A determination of optimal ship forms based on Michell's wave resistance

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Outline

1. Michell's wave resistance formula
2. Formulation of the optimization problem
3. Theoretical results
4. Numerical results
5. About the case $\epsilon = 0$
6. Conclusion and perspectives

Traditionally, the resistance of water to the motion of a ship is represented as

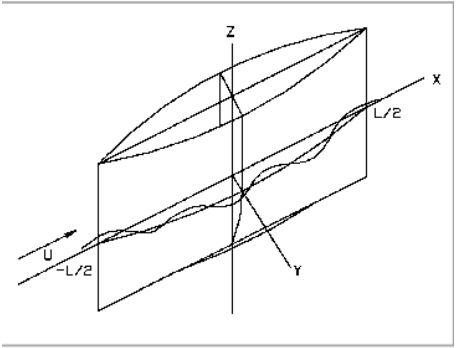
$$R_{water} = R_{wave} + R_{viscous},$$

with

$$R_{viscous} = R_{frictional} + R_{eddy}.$$

(AFP / N. Lambert photography)

(Shutterstock.com/ AlexKol photography)



Z
X
L/2
U
-L/2
Y

Consider a ship moving with constant velocity U on the surface of an unbounded fluid.

- coordinates xyz are fixed to the ship
- the xy-plane is the water surface, z is vertically downward

The (half-)immerged hull surface is represented by a continuous nonnegative function

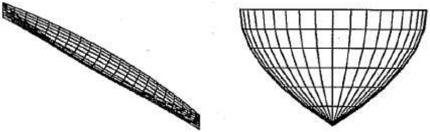
$$y = f(x,z) \geq 0, \quad x \in [-L/2, L/2], \quad z \in [0, T],$$

where L is the length and T is the draft of the ship. We also assume

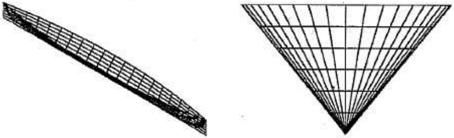
$$f(\pm L/2, z) = 0 \quad \forall z \text{ and } \quad f(x, T) = 0 \quad \forall x.$$

Example: for a Wigley hull with beam B, we have $f(x,z) = (B/2)S(z)(1 - 4x^2/L^2)$ with

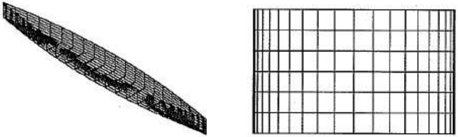
$$S(z) = \begin{cases} 1 - (z/T)^2 & \text{(parabolic cross section)} \\ 1 - z/T & \text{(triangular cross section)} \\ 1 & \text{(rectangular cross section).} \end{cases}$$



(a) Parabolic in shape



(b) Triangular in shape



(c) Rectangular in shape

Michell's formula (1898) reads:

$$R_{Michell} = \frac{4\rho g^2}{\pi U^2} \int_1^\infty (I(\lambda)^2 + J(\lambda)^2) \frac{\lambda^2}{\sqrt{\lambda^2 - 1}} d\lambda, \qquad (1)$$

with

$$I(\lambda) = \int_{-L/2}^{L/2} \int_0^T \frac{\partial f(x,z)}{\partial x} \exp\left(-\frac{\lambda^2 gz}{U^2}\right) \cos\left(\frac{\lambda gx}{U^2}\right) dxdz, \qquad (2)$$

$$J(\lambda) = \int_{-L/2}^{L/2} \int_0^T \frac{\partial f(x,z)}{\partial x} \exp\left(-\frac{\lambda^2 gz}{U^2}\right) \sin\left(\frac{\lambda gx}{U^2}\right) dxdz. \qquad (3)$$

