Partial differential equations, optimal design and numerics

Finite-time stabilization of strings connected by point mass and the SMB chromatography

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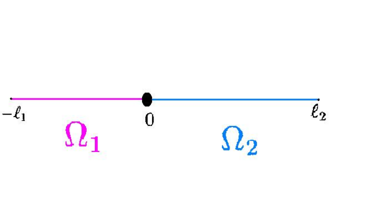
Outline

Problem description



- The deformations of the first and second string will be described respectively by the functions :

$$\begin{array}{lll} u = u(x,t), & x \in \Omega_1, & t > 0, \\ v = v(x,t), & x \in \Omega_2, & t > 0. \end{array}$$

- The position of the mass $M > 0$ attached to the strings at the point $x = 0$ is described by the function $z = z(t)$ for $t > 0$.

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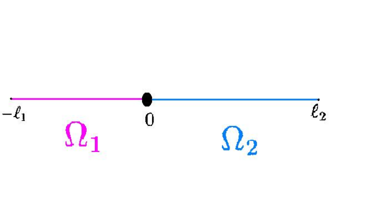

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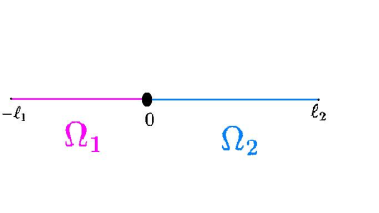

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- The position of the mass $M > 0$ attached to the strings at the point $x = 0$ is described by the function $z = z(t)$ for $t > 0$.

The unforced system is given by :

$$\begin{cases} \rho_1 u_{tt} = \sigma_1 u_{xx}, & x \in \Omega_1, \quad t > 0, \\ \rho_2 v_{tt} = \sigma_2 v_{xx}, & x \in \Omega_2, \quad t > 0, \\ M z_{tt}(t) + \sigma_1 u_x(0,t) - \sigma_2 v_x(0,t) = 0, & \\ u(-\ell_1, t) = v(\ell_2, t) = 0, & t > 0, \\ u(0,t) = v(0,t) = z(t), & t > 0, \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x), & x \in \Omega_1, \\ v(x,0) = v^0(x), \quad v_t(x,0) = v^1(x), & x \in \Omega_2, \\ z(0) = z^0, \qquad z_t(0) = z^1. & \end{cases} \tag{2.1}$$

▶ The energy of the system (2.1) is given by :

$$E_M(t) = \frac{1}{2}\int_{-\ell_1}^{0} [\rho_1 |u_t(x,t)|^2 + \sigma_1 |u_x(x,t)|^2]\, dx + \frac{M}{2}|z_t(t)|^2 + \frac{1}{2}\int_{0}^{\ell_2} [\rho_2 |v_t(x,t)|^2 + \sigma_2 |v_x(x,t)|^2]\, dx.$$

▶ (Hansan and Zuazua, SIAM 95) : System (2.1) is stable, in particular E_M *is* **conserved**.

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Finite-time stabilization with two boundary controls

Definition

Let $f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{X}$ smooth function, f(0,0)=0.
The control system $\dot{x} = f(x,u)$ is said finite-time stabilizable if there exists an admissible feedback u for which $\dot{x} = f(x, u(x))$ is F.T.S in the sense that $\exists\ r > 0 :\ |x(0)| < r$ such that $x(t) = 0$ in finite-time.

Example

$$\dot{x} = -x^{\frac{1}{3}}, \qquad x(0) = x_0$$

$$x(t) = \begin{cases} sgn(x_0)(x_0^{\frac{2}{3}} - \frac{2}{3}t)^{\frac{3}{2}} & if\ \ 0 \leq t \leq \frac{3}{2}|x_0|^{\frac{3}{2}} \\ 0 & if\ \ t \geq \frac{3}{2}|x_0|^{\frac{3}{2}}. \end{cases}$$

Problematic

- Our objective is to build proper feedbacks (boundary or internal) such that the solution of our system vanishes in finite-time.
- This means to attenuate vibrations of the strings.
- This requires to define the appropriate functional space of state and control such that the solutions in closed loop are defined.
- $L^2(X)$ denotes the space square integrable functions $f : X \to \mathbb{R}$.

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Strategy of Stabilization

▶ We start by transforming the two wave equations to first order, hence we introduce the Riemann invariants.

•The equation $u_{tt} - \frac{\sigma_1}{\rho_1} u_{xx} = 0$ is converted to

$$\begin{cases} \partial_t u_1 + \sqrt{\frac{\sigma_1}{\rho_1}} \partial_x u_1 = 0, \\ \partial_t u_2 - \sqrt{\frac{\sigma_1}{\rho_1}} \partial_x u_2 = 0, \end{cases} \quad \text{with} \quad \begin{cases} u_1 = r_1 - \sqrt{\frac{\sigma_1}{\rho_1}} s_1, \\ u_2 = r_1 + \sqrt{\frac{\sigma_1}{\rho_1}} s_1, \end{cases}$$

where $(r_1, s_1) = (\partial_t u, \partial_x u)$.

