimeter + direct methods
$\Rightarrow$ existence of E minimizer such that its 'essential' boundary is a hypersurface in a 'weak geometric measure sense'.

Main goal: establish the suitable PDE satisfied by the equilibrium state of the liquid and the contact equation at the boundary of the (possible) adhesion set (Young's law)

$\hookrightarrow$ Regularity issue for the free boundary of the contact set

$\hookrightarrow$ Further regularity hypotheses on $\partial\Omega$ and some compatibility condition between the anisotropic surface tension density $\varphi(x, \nu)$ and the adhesion coefficent σ

Rem. For $\varphi(x, \nu) = |\nu|$, $g = 0 = \sigma$ one recovers the relative isoperimetric inequality in Ω, hence the regularity cannot improve that of minimal surfaces.

Trasversality conditions and contact equations

Theorem(Maggi-De Philippis 2015)
If $\partial\Omega$ is $C^{1,1}$, φ is regular elliptic, $g \in L^\infty(\Omega)$, $\sigma \in \mathrm{Lip}(\Omega)$ and

$$-\varphi(x, -\nu_\Omega(x)) < \sigma(x) < \varphi(x, \nu_\Omega(x)) \qquad \forall x \in \partial\Omega$$

Then E is an open set with $\partial E \cap \partial\Omega$ a set of finite perimeter in $\partial\Omega$. Moreover $\exists$ a set Σ with $\mathcal{H}^{n-2}(\Sigma) = 0$ and $\overline{\partial E \cap \Omega} \setminus \Sigma$ is a $C^{1,1/2}$ hypersurface with boundary and it holds

$$\mathrm{div}(\nabla\varphi(x, \nu_E(x))) + \nabla\varphi(x, \nu_E(x)) \cdot \nu_\Omega(x) = -g(x) + \mathit{constant}$$
$$\nabla\varphi(x, \nu_E(x)) \cdot \nu_\Omega(x) = \sigma(x) \quad \text{for any } x \text{ in } \overline{\partial E \cap \Omega} \cap \partial\Omega \setminus \Sigma$$

For $\varphi(x, \nu) = |\nu|$ and $n = 3$ the contact equations rewrites as

$$H(x) = -g(x) + \mathit{constant} \qquad \text{for any } x \text{ in } \overline{\partial E \cap \Omega} \cap \Omega$$
$$\nu_E(x) \cdot \nu_\Omega(x) = \sigma(x) \qquad \text{for any } x \text{ in } \overline{\partial E \cap \Omega} \cap \partial\Omega$$

Nonlocal fractional perimeters

Capillarity surfaces are determined by the balance between adhesive and cohesive forces, depending in turn by the intermolecular interactions.

Main idea: substitute the *local* nature of these interactions admitting in the model *long-range* interactions among particles producing surface tension effects both in the cohesive and adhesive term.

$\hookrightarrow$ Nonlocal fractional perimeters

For $s \in (0,1)$ and Ω regular bounded open set let

$$P_s(E,\Omega) := \int_{E\cap\Omega}\int_{\Omega\setminus E} \frac{1}{|x-y|^{n+s}}\,dy\,dx$$

$$+ \int_{E\cap\Omega}\int_{\Omega^c\setminus E} \frac{1}{|x-y|^{n+s}}\,dy\,dx + \int_{E\setminus\Omega}\int_{\Omega\setminus E} \frac{1}{|x-y|^{n+s}}\,dy\,dx$$

Main features

• different scaling law:
For any $\lambda > 0$, E, Ω it holds $P_s(\lambda E, \lambda\Omega) = \lambda^{n-s} P_s(E, \Omega)$

• weaker than the euclidean perimeter if Ω regular
$\exists\ C = C(n, s, \Omega) > 0$ such that $P_s(E, \Omega) \leq C\mathrm{Per}(E, \Omega) + C$

• finiteness on sets F whose boundary has Hausdorff dimension $n - s > n - 1$

• compact embedding in $L^1(\Omega)$, continuous in $L^{n/n-1}(\Omega)$
$\forall (E_k)$ with $P_s(E_k, \Omega) \leq C\ \exists\ E_{k_h}, E$ with $\|E_{k_h} - E\|_{L^1(\Omega)} \to 0.$

• 'approximation' of euclidean perimeter in a variational sense

Theorem(Ambr.-De Phil. Mart., Caffa.-Roquej.-Savin 2011)

$$\Gamma\text{-}\lim_{s\to 1-} (1 - s) P_s(E, \Omega) = w_{n-1} \mathrm{Per}(E, \Omega), \qquad w_{n-1} = \mathcal{H}^{n-1}(\mathcal{S}^{n-1})$$

Nonlocal free energies

Let $\sigma \in C^0(\bar{\Omega})$; for $E \subset \Omega$ we define

$$\mathcal{F}_s^\sigma(E) := \int_E \int_{\Omega\setminus E} \frac{1}{|x-y|^{n+s}}\, dy\, dx + \int_E \int_{\Omega^c} \frac{\sigma(x)}{|x-y|^{n+s}}\, dy\, dx$$

Goal: Analyze these new nonlocal surface and wetting energies in capillarity problems.

