The obstacle problem for the porous medium equation

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The porous medium equation

Porous medium equation (PME):

$$\partial_t u - \Delta u^m = 0, \qquad m > 0,$$

with $u = u(x,t) \geq 0$.

- Model for the motion of a fluid in a porous medium
- Model for an explosion (like a bomb)
- Useful for creating a perfect coffee (project of illy with the university of Florence)

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The Espresso Coffee Problem

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ABSTRACT. We review the results of a long research project on the espresso coffee brewing process, carried out jointly by the industrial mathematics research group at the Department of Mathematics "U. Dini" of the University of Florence and the Italian company illycaffè s. p. a. (Trieste).

We describe the main experimental steps of the research and present the mathematical models developed in order to interpret the data correctly. The models are of increasing complexity, the first being confined to the mechanical phenomena (experiments performed with cold water), while the most comprehensive includes the influence of dissolution. Particular emphasis is put on the fact that the process deviates significantly from usual filtration in standard porous media, although the classical Darcy's law is assumed as the fundamental flow mechanism.

The porous medium equation

Porous medium equation (PME):

$$\partial_t u - \Delta \underbrace{u^m}_{=u^{m-1}u} = 0, \qquad m > 0.$$

Behavior of the solution depends on m in a crucial way:

- $m = 1$: classical heat equation (smooth solutions)
- $m > 1$ (slow diffusion equation):
  - fundamental solution has compact support at any time
  - degenerate behavior: $|u|^{m-1} \to 0$ as $u \to 0$
- $m < 1$ (fast diffusion equation):
  - fundamental solution has no comp. supp. for any time $t > 0$
  - singular behavior: $|u|^{m-1} \to \infty$ as $u \to 0$

Some known results for the PME

- Existence of weak and very weak solutions
- Asymptotic behavior (decay when $t \to \infty$)
- Hölder continuity of solutions
- Harnack inequality
- Behavior of the free boundary
- u^{m-1} is differentiable (even $C^{1,\alpha}$) after some waiting time

List of authors (not exhaustive): Barenblatt, Aaronson, Caffarelli, Wolanski, Vázquez, DiBenedetto, Gianazza, Vespri, Kinnunen, Lindqvist, Liskevich, Skrypnik, Duzaar

Obstacle problem for the PME

Obstacle: $\psi \colon \Omega_T \to \mathbb{R}_+$

Lateral boundary datum: $g \colon \Omega_T \to \mathbb{R}_+, \quad g \geq \psi$ on Ω_T

Initial boundary datum: $u_o \colon \Omega \to \mathbb{R}_+, \quad u_o \geq \psi(\cdot, 0)$ on Ω

The obstacle problem:

$$\begin{cases} u \geq \psi \ \text{ on } \Omega_T, \\ u \text{ supersolution to the PME, i.e. } \partial_t u - \Delta u^m \geq 0 \ \text{ on } \Omega_T, \\ u \text{ solution to the PME on } \{u > \psi\}. \end{cases}$$

Formally the problem reads as:

$$\begin{cases} \max\{-(\partial_t u - \Delta u^m), \psi - u\} = 0 & \text{in } \Omega_T, \\ u = g & \text{on } \partial\Omega \times (0, T). \\ u(\cdot, 0) = u_o & \text{on } \Omega. \end{cases}$$

Known existence results

- Alt-Luckhaus: double obstacle problem for $\partial_t b(u) - \Delta u = 0$, with $\psi_1 \leq u \leq \psi_2$, and $\psi_1, \psi_2 \in L^\infty$;
- Brezis, Lions: obstacle problem for the parabolic p-Laplace $\partial_t u - \mathrm{div}(|\nabla u|^{p-2} \nabla u) = 0$, with ψ not increasing in time;
- B.-Duzaar-Mingione, Scheven, Lindqvist-Parviainen: obstacle problem for the parabolic p-Laplace, no monotonicity assumption on ψ.

Variational inequality

$u \geq \psi$ satisfies

$$\int_{\Omega_T} \left[\partial_t u (v^m - u^m) + \nabla u^m \cdot (\nabla v^m - \nabla u^m) \right] dz \geq 0$$

for any $v \geq \psi$ with $v = u$ on $\partial_{\text{par}} \Omega_T$.

- u supersolution: $v^m = u^m + \varphi$, with $\varphi \geq 0$, $\varphi = 0$ on $\partial_{\text{par}} \Omega_T$

$$\Rightarrow \underbrace{\int_{\Omega_T} \left[\partial_t u \varphi + \nabla u^m \cdot \nabla \varphi \right] dz}_{= \int_{\Omega_T} (\partial_t u - \Delta u^m) \varphi \, dz} \geq 0$$

- u solution on $\{u > \psi\}$: $v^m = u^m \pm \varphi$, with $\operatorname{spt} \varphi \subset \{u > \psi\}$, $u^m \pm \varphi \geq \psi^m$

$$\Rightarrow \underbrace{\int_{\Omega_T} \left[\partial_t u \varphi + \nabla u^m \cdot \nabla \varphi \right] dz}_{= \int_{\Omega_T \cap \operatorname{spt} \varphi} (\partial_t u - \Delta u^m) \varphi \, dz} = 0$$

Strong solutions

Function space for solutions:

$$K_\psi(\Omega_T) := \Big\{ v \in C^0\big([0,T]; L^{m+1}(\Omega)\big) : v^m \in L^2\big(0,T; H^1(\Omega)\big), \\ v \geq \psi \text{ a.e. on } \Omega_T \Big\}.$$

Definition: $u \in K_\psi(\Omega_T)$ with $u = g$ on $\partial\Omega \times (0,T)$, $u(\cdot, 0) = u_o$ on Ω and $\partial_t u \in L^2(0,T; H^{-1}(\Omega))$ is a strong solution to the obstacle problem if

$$\int_{\Omega_T} \big[\langle \partial_t u, v^m - u^m \rangle + \nabla u^m \cdot (\nabla v^m - \nabla u^m) \big] \, dz \geq 0$$

holds true for any $v \in K_\psi(\Omega_T)$ with $v = g$ on $\partial\Omega \times (0,T)$.