• The equation $v_{tt} - \frac{\sigma_2}{\rho_2} v_{xx} = 0$ is converted to

$$\begin{cases} \partial_t v_1 + \sqrt{\frac{\sigma_2}{\rho_2}} \partial_x v_1 = 0, \\ \partial_t v_2 - \sqrt{\frac{\sigma_2}{\rho_2}} \partial_x v_2 = 0. \end{cases} \quad \text{with} \quad \begin{cases} v_1 = r_2 - \sqrt{\frac{\sigma_2}{\rho_2}} s_2, \\ v_2 = r_2 + \sqrt{\frac{\sigma_2}{\rho_2}} s_2, \end{cases}$$

where $(r_2, s_2) = (\partial_t v, \partial_x v)$.

Hence, for $i = 1, 2$, we get the following hybrid system :

$$\begin{cases} \partial_t u_i(x,t) + \lambda_i \partial_x u_i(x,t) = 0, & (x,t) \in (-\ell_1, 0) \times (0,\infty); \\ \partial_t v_i(x,t) + \mu_i \partial_x v_i(x,t) = 0, & (x,t) \in (0, \ell_2) \times (0,\infty); \\ 2M z_{tt}(t) + \sqrt{\sigma_1 \rho_1}\big(u_2(0,t) - u_1(0,t)\big) & \\ \qquad -\sqrt{\sigma_2 \rho_2}\big(v_2(0,t) - v_1(0,t)\big) = 0; & \\ 2 z_t(t) = u_1(0,t) + u_2(0,t) = v_1(0,t) + v_2(0,t); & \\ u_i(x,0) = u_i^0(x); \quad v_i(x,0) = v_i^0(x). & \end{cases} \tag{3.1}$$

With $\lambda_1 \geq c_1 > 0 > -c_1 > \lambda_2,$

and

$\mu_1 \geq c_2 > 0 > -c_2 > \mu_2.$

As in the work of (**Perrollaz-Rosier** 2014), we assume that the boundary conditions satisfy the ODE , for example

$$\frac{d}{dt}u_1(-\ell_1,t) = -k\ sgn\big(u_1(-\ell_1,t)\big)\big|u_1(-\ell_1,t)\big|^{\gamma}, \tag{3.2}$$

$$\frac{d}{dt}v_2(\ell_2,t) = -k\ sgn\big(v_2(\ell_2,t)\big)\big|v_2(\ell_2,t)\big|^{\gamma}. \tag{3.3}$$

Where $(k,\gamma) \in (0,\infty) \times (0,1)$,

and $\text{sgn} = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$

Definition

We call **settling time** or **response time** of the transformed system (3.1), the critical time $T(u_i^0, v_i^0)$ such that

$$(u_i, v_i)(x,t) = 0 \qquad \forall t \geq T.$$

$$T_1 = \frac{|u_1^0(-\ell_1)|^{1-\gamma}}{(1-\gamma)k} + \frac{1}{c_1},$$

$$T_2 = \frac{|v_2^0(\ell_2)|^{1-\gamma}}{(1-\gamma)k} + \frac{1}{c_2}.$$

Let $\quad T^* = max(T_1, T_2) \quad$ and $\quad c^* = min(c_1, c_2),$

$$\forall t \geq T^* - \frac{1}{c^*}, \qquad u_1(-\ell_1, t) = v_2(\ell_2, t) = 0. \qquad (3.4)$$

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$$\begin{cases} \frac{\rho_1}{\sigma_1}\partial_t u_1(x,t) + \lambda_1 \partial_x u_1(x,t) = 0, & (x,t) \in (-\ell_1, 0) \times (0,\infty), \\ u_1(x,0) = u_1^0(x), \\ u_1(-\ell_1, t) = u_{-\ell_1}(t), \end{cases} \tag{3.5}$$

$$2Mz_{tt}(t) + \sqrt{\sigma_1\rho_1}\big(u_2(0,t) - u_1(0,t)\big) - \sqrt{\sigma_2\rho_2}\big(v_2(0,t) - v_1(0,t)\big) = 0;$$
$$2z_t(t) = u_1(0,t) + u_2(0,t) = v_1(0,t) + v_2(0,t);$$

$$\begin{cases} \frac{\rho_2}{\sigma_2}\partial_t v_2(x,t) + \mu_2 \partial_x v_2(x,t) = 0, & (x,t) \in (0, \ell_2) \times (0,\infty), \\ v_2(x,0) = v_2^0(x), \\ v_2(\ell_2, t) = v_{\ell_2}(t). \end{cases} \tag{3.6}$$

- Let ϕ_{λ_1} the flow associated with λ_1 which is defined on a subinterval $[e_{\lambda_1}(t,x), f_{\lambda_1}(t,x)]$ of $[0, T^*]$.
- ϕ_{λ_1} denote the $\mathcal{C}^1$ maximal solution to the Cauchy problem

$$\begin{cases} \partial_s \phi_{\lambda_1}(s,x,t) = \lambda_1, \\ \phi_{\lambda_1}(t,x,t) = x. \end{cases}$$

- The domain of ϕ_{λ_1} is denoted by : $D_1 = \{(s,x,t); (x,t) \in [-\ell_1, 0] \times [0, T_1], s \in [e_{\lambda_1}, f_{\lambda_1}](x,t)\}$.