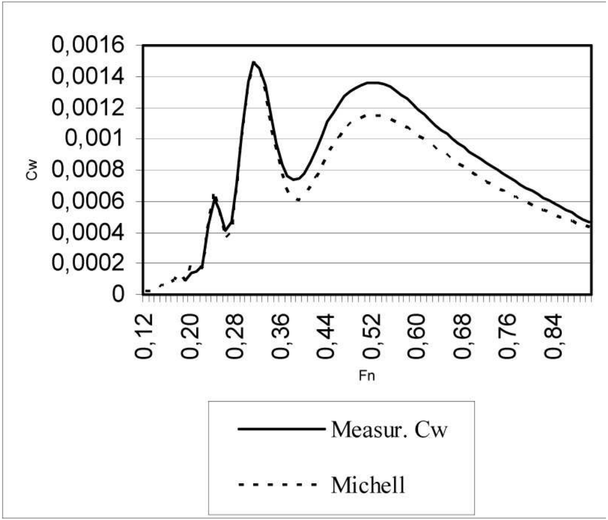
- U (in $\mathrm{m} \cdot \mathrm{s}^{-1}$) is the speed of the ship
- ρ (in $\mathrm{kg} \cdot \mathrm{m}^{-3}$) is the (constant) density of the fluid
- g (in $\mathrm{m} \cdot \mathrm{s}^{-2}$) is the standard gravity.

$R_{Michell}$ has the dimension of a force. λ has no dimension and $\lambda = 1/\cos\theta$ where θ is the angle at which the wave is propagating.

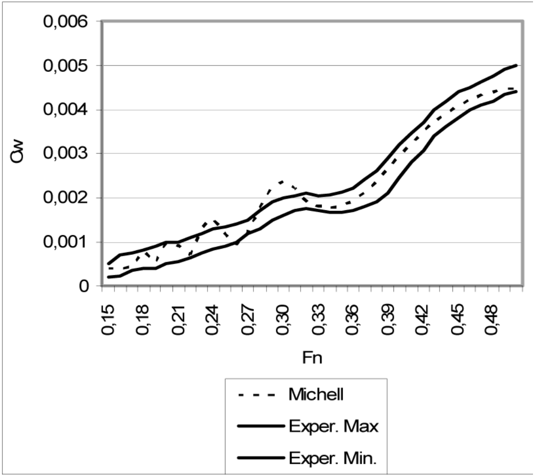
- The fluid is incompressible, inviscid, the flow is irrotational
- A steady state has been reached
- Linearized theory (flow potential with linearized boundary conditions)
- Thin ship assumptions: $|\partial_x f| << 1$, $|\partial_z f| << 1$.

Experiments starting in the 1920's (**Wigley, Weinblum**): reasonable good agreement between theory and experiment (**Gotman'02**). Typical values for Wigley: $L/B \approx 10$ and $T/B = 1.5$.

The following figures represent the **wave coefficient** $C_W = 2R_{wave}/(\rho U^2 A)$ (with A the wetted surface of the hull) in terms of the **Froude number** $F = U/\sqrt{gL}$.



Comparison Michell and experimental data (**Weinblum'52**)



Comparison Michell and experimental data (parabolic Wigley model, **Bai'79**)

Derivation of Michell's formula (sketch)

In the coordinates xyz fixed to the ship, we have $\bar{U} = -U + u$, where u is the perturbed velocity flow. We seek a potential flow Φ (i.e. with $u = \nabla\Phi$), even with respect to y, which satisfies in $D = \mathbf{R}_x \times (\mathbf{R}_+)_y \times (\mathbf{R}_+)_z$:

$$\Delta\Phi = 0 \text{ in } D \quad (4)$$

$$\partial_{xx}\Phi - (g/U^2)\partial_z\Phi = 0, \quad z = 0 \quad (5)$$

$$\partial_y\Phi = -Uf_x, \quad y = 0^+ \quad (6)$$

$$\nabla\Phi \to 0 \text{ as } x \to +\infty. \quad (7)$$

Φ can be computed explicitly by means of Green functions and Fourier transform.

Let $\Omega = (-L/2, L/2) \times (0, T)$. Then the wave resistance reads

$$R_{wave} = -2 \int_{\Omega} \delta p f_x(x, z) dxdz,$$

where δp is the difference of pressure due to the ship. (Notice that R_{wave} is the drag force in this linearized model).
From Φ, we derive δp so that

$$R_{wave} = -2\rho U \int_{\Omega} \Phi_x(x, 0, z) f_x(x, z) dxdz.$$

Computing, we obtain $R_{wave} = R_{Michell}$ as given by (1).

Formulation of the optimization problem

1st idea: finding a ship of minimal wave resistance among admissible functions $f : \Omega \to \mathbf{R}_+$, for a constant speed U and a given volume V of the hull.
$f \mapsto R_{Michell}(f)$ is a positive semi-definite quadratic functional, **but** the problem above is ill-posed (**Sretensky'35, Krein'52**). In particular, it is underdetermined.