$\langle \cdot, \cdot \rangle$: duality pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$

Existence of strong solutions

Theorem (B., Lukkari, Scheven).

Let $m > \frac{n-2}{n}$ and $\psi, \psi^m, g, g^m, u_o, u_o^m$ be smooth. Then, there exists a strong solution to the obstacle problem which is locally Hölder continuous.

Nonsmooth obstacles

$$\psi^m \in L^2(0,T;H^1(\Omega)), \quad \partial_t \psi^m \in L^{\frac{m+1}{m}}, \quad \psi^m(\cdot,0) \in H^1(\Omega)$$

Definition: $u \in K_\psi(\Omega_T)$ with $u = g$ on $\partial\Omega \times (0,T)$ is a weak solution to the obstacle problem if

$$\begin{aligned}
&\int_{\Omega_T} \left[\nabla u^m \cdot (\nabla v^m - \nabla u^m) + (v-u)\partial_t v^m\right] dz \\
&\quad \geq \int_{\Omega\times\{T\}} \left[u v^m - \tfrac{1}{m+1}\left(u^{m+1} + v^{m+1}\right)\right] dx \\
&\qquad - \int_{\Omega} \left[u_o v^m(0) - \tfrac{1}{m+1}\left(u_o^{m+1} + v^{m+1}(0)\right)\right] dx
\end{aligned}$$

for any $v \in K_\psi(\Omega_T)$ with $v = g$ on $\partial\Omega \times (0,T)$ and $\partial_t v^m \in L^{\frac{m+1}{m}}(\Omega_T)$.

Existence of weak solutions

Theorem (B., Lukkari, Scheven).

Let $m > \frac{n-2}{n}$ and ψ, g, u_o satisfy

$$\psi^m, g^m \in L^2(0,T;H^1(\Omega)), \quad \partial_t \psi^m, \partial_t g^m \in L^{\frac{m+1}{m}}(\Omega_T),$$
$$\psi^m(\cdot,0), u_o^m \in H^1(\Omega).$$

Then, there exists a weak solution to the obstacle problem.

Regularity

Question: How regular are solutions to the obstacle problem?

Heuristics:

- On $\{u = \psi\}$: u is as regular as the obstacle
- On $\{u > \psi\}$: u is a solution to the PME

$\Rightarrow$ Hölder continuity is the best regularity expected

Regularity result

Theorem (B., Lukkari, Scheven).

Let $m \geq 1$ and ψ be Hölder continuous and u be a weak solution to the obstacle problem. Then, u is locally Hölder continuous.

Proof (local boundedness)

u is locally bounded on Ω_T with the estimate

$$\sup_{Q_{\rho,\theta}(z_o)} u \le c \max\left\{ \sup_{Q_{2\rho,2\theta}(z_o)} \psi, \left(\frac{1}{\rho^{n+2}} \int_{Q_{2\rho,2\theta}(z_o)} u^{2m}\, dz \right)^{\frac{1}{m+1}}, \left(\frac{\rho^2}{\theta} \right)^{\frac{1}{m-1}} \right\}.$$

Proof (intrinsic geometry)

Intrinsic geometry

- $$\partial_t u - \underbrace{\Delta(u^{m-1}u)}_{\sim \mu^{m-1}\Delta u} = 0, \qquad \text{on } Q_{\rho,s}$$
- Assume $u \sim \mu > 0$ on $Q_{\rho,s}$ (for instance $\frac{1}{2}\mu \le u \le 2\mu$)
- Rescale the solution on $Q_{1,1}$:

$$v(y,\tau) := u(\rho y, s\tau), \qquad \text{on } Q_{1,1}$$

$$\Rightarrow \partial_\tau v = s\partial_t u \sim s\mu^{m-1}\Delta u = \frac{s\mu^{m-1}}{\rho^2}\Delta v \quad \Rightarrow \quad s = \mu^{1-m}\rho^2$$

- $\Rightarrow$ Homogeneous behavior on intrinsic cylinders of the form $Q = Q_{\rho,\mu^{1-m}\rho^2}$, provided $u \sim \mu$ on Q.

Proof (Two regimes)

Fix $z_o \in \Omega_T$ and consider cylinders with vertex z_o. Distinguish between two types of cylinders.

- **Nondegenerate regime:** $Q = Q_{\rho,\mu^{1-m}\rho^2}(z_o)$ satisfies (NR) if

$$\sup_Q u \le \mu \le 2 \inf_Q u \qquad (*)$$

$\overset{intr.\ geometry}{\Longrightarrow}$ u behaves like a solution to the obstacle problem for the heat equation on Q

- **Degenerate regime:** If $(*)$ is not satisfied, construct a sequence of nested clinders $Q_i = Q_{\rho_i,\theta_i\rho_i^2}(z_o)$ such that either

  there exists $i_o \in \mathbb{N}_0$ such that Q_{i_o} satisfies (NR)

  or

  $\operatorname{osc}_{Q_i} u \to 0,$ as $i \to \infty$ in a quantified way

  $(u(z_o) = 0$ in this case)