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Consider the characteristic lines

$$P_1 = \{(s, \phi_{\lambda_1}(s, 0, 0));\ s \in [0, f_{\lambda_1}(0, 0)]\},$$

$$I_1 = \{(t, x) \in [0, T] \times [-\ell_1, 0]; e_{\lambda_1}(t, x) = 0\},$$

$$J_1 = \{(t, x) \in [0, T] \times [-\ell_1, 0]; \phi_{\lambda_1}(e_{\lambda_1}(t, x), t, x) = -\ell_1\}.$$

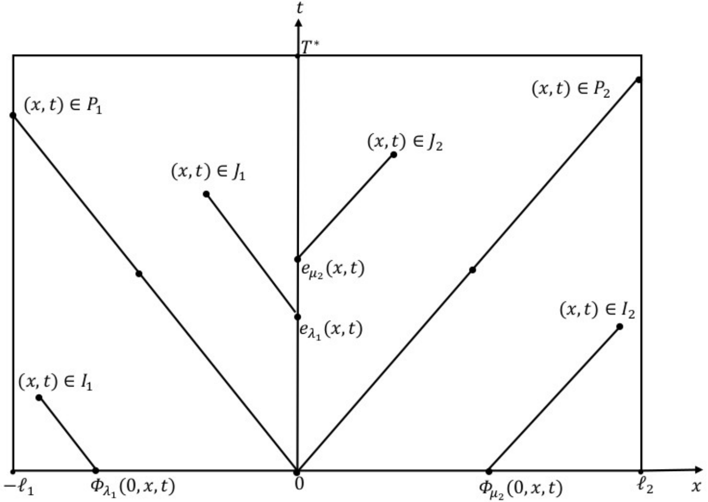

figure 2 : Partition of $[-\ell_1, \ell_2] \times [0, T^*]$ into $I_1 \bigcup P_1 \bigcup J_1 \bigcup I_2 \bigcup P_2 \bigcup J_2$

$$\begin{cases} \frac{\rho_1}{\sigma_1}\partial_t u_1 + \lambda_1 \partial_x u_1 = 0, \\ u_1(x,0) = u_1^0(x), \\ u_1(-\ell_1, t) = u_{-\ell_1}(t), \end{cases} \quad (3.5) \qquad \begin{cases} \frac{\rho_2}{\sigma_2}\partial_t v_2 + \mu_2 \partial_x v_2 = 0, \\ v_2(x,0) = v_2^0(x), \\ v_2(\ell_2, t) = v_{\ell_2}(t). \end{cases} \quad (3.6)$$

Proposition

We suppose that u_1^0 et $u_{-\ell_1}$ (resp v_2^0 et v_{ℓ_2}) are uniformly Lipschitz continuous, and $u_1^0(-\ell_1) = u_{-\ell_1}(0)$ (resp $v_2^0(\ell_2) = v_{\ell_2}(0)$). Then

$$u_1(x,t) = \begin{cases} u_{-\ell_1}(e_{\lambda_1}(x,t)) & \text{if } (x,t) \in J_1, \\ u_1^0(\phi_{\lambda_1}(0,x,t)) & \text{if } (x,t) \in I_1 \cup P_1, \end{cases} \quad (3.7)$$

$$\left(resp \quad v_2(x,t) = \begin{cases} v_{\ell_2}(e_{\mu_2}(x,t)) & \text{if } (x,t) \in J_2, \\ v_2^0(\phi_{\mu_2}(0,x,t)) & \text{if } (x,t) \in I_2 \cup P_2, \end{cases} \right)$$

is the unique weak solution of (3.5) (resp (3.6)) in the class $L^2([-\ell_1, 0] \times [0, T_1])$ (resp $L^2([0, \ell_2] \times [0, T_2])$).

Idea of the proof

Step 1. Existence of solution

$$\Omega = \Big\{(u_1, v_2)/ \;\; u_1 \in L^2([-\ell_1, 0] \times [0, T_1]), \;\; v_2 \in L^2([0, \ell_2] \times [0, T_2]),$$

$$\text{and } u_1, v_2 \text{ with the same Lipschitz constant}\Big\}$$ equipped with the topology of the uniform convergence. We show

- ▷ By Ascoli-Arzela theorem, Ω is a compact set in $C^0([-\ell_1, 0] \times [0, T_1]) \times C^0([0, \ell_2] \times [0, T_2])$.
- ▷ Ω is a convex set.

For $(\widetilde{u}_1, \widetilde{v}_2) \in \Omega$, we define $(u_1, v_2) = F(\widetilde{u}_1, \widetilde{v}_2)$, and we show that

- ▷ F is continuous on Ω.
- ▷ It follows from Schauder fixed-point theorem that F has a fixed-point

Step 2. Uniqueness

Let $u_1^0 \in L^2([-\ell_1, 0])$ and $v_2^0 \in L^2([0, \ell_2])$. We assume that $u_1, u_1^{'}$ (resp $v_2, v_2^{'}$) are two weak solutions of system (3.5) (resp (3.6)).