- study the asymptotics as $s \to 1-$ of $\mathcal{F}_s^\sigma(E)$;
- study existence of minimizers for $\mathcal{F}_s^\sigma(E) + \int_E g_s$ with $|E| = m$;
- establish suitable regularity properties of minimizers;
- deduce non local contact equations and relative contact angles;
- compare the information derived by the local and non-local model.

Γ-convergence analysis

Theorem(G.-Novaga)

Assume Ω regular and $\sigma \in C^0(\bar{\Omega})$, then $(1-s)\mathcal{F}_s^\sigma \to \mathcal{F}^\sigma$ as $s \to 1-$ with respect to the L^1 convergence in Ω, where $\mathcal{F}^\sigma$ is defined by

$$\mathcal{F}^\sigma(E) = w_{n-1}\Big(\mathrm{Per}(E,\Omega) + \int_{\partial^* E\cap\partial\Omega} (-1)\vee\sigma\wedge 1\, d\mathcal{H}^{n-1} + \int_{\{\sigma<-1\}\cap\partial\Omega} (1+\sigma)\, d\mathcal{H}^{n-1}\Big)$$

Approximation of local capillarity problems

Theorem(G.-Novaga)

Let $m \in (0, |\Omega|)$ and $g \in L^\infty(\Omega)$. Assume that

$$-1 \leq \sigma(x) \leq 1 \text{ for any } x \in \Omega,$$

then the energies

$$(1-s)\mathcal{F}_s^\sigma + \int_E g(x)$$

defined for sets $E \subseteq \Omega$ with $|E| = m$, Γ-converge as $s \to 1-$, with respect to the L^1 convergence in Ω, to

$$w_{n-1}\Big(\mathrm{Per}(E,\Omega) + \int_{\partial^* E \cap \partial\Omega} \sigma \, d\mathcal{H}^{n-1}\Big) + \int_E g(x)\, dx$$

defined for sets $E \subseteq \Omega$ with $|E| = m$.

Relaxation of Gauss' free energy

Proposition(G.-Novaga)

Assume Ω regular and $\sigma \in C^0(\bar{\Omega})$. For $E \subset \Omega$ define F^σ as

$$F^\sigma(E) = w_{n-1}\Big(\mathrm{Per}(E,\Omega) + \int_{\partial^* E \cap \partial\Omega} \sigma \, d\mathcal{H}^{n-1}\Big).$$

Then its lower semicontinuous envelope with respect to the L^1 convergence in Ω is

$$\mathcal{F}^\sigma(E) = w_{n-1}\Big(\mathrm{Per}(E,\Omega) + \int_{\partial^* E \cap \partial\Omega} (-1) \vee \sigma \wedge 1 \, d\mathcal{H}^{n-1} + \int_{\{\sigma < -1\} \cap \partial\Omega} (1+\sigma) \, d\mathcal{H}^{n-1}\Big)$$

Asymptotics of P_s with an additional constraint

If $\sigma \equiv 1$ we recover the analogous of Ambrosio-De Philippis-Martinazzi result with the additional condition that admissible sets must lie in Ω:

Proposition Define $\tilde{P}_s(E,\Omega)$ for any measurable set $E \subset \mathbb{R}^n$ as

$$\tilde{P}_s(E,\Omega) = \begin{cases} P_s(E,\Omega) & \textit{if } E \subseteq \Omega \\ +\infty & \textit{otherwise.} \end{cases}$$

Then $(1-s)\tilde{P}_s(E,\Omega)$ Γ-converge as $s \to 1-$ with respect to the L^1_{loc} convergence in $\mathbb{R}^n$ to

$$w_{n-1}\text{Per }(E) = w_{n-1}\Big(\text{Per }(E,\Omega) + \mathcal{H}^{n-1}(\partial^* E \cap \partial\Omega)\Big).$$

Rem. For sets touching the boundary the recovery sequence is obtained in the Ambr.-De Phil.-Mart.'s setting by sets having boundary transversal to $\partial\Omega$.

Existence and compactness of s-minimizers

Proposition

Let $m \in (0, |\Omega|)$ and $g_s \in L^\infty(\Omega)$ then there exists at least a minimizer E_s of $\mathcal{F}_s^\sigma(E) + \int_E g_s \, dx$ among sets $E \subseteq \Omega$ with $|E| = m$.

Ω regular $\Rightarrow P_s(\Omega, \mathbb{R}^n) < +\infty \Rightarrow$ l.s.c. of $\mathcal{F}_s^\sigma(E)$. The Sobolev fractional embedding allows to conclude.

Proposition

For $s \in (0, 1)$ let E_s be as above with $g_s = g/(1-s)$. Then there exists a set E limit point of E_s in L^1. Moreover E is a minimizer for $\mathcal{F}^\sigma(E) + \int_E g \, dx$ among sets $E \subseteq \Omega$ with $|E| = m$.

We can prove a priori estimates on $(1-s)\mathcal{F}_s^\sigma(E)$ and deduce the validity of the Frechet-Kolmogorov compactness criterion. The last part follows by the Γ-convergence.