# On the gaps in spectrum of the periodic Maxwell Operator: applications to Photonic Crystal Fibre design

Shane Cooper
University of Bath

Joint with Ilia Kamotski (UCL) and Valery Smyshlyaev (UCL)

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# Photonic crystal fibres: guiding light by confinement

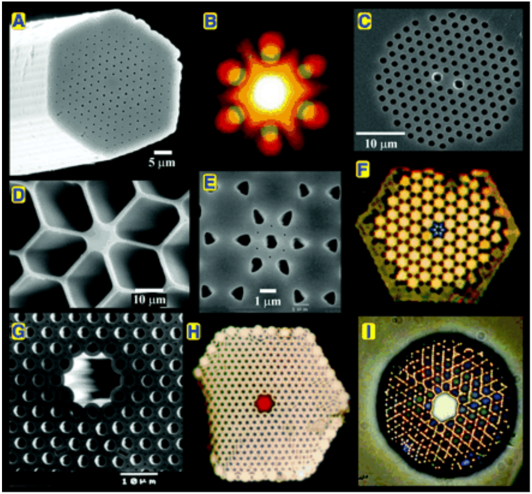
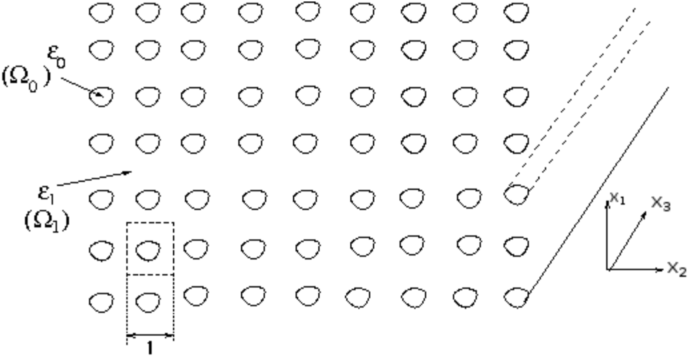



Figure: Taken from "Photonic Crystal Fibres" Phillip Russell, Science, 2015

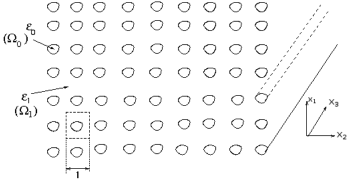
# Photonic crystals: Problem Formulation




$$\nabla \times E = -\mu \frac{\partial H}{\partial t}, \qquad \nabla \times H = \epsilon \frac{\partial E}{\partial t},$$
$$\nabla \cdot (\epsilon E) = 0, \qquad \nabla \cdot H = 0,$$

$$\epsilon = \epsilon_0 \chi_0(x) + \epsilon_1 \chi_1(x), \quad \epsilon_0 \neq \epsilon_1, \quad \mu \text{ constant} \quad (\mu = 1)$$

$$E = E(x_1, x_2) \exp\big(\mathrm{i}(kx_3 + \omega t)\big), \qquad H = H(x_1, x_2) \exp\big(\mathrm{i}(kx_3 + \omega t)\big)$$




In each phase $E_3$ and $H_3$ satisfy the following equations

$$\Delta E_3 + (\omega^2 \epsilon_1 - k^2)E_3 = 0, \quad \Delta H_3 + (\omega^2 \epsilon_1 - k^2)H_3 = 0 \quad \text{in } \Omega_1$$

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$E_3$ and $H_3$ coupled across interface $\Gamma = \partial\Omega_0$:

$$\omega\left[\frac{\epsilon}{a}\nabla E_3 \cdot n\right] = -k\left[\frac{1}{a}\nabla H_3 \cdot n^{\perp}\right], \qquad k\left[\frac{1}{a}\nabla E_3 \cdot n^{\perp}\right] = \omega\left[\frac{1}{a}\nabla H_3 \cdot n\right]$$

where $a = \omega^2\epsilon(x) - k^2$ discontinuous on $\Gamma$.

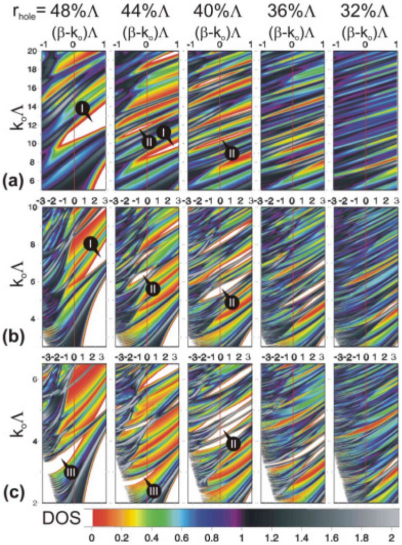


Figure: From J.M.Pottage, D.M.Bird, T.D.Hedley, T.A.Birks, J.C.Knight and P.St.J. Russell, Optics Express, 2003