$$\text{Assume} \quad \begin{aligned} \widehat{u}_1 &= u_1 - u_1^{'}, \\ \widehat{v}_2 &= v_2 - v_2^{'}. \end{aligned}$$

$$\text{Let } \widehat{u}_1 \in L^2([-\ell_1, 0] \times [0, T_1]), \widehat{v}_2 \in L^2([0, \ell_2] \times [0, T_2]),$$

$$\text{verify} \qquad \partial_t \widehat{u}_1 (x,t) + \lambda_1 \partial_x \widehat{u}_1(x,t) = 0, \tag{3.8}$$

$$\partial_t \widehat{v}_2(x,t) + \mu_1 \partial_x \widehat{v}_2(x,t) = 0, \tag{3.9}$$

$$\widehat{u}_1(-\ell_1, t) = \widehat{u}_1(x, 0) = \widehat{u}_1(0,t) = 0, \tag{3.10}$$

$$\widehat{v}_2(\ell_2, t) = \widehat{v}_2(x, 0) = \widehat{v}_2(0,t) = 0. \tag{3.11}$$

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$$\widehat{u}_1(-\ell_1,t) = \widehat{u}_1(x,0) = \widehat{u}_1(0,t) = 0, \tag{3.10}$$
$$\widehat{v}_2(\ell_2,t) = \widehat{v}_2(x,0) = \widehat{v}_2(0,t) = 0. \tag{3.11}$$

Multiplying in (3.8) $\big($resp (3.9)$\big)$ by $2\widehat{u}_1$ $\big($resp $2\widehat{v}_2\big)$, integrating over $(-\ell_1,0)\times(0,t)$ $\big($resp $(0,\ell_2)\times(0,t)\big)$. Thus, adding the two equations and using (3.10) and (3.11), gives

$$\|\widehat{u}_1(x,t)\|^2_{L^2\left((-\ell_1,0)\times(0,T^*)\right)} + \|\widehat{v}_2(x,t)\|^2_{L^2\left((0,\ell_2)\times(0,T^*)\right)} = 0$$

which proves the uniqueness of solutions.

$$\forall t \geq T^* - \frac{1}{c^*} \qquad u_1(-\ell_1, t) = v_2(\ell_2, t) = 0, \tag{3.4}$$

$$u_1(x,t) = \begin{cases} u_{-\ell_1}(e_{\lambda_1}(x,t)) & \text{if } (x,t) \in J_1, \\ u_1^0(\phi_{\lambda_1}(0,x,t)) & \text{if } (x,t) \in I_1 \cup P_1, \end{cases} \tag{3.7}$$

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Proposition

$$u_1(x, T^*) = 0, \qquad x \in [-\ell_1, 0], \tag{3.12}$$

$$v_2(x, T^*) = 0, \qquad x \in [0, \ell_2]. \tag{3.13}$$

In a second step, we show the finite-time stability of u_2 and v_1. The idea is to find a relation between u_1 and u_2 (resp v_1 and v_2) then deduce the stability of one from the other.
More precisely, we will try to find a function h_1 (resp h_2) such that at the point mass $(x = 0)$ we have

$$u_2(0, t) = h_1\big(u_1(0, t)\big),$$

and

$$v_1(0, t) = h_2\big(v_2(0, t)\big).$$

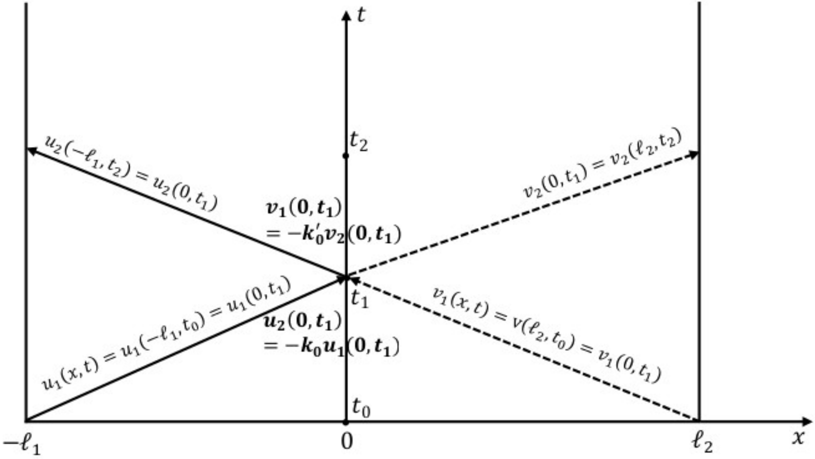

figure 3 : The invariants along the characteristic curves

- This implies that, for any arbitrary t there is an increasing sequence of time instants t_i, i=0,1,2,... such that

$$\begin{cases} u_1(-\ell_1, t_i) = u_1(0, t_{i+1}), \\ u_2(-\ell_1, t_i) = u_2(0, t_{i-1}), \end{cases} \qquad \begin{cases} v_1(\ell_2, t_i) = v_1(0, t_{i+1}), \\ v_2(\ell_2, t_i) = v_2(0, t_{i-1}). \end{cases}$$

- By the same technique used by (Coron, d'Andréa-Novel and Bastin, 2007) for some hyperbolic systems, we get implicitly the following compatibility conditions

$$u_2(0, t) = -k_0 u_1(0, t), \tag{3.14}$$

and

$$v_1(0, t) = -k_0^{'} v_2(0, t). \tag{3.15}$$

Where k_0, $k_0^{'}$ two positive constants.