Most authors proposed to add conditions and/or to work in finite dimension (**Weinblum'56, Kostyukov'68,...**)
Another approach, that we chose: add a **regularizing** term which represents the viscous resistance (**Lian-en'84, Michalski et al'87**)

We define

$$v = g/U^2 > 0 \quad \text{and} \quad T_f(v,\lambda) = I(\lambda) - iJ(\lambda),$$

where I and J are given by (2)-(3). Then

$$T_f(v,\lambda) = \int_{-L/2}^{L/2} \int_0^T \partial_x f(x,z) e^{-\lambda^2 vz} e^{-i\lambda vx} dxdz, \tag{8}$$

and $R_{Michell}$ can be written

$$R(v,f) = \frac{4\rho g v}{\pi} \int_1^\infty |T_f(v,\lambda)|^2 \frac{\lambda^2}{\sqrt{\lambda^2 - 1}} d\lambda. \tag{9}$$

For the numerical computation, we let $\Lambda >> 1$ and consider

$$R^\Lambda(v, f) = \frac{4\rho g v}{\pi} \int_1^\Lambda |T_f(v, \lambda)|^2 \, d\mu(\lambda), \tag{10}$$

where μ is a nonnegative and finite borelian measure on $[1, \Lambda]$. Typically,

$$d\mu(\lambda) = \frac{\lambda^2}{\sqrt{\lambda^2 - 1}} d\lambda,$$

or a numerical integration of this weight.

For the viscous resistance, we propose

$$R_{viscous} = \frac{1}{2}\rho U^2 C_{vd} A,$$

where C_{vd} is the (constant) viscous drag coefficient, and A is the wetted surface area given by

$$A = 2\int_{\Omega}\sqrt{1 + |\nabla f(x,z)|^2}\,\mathrm{d}x\mathrm{d}z\,.$$

For instance, the ITTC 1957 model-ship correlation line gives

$$C_{vd} = 0.075/(\log_{10}(Re) - 2)^2,$$

where $Re = UL/\nu$ is the Reynolds number and ν the kinematic viscosity of water.
For small ∇f (thin ship assumption)

$$R_{viscous} \approx \rho U^2\, C_{vd}\, \left(\int_{\Omega} dxdz + \frac{1}{2}\int_{\Omega}|\nabla f(x,z)|^2\,\mathrm{d}x\mathrm{d}z\right)\,.$$

By setting

$$\epsilon = \frac{1}{2}\,\rho U^2\, C_{vd}\,, \tag{11}$$

and dropping the constant term, we obtain

$$R^*_{viscous} = \epsilon \int_\Omega |\nabla f(x,z)|^2\,\mathrm{d}x\mathrm{d}z\,.$$

The **total water resistance** functional $N^{\Lambda,\epsilon}(v,\cdot)$ is

$$N^{\Lambda,\epsilon}(v,f) := R^\Lambda(v,f) + \epsilon \int_\Omega |\nabla f(x,z)|^2 dxdz.$$

The function space is

$$H = \left\{ f \in H^1(\Omega) \ : f(\pm L/2, \cdot) = 0 \text{ and } f(\cdot, T) = 0 \text{ a.e. } \right\},$$

Let $V > 0$ be the (half-)volume of an immerged hull. The set of admissible functions is

$$C_V = \left\{ f \in H \ : \ \int_\Omega f(x,z)dxdz = V \text{ and } f \geq 0 \text{ a.e. in } \Omega \right\}.$$

Notice that C_V is a closed convex subset of H.
NB: the volume is proportional to the *displacement* of the ship.

The optimization problem

Our **optimization problem** $\mathcal{P}^{\Lambda,\epsilon}$ reads: for a given Kelvin wave number v and for a given volume $V > 0$, find the function $f^\star$ which minimizes the total resistance $N^{\Lambda,\epsilon}(v, f)$ among functions $f \in C_V$.

Recall that

$$N^{\Lambda,\epsilon}(v, f) := R^{\Lambda}(v, f) + \epsilon \int_{\Omega} |\nabla f(x, z)|^2 dxdz$$

and

$$v = g/U^2.$$

In short, "minimize the (total) drag for a given displacement".

Well-posedness

The parameters $\rho > 0$, $g > 0$, $V > 0$, $\Lambda > 0$, $v > 0$ and $\epsilon > 0$ are fixed (unless otherwise stated).

Theorem (Dambrine, P. & Rousseaux (to