# Maxwell's equations as elliptic system: Strong and Weak formulation

$$\partial_1 \left( \frac{i\omega\epsilon}{a} E_{3,1} \right) + \partial_2 \left( \frac{i\omega\epsilon}{a} E_{3,2} \right) + \partial_1 \left( \frac{ik}{a} H_{3,2} \right) - \partial_2 \left( \frac{ik}{a} H_{3,1} \right) = -i\omega\epsilon E_3$$

$$\partial_1 \left( \frac{ik}{a} E_{3,2} \right) - \partial_2 \left( \frac{ik}{a} E_{3,1} \right) - \partial_1 \left( \frac{i\omega}{a} H_{3,1} \right) - \partial_2 \left( \frac{i\omega}{a} H_{3,2} \right) = i\omega H_3,$$

Find $u = (E_3, H_3)$ such that

$$\int_{\mathbb{R}^2} \frac{\omega}{a} \left( \epsilon \nabla u_1 \cdot \overline{\nabla \phi_1} + \nabla u_2 \cdot \overline{\nabla \phi_2} \right) + \frac{k}{a} \left( \{ \overline{\phi_1}, u_2 \} + \{ u_1, \overline{\phi_2} \} \right) \, \mathrm{d}x$$

$$= \omega \int_{\mathbb{R}^2} \epsilon(x) u_1 \overline{\phi_1} \, \mathrm{d}x + u_2 \overline{\varphi_2} \quad \forall \phi \in C_0^\infty(\mathbb{R}^2)$$

$\{f, g\} := f_{x_1} g_{x_2} - g_{x_1} f_{x_2}$.
The above form is symmetric, and positive if $k^2 < \omega^2 \min\{\epsilon_0, \epsilon_1\}$.

# Oblique incidence (Case of PCF): $k \neq 0$

If $k = \omega\kappa$, $\kappa \geq 0$ gives **usual spectral problem**: Find $u$ such that

$$\int_{\mathbb{R}^2} \frac{1}{\epsilon(x) - \kappa^2} \big(\epsilon(x)\nabla u_1 \cdot \overline{\nabla\phi_1} + \nabla u_2 \cdot \overline{\nabla\phi_2}\ \big) +$$

$$+ \int_{\mathbb{R}^2} \frac{\kappa}{\epsilon(x) - \kappa^2} \left(\left\{\overline{\phi_1}, u_2\right\} + \left\{u_1, \overline{\phi_2}\right\}\right)\ \mathrm{d}x = \omega^2 \int_{\mathbb{R}^2} \epsilon(x) u_1 \overline{\phi_1}\ \mathrm{d}x + u_2\overline{\phi_2}$$

$$\forall \phi \in C_0^\infty(\mathbb{R}^2), \quad a(x) = \omega^2\epsilon(x) - k^2$$

The above form is symmetric, and positive if $\kappa^2 < \min\{\epsilon_0, \epsilon_1\}$.

# Anti-resonant reflecting optical waveguide (ARROW)

Assume $\epsilon_0 > \epsilon_1 = 1$.

$$B_\kappa[u] := \int_{\Omega_1} \frac{\epsilon_0 - 1}{1-\kappa^2}|\partial u|^2 + \frac{\epsilon_0+\kappa}{1+\kappa}|\nabla u|^2 dx + \int_{\Omega_0} \epsilon_0|\nabla u_1|^2 + |\nabla u_2|^2 dx$$

where

$$|\partial u|^2 = |\partial_{x_1} u_1 + \partial_{x_2} u_2|^2 + |\partial_{x_2} u_1 - \partial_{x_1} u_2|^2$$

Scalar product is

$$A[u] := \int_{\Omega_1} |u_1|^2 dx + \int_{\Omega_0} \epsilon_0|u_1|^2 + |u_2|^2 dx$$

## Spectral problem

$$B_\kappa(u,\phi) = \lambda A(u,\phi), \qquad \forall \phi \in C_0^\infty(\mathbb{R}^2),$$

here $\lambda^2 = \omega^2(\epsilon_0 - \kappa^2)$.

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- $\epsilon_0 \leq \kappa^2$ : No solutions with $\omega \in \mathbb{R}\backslash\{0\}$
- $1 < \kappa < \epsilon_0$ : $B_\kappa$ is sign-indefinite
- $\kappa < 1$ : Form positive

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Goal Analytically study spectrum as $\kappa \to 1$.