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$$u_2(0, t) = -k_0 u_1(0, t), \tag{3.14}$$

and

$$v_1(0, t) = -k_0^{'} v_2(0, t). \tag{3.15}$$

Where k_0, $k_0^{'}$ two positive constants.

Corollary

Assume that u_2 and v_1 satisfy (3.14) and (3.15). Then

$$u_2(0,t) = v_1(0,t) = 0 \qquad \text{for } t \geq T^*. \tag{3.16}$$

This result is a direct application of Greenberg and Li's theorem (1984).

Remark

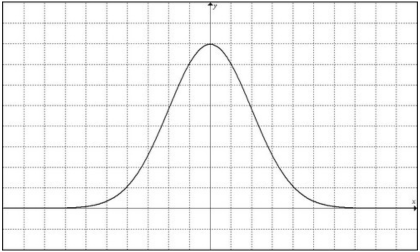
Rather than taking the compatibility conditions we apply a control p in $x = 0$, for example

$$p = -k_1 sgn(\dot{z})|\dot{z}|^{\alpha} - k_2 sgn(z)|z|^{\frac{\alpha}{2-\alpha}},$$

then, we get $z = 0$ in F.T.
In this case, by $2\dot{z}(t) = u_1(0,t) + u_2(0,t) = v_1(0,t) + v_2(0,t)$, one easily show (3.16).

Theorem

For every $t \geq T^*$, $\mathbf{z(t)}$ is equal to a constant that depends on the initials data. Moreover, the energy E_M is constant for t is large enough.



Apparition of Synchronization Phenomena

$$u_1(x, T^*) = 0, \qquad x \in [-\ell_1, 0], \qquad (3.12)$$
$$v_2(x, T^*) = 0, \qquad x \in [0, \ell_2], \qquad (3.13)$$
$$u_2(0, t) = v_1(0, t) = 0 \qquad \text{for } t \geq T^*. \qquad (3.16)$$

Idea of the proof

- $\dot{E}_M = \int_{-\ell_1}^{0} \rho_1 u_t u_{tt}\, dx + \int_{-\ell_1}^{0} \sigma_1 u_x u_{xt}\, dx + \int_{0}^{\ell_2} \rho_2 v_t v_{tt}\, dx + \int_{0}^{\ell_2} \sigma_2 v_x v_{xt}\, dx + M z_t z_{tt},$

- recall that

$$Mz_{tt} = \frac{\sqrt{\sigma_2\rho_2}}{2}\left(v_2(0,t) - v_1(0,t)\right) - \frac{\sqrt{\sigma_1\rho_1}}{2}\left(u_2(0,t) - u_1(0,t)\right),$$

- we deduce thanks to (3.12), (3.13) and (3.16) that

$$z_{tt} = 0 \qquad \forall t \geq T^* \qquad (3.17)$$

In replacing by the Riemann Invariants, we find

$$u_x(-\ell_1, t) = v_x(\ell_2, t) = 0 \qquad \text{for } t \geq T^*$$

by integration by parts, and taking into account (3.17) we deduce that, for every $t \geq T^*$

$$\dot{E}_M = \int_{-\ell_1}^{0} u_t(\rho_1 u_{tt} - \sigma_1 u_{xx})\, dx + \int_0^{\ell_2} v_t(\rho_2 v_{tt} - \sigma_2 v_{xx})\, dx = 0.$$

So, the energy E_M is conserved for $t \geq T^*$, in particular $E = E(T^*)$. Then

$$\dot{z}(t) = 0 \qquad \forall\ t \geq T^*.$$

Interpretation

Attenuation of the vibration of the two strings on either side of the mass point, while ignoring the mass position.

Since

$$z \longrightarrow z_c$$
$$t \to T^*,$$

where z_c est constante.

Problem : Synchronization phenomena

How we can change the feedbacks obtained so that

$$|z_c| \leq \varepsilon,$$

with $\varepsilon > 0$ fixed in advance.

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Attenuation of the vibration of the two strings on either side of the mass point, while ignoring the mass position.

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Finite-time stabilization by acting on the point mass

Lemma

Let the scalar control system $\dot{x} = u + g(t)$, where $g(.)$ is a finite-time perturbation. Then for t is large enough, $\dot{x} = u$ is F.T.S by choosing

$$u = -k\ sgn(x)|x|^{\alpha}, \qquad (k, \alpha) \in ((0, \infty) \times (0, 1)).$$

We will be interested in the finite-time stability of the following system

$$\begin{cases} \rho_1 u_{tt} = \sigma_1 u_{xx}, & x \in \Omega_1, \quad t > 0, \\ \rho_2 v_{tt} = \sigma_2 v_{xx}, & x \in \Omega_2, \quad t > 0, \\ z_{tt}(t) + \sigma_1 u_x(0,t) - \sigma_2 v_x(0,t) = 0, & t > 0, \\ u(-\ell_1, t) = v(\ell_2, t) = 0, & t > 0. \end{cases}$$

with the following feedbacks at the point mass $x = 0$

$$\frac{d}{dt} v_x(0,t) = -k \ sgn\big(v_x(0,t)\big)\big|v_x(0,t)\big|^{\gamma}, \tag{4.1}$$

$$p := -\sigma_1 u_x(0,t) = -k_1 sgn(\dot{z})|\dot{z}|^{\alpha} - k_2 sgn(z)|z|^{\frac{\alpha}{2-\alpha}}. \tag{4.2}$$

Then by the precedent lemma and from the equation

$$z_{tt} = \frac{1}{M}\left[\sigma_2 v_x(0,t) - \sigma_1 u_x(0,t)\right],$$

we get $z(t) = 0$ in finite time.
Let T_* the settling time of z.

Theorem

Under the family of homogeneous continuous controllers (4.1) and (4.2), the energy E_M of the system vanishes in **finite time**. More precisely, we get $u = v = 0$ in **finite time**.

Idea of the proof

- u and v are given, respectively, in terms of the initial data by d'Alembert's formula as follows

$$u(x,t) = \frac{1}{2}\big[z(t-d_1x)+z(t+d_1x)\big] + \frac{1}{2d_1}\int_{t-d_1x}^{t+d_1x} u_x(0,s)ds,$$

$$v(x,t) = \frac{1}{2}\big[z(t-d_2x)+z(t+d_2x)\big] + \frac{1}{2d_2}\int_{t-d_2x}^{t+d_2x} v_x(0,s)ds,$$

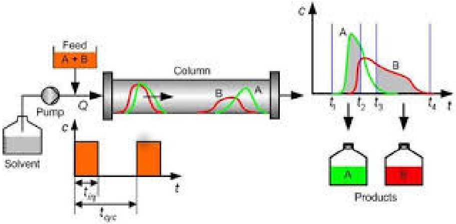
with $d_i = \dfrac{\sigma_i}{\rho_i}$ for $i = 1,2$.

- Using (4.1) and (4.2) it is easily seen that u and v vanish in finite time.
- A simple calculation of the system energy, allows us to conclude that E_M vanishes in finite time.

Simulated moving bed (SMB) chromatography

Definition

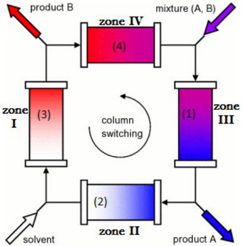
The SMB chromatography is a technique used to separate particles that would be difficult or impossible to resolve otherwise.

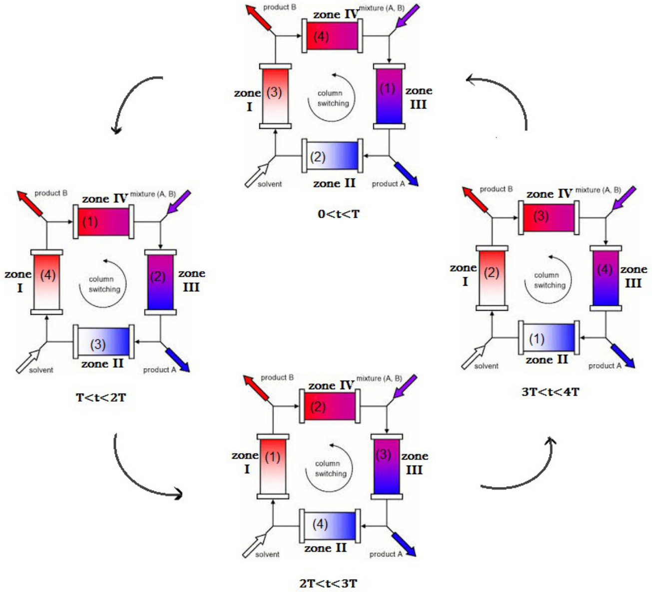


The use of many columns allows for a continuous separation with a better performance than the discontinuous single-column chromatography.

System Description

- The SMB system is divided in four zones $\{I, II, III, IV\}$.
- Each zone contains one chromatography column $i \in \{1, 2, 3, 4\}$.
- Between each zone there will be provision for 4 process steams :
  - ▷ **Two inlets** : - Feed mixture (A, B),
    - Incoming solvent.
  - ▷ **Two outlets** : - The less absorbed component (A),
    - The more absorbed component (B).




product B
zone IV
mixture (A, B)
(4)
zone I
(3)
column switching
(1)
zone III
(2)
solvent
zone II
product A
0<t<T
(1)
(4)
(2)
(3)
T<t<2T
(3)
(2)
(4)
(1)
3T<t<4T
(2)
(1)
(3)
(4)
2T<t<3T

- The switching time period is **T**
- The column length is **L**.
- $C_i^\ell(t,x) \geq 0$ is the concentration of species $\ell \in \{A, B\}$ in the column $i \in \{1,2,3,4\}$.

with
$$0 \leq x \leq L,$$
$$t \geq 0.$$

- V_1 (resp V_2) is the fluid velocity in the columns located in zones **I** and **III** (resp **II** and **IV**).
- $h^\ell > 0$ denotes the Henry coefficient.
- $V_F > 0$ is the constant fluid velocity while C_F^A, $C_F^B > 0$ are the constant species concentrations in the input flow.