## Floquet-Bloch decomposition

Fixed $\theta \in [-\pi, \pi)^3$. Find $u \in H^1_\theta(Q)$ ( $u(y) = e^{i\theta\cdot y} v(y)$, $v$ Q-periodic) such that

$$B_\kappa(u, \phi) = \lambda A(u, \phi), \qquad \forall \phi \in C_0^\infty(\mathbb{R}^2), \quad \forall \phi \in V(\theta).$$

Spectrum:

$$0 \leq \lambda_1(\kappa, \theta) \leq \lambda_2(\kappa, \theta) \leq \ldots \leq \lambda_n(\kappa, \theta) \leq \ldots$$

For $\kappa < 1$. Let

$$\Sigma_\theta^\kappa = \sum_{i=1}^\infty \lambda_i(\kappa, \theta).$$

# Asymptotic behaviour of spectra near $\kappa = 1$

## Theorem

'

$$\lim_{\kappa \nearrow 1} \cup_\theta \Sigma_\theta^\kappa = \cup_\theta \Sigma_\theta^1,$$

*where*

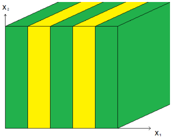
$$\Sigma_\theta^1 = \sum_{i=1}^{\infty} \lambda_i(1,\theta)$$

*and* $\lambda_i(1,\theta)$ *are eigenvalues of*

$$B[u] := \int_Q \epsilon_0 |\nabla u_1|^2 + |\nabla u_2|^2$$

*with domain* $V = \{u \in H^1_\theta(\mathbb{Q}) : \partial u = 0 \text{ in } Q_1\}$ *and scalar product*

$$a[u] := \int_Q |u_1|^2 + (\epsilon_0 - 1)|u_2|^2$$



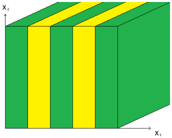

$$f_1(z) := \cos\sqrt{z(\epsilon_0-\epsilon_1)}(b-a)$$

$$f_2(z) := \frac{1-b+a}{2}\sqrt{z(\epsilon_0-\epsilon_1)}\sin\sqrt{z(\epsilon_0-\epsilon_1)}(b-a)$$

TM polarised EM-field $u=(v,0)$: $\cos\theta = f_1(\lambda) - \frac{\epsilon_1}{\epsilon_0} f_2(\lambda)$

TE polarised EM-field $u=(0,v)$: $\cos\theta = f_1(\lambda) - f_2(\lambda)$

# Example: 1-dimensional Photonic crystal fibre




$$f_1(z) := \cos\sqrt{z(\epsilon_0 - \epsilon_1)}(b - a)$$

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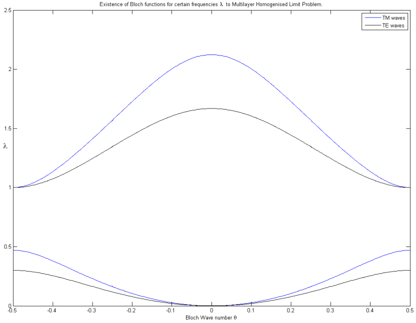
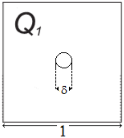
TM polarised EM-field $u = (v, 0)$: $\cos\theta = f_1(\lambda) - \frac{\epsilon_1}{\epsilon_0} f_2(\lambda)$

TE polarised EM-field $u = (0, v)$: $\cos\theta = f_1(\lambda) - f_2(\lambda)$






$Q_0 = \delta\Omega$ for some smooth, open bounded $\Omega$

$$\lambda_2(1,\theta) \leq -\frac{c_1}{\delta^2 \ln \delta} \qquad \& \qquad \lambda_3(1,\theta) \geq c_2\delta^{-2}$$

## Theorem

*Let* $\Sigma_\theta^\delta = \sum_{i=1}^\infty \lambda_i(1,\theta)$.
*Then,*

$$\lim_{\delta\to 0} \cup_\theta \delta^2 \ln \delta \Sigma_\theta^\delta = [0, \Lambda^\star], \qquad \lim_{\delta\to 0} \cup_\theta \delta^2 \Sigma_\theta^\delta = \{0, \Lambda_1, \Lambda_2, \ldots, \Lambda_3\},$$

*where* $\Lambda_i$ *are eigenvalues of*

$$B[u] := \int_\Omega \epsilon_0 |\nabla u_1|^2 + |\nabla u_2|^2$$

*with domain* $V = \{u \in H^1_{\mathrm{loc}}(\mathbb{R}^{\not\geq}) : \partial u = 0 \text{ in } \mathbb{R}^2 \backslash \Omega\}$ *and scalar product*

$$a[u] := \int_{\mathbb{R}^2} |u_1|^2 + (\epsilon_0 - 1)|u_2|^2$$

Thank you for listening