The general form for the hyperbolic system of conservation laws describing the periodic SMB chromatography is given for $mT \leq t < (m+1)T, \quad m = 0, 1, 2, \ldots$ by

$$\begin{aligned} &(1+h)\partial_t C^\ell + (P^m)\Upsilon(P^m)^T \partial_x C^\ell = 0, \\ &C^\ell(t,0) = P^m K (P^m)^T C^\ell(t,L) + (P^m) U^\ell, \\ &C^\ell(0,x) = C_0^\ell(x). \end{aligned} \tag{5.1}$$

with

$$C^\ell(t,x) = \left(C_1^\ell, C_2^\ell, C_3^\ell, C_4^\ell\right)^T,$$

$$U^\ell = \left((V_F/V_1)C_F^\ell, 0, 0, 0\right)^T, \quad \Upsilon = diag\{V_1, V_2, V_1, V_2\},$$

$$K = \begin{pmatrix} 0 & 0 & 0 & V_2/V_1 \\ 1 & 0 & 0 & 0 \\ 0 & V_2/V_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Proposition

The system (5.1) has a single quasi-periodic time solution $C^* = (C^{*A}, C^{*B})$ such that $C^*(t,x) = C^*(t+4T,x)$, $x \in [0,L]$, $t \geq 0$ provided that $V_F C_F^A$ and $V_F C_F^B$ are sufficiently small.

Idea of the proof

- **Existence** : Schauder fixed-point theorem.
- **Uniqueness** : Assume that (5.1) admits two solutions C^* and $\widetilde{C}^*$, then we prove that for $\widehat{C} = C^* - \widetilde{C}^*$

$$\|\widehat{C}(t,x)\|^2_{L^2\left((mT,(m+1)T)\times(0,L)\right)} = 0.$$

- **Periodicity** : For $z(t,x) = C^*(t,x) - C^*(t+4T,x)$ we have shown without difficulty that under some conditions

$$z \equiv 0 \quad \forall (t,x) \in [mT,(m+1)T[\times[0,L].$$

Controllability around quasi-periodic trajectories

Let

$$z(t,x) = (z_1, z_2, z_3, z_4)^T$$

such that

$$z(t, x) = C(t, x) - C(t + 4T, x)$$

be the solution for the following problem

$$\partial_t z + A_m \partial_x z = 0, \tag{5.2}$$

$$z(t, 0) = P^m K (P^m)^T z(t, L) := u_m(t), \tag{5.3}$$

$$z(0, x) = 0, \quad z_t(0, x) = z^1(x). \tag{5.4}$$

with

$$u_m(t) = \begin{cases} u^1(t) & \text{if m even,} \\ u^2(t) & \text{if m odd,} \end{cases}$$

$$u^1(t) = (v_1, v_2, v_3, v_4)^T \text{ and } u^2(t) = (\overline{v}_1, \overline{v}_2, \overline{v}_3, \overline{v}_4)^T.$$

Theorem

The control system (5.2)-(5.4) is controllable in time if and only if

$$T \geq \frac{L}{4\lambda_2}.$$

Idea of the Proof

▶**Step 1.** Choose two measurable functions with values assumed to switch so that, the control activates alternating in a manner that, in each time t, only one control is active.

$$\gamma_v^m(t) = \begin{cases} 1 & \text{if m even,} \\ 0 & \text{if m odd,} \end{cases} \qquad \gamma_{\overline{v}}^m(t) = \begin{cases} 0 & \text{if m even,} \\ 1 & \text{if m odd.} \end{cases}$$

$\Rightarrow$ $$u_m(t) = u^1(t)\gamma_v^m(t) + u^2(t)\gamma_{\overline{v}}^m(t).$$

▶ **Step 2. Write the system in the characteristic form** :

- Let define the Riemann invariants of (5.2)-(5.4) by

$$w_i(t,x) = L_i z(t,x),$$

where L_i is the left eigenvector satisfying

$$L_i.A_m = \mu_i L_i,$$

- The eigenvalue μ_i is expressed as follows

$$\mu = \lambda_1 \gamma^1_{i,m}(t) + \lambda_2 \gamma^2_{i,m}(t),$$

with

$$\gamma^1_{i,m}(t) = \begin{cases} 1 & \text{if } t \in [mT,(m+1)T[\ \chi_{\{S_1 \cup S_4\}}, \\ 0 & \text{if } t \in [mT,(m+1)T[\ \chi_{\{S_2 \cup S_3\}}, \end{cases}$$

$$\gamma_{i,m}^2(t) = \begin{cases} 0 & \text{if } t \in [mT,(m+1)T[\ \chi_{\{S_1 \cup S_4\}}, \\ 1 & \text{if } t \in [mT,(m+1)T[\ \chi_{\{S_2 \cup S_3\}}. \end{cases}$$

$S_1 = \{m \text{ even }, i \text{ odd}\}$, $S_3 = \{m \text{ odd }, i \text{ odd }\}$,

$S_2 = \{m \text{ even }, i \text{ even}\}$, $S_4 = \{m \text{ odd }, i \text{ even }\}$.

$\Rightarrow$ Each Riemann invariant $w_i(t,x)$ is a solution of the scalar advection problem

$$\partial_t w_i + \mu_i \partial_x w_i = 0, \tag{5.5}$$
$$w_i(t,0) = L_i(v_i(t)\gamma_v^m(t) + \overline{v}_i(t)\gamma_{\overline{v}}^m(t)), \tag{5.6}$$
$$w_i(0,x) = w_i^0(x) = L_i z_i^0(x) = 0, \tag{5.7}$$
$$\partial_t w_i(x,0) = w_i^1(x) = L_i z_i^1(x). \tag{5.8}$$

▶ **Step 3. Give the explicit solution of the problem** (5.5)-(5.8) for every $t \in [mT, (m+1)T[$.

- If m is **even**, the solution is

$$w_i(t,x) = \begin{cases} w_i^0(x-\mu_i t) & \text{if } x-\mu_i t \geq 0, \\ L_i.v_i(t-\frac{x}{\mu_i}) & \text{if } x-\mu_i t < 0. \end{cases} \quad (5.9)$$

- In the case when m is **odd** the solution expression is

$$w_i(t,x) = \begin{cases} w_i^0(x-\mu_i t) & \text{if } x-\mu_i t \geq 0, \\ L_i.\overline{v}_i(t-\frac{x}{\mu_i}) & \text{if } x-\mu_i t < 0. \end{cases} \quad (5.10)$$

Step 4. The time of controllability

- For the case $m = 0$ i.e $0 \leq t < T$, the explicit solution of w_i, for $i = 1, 2, 3, 4$ is given by (5.9).
  Hence, based on Coron's proof, it is easily shown that

$$w_i(T, x) = v_i(T - \frac{x}{\mu}) = w_i^1(x) \qquad \text{for} \qquad T \geq \frac{L}{\lambda_2}.$$

- We treat the case $m = 1$ $(T \leq t < 2T)$. By the same aspect we have

$$w_i(T, x) = \overline{v}_i(T - \frac{x}{\mu}) = w_i^1(x) \qquad \text{for} \qquad T \geq \frac{L}{2\lambda_2}.$$

- By iterative manner, for $t \in [mT, (m+1)T[$, it is easily shown that , the system is controllable for $T \geq \dfrac{L}{(m+1)\lambda_2}$.

- The **periodicity** of the solution $(w_i(t,x) = w_i(t+4T, x))$ **intervenes** to **reduce** the **time of controllability**, thus

$$T \geq \frac{L}{4\lambda_2}.$$

Finite-time stabilizability of the SMB chromatography

In order to show the F.T.S around the trajectory C^*, we define the Riemann coordinates as follows

$$R_i = (1+h)(C_i - C_i^*), \qquad 1 \leq i \leq 4.$$

Then, the quasi-periodic linear system (5.1) is written

$$\partial_t R + \Lambda_m \partial_x R = 0, \tag{5.11}$$

$$R(t,0) = K_m R(t,L), \tag{5.12}$$

$$R(0,x) = R^0(x). \tag{5.13}$$

with

$$R(t,x) = (R_1, R_2, R_3, R_4)^T$$

$$\Lambda_m = (P^m)\Lambda(P^m)^T, \;\; K_m = (P^m)K(P^m)^T, \;\; \Lambda = diag\{\lambda_1, \lambda_2, \lambda_1, \lambda_2\}.$$

- To get this F.T.S, we will add boundary feedbacks as follows

$$\frac{d}{dt}R_1(t,0) = -ksgn(R_1(t,0))|R_1(t,0)|^{\alpha}, \qquad (5.14)$$
$$\frac{d}{dt}R_3(t,0) = -ksgn(R_3(t,0))|R_3(t,0)|^{\alpha}. \qquad (5.15)$$

Let $T^* = max(\frac{|R_j^0(0)|^{1-\alpha}}{(1-\alpha)K}), \quad j = 1, 3.$

Thus,

$$R_j(t,0) = 0 \qquad \forall t \geq T^*.$$

with $\quad (K,\alpha) \in (0,\infty) \times (0,1).$

Proposition

Under the feedback laws (5.14)-(5.15) that can be also **discontinuous** or **bounded**, the periodic solution $C^*(t,x)$ of the system (5.1) vanishes for $t \geq T^*$.

Idea of the proof

- Using the **characteristic method** in particular the **explicit solution** of (5.11)-(5.13), we prove that

$$R_j(t,x) = 0 \qquad \forall t \geq T^*.$$

- The switching boundary condition (5.12) is equivalent to

$$R_1(t,0) = (P^m)\frac{V_2}{V_1}(P^m)^T R_4(t,L), \tag{5.16}$$
$$R_2(t,0) = R_1(t,L), \tag{5.17}$$
$$R_3(t,0) = (P^m)\frac{V_2}{V_1}(P^m)^T R_2(t,L), \tag{5.18}$$
$$R_4(t,0) = R_3(t,L). \tag{5.19}$$

- We have from the feedback (5.14), and the equality (5.17) :

$$R_2(t,0) = 0 \quad \forall t \geq T^*, \tag{5.20}$$

$\Rightarrow$ From the characteristic method which says that each solution R_i of (5.11)-(5.13) is constant along its characteristic curves. Using (5.20) one can deduce that

$$R_2(t,x) = 0 \qquad \forall t \geq T^*.$$

- By similar way, from (5.19) and the feedback (5.15) we show that

$$R_4(t,x) = 0 \ \ \forall t \geq T^*.$$

Thank you for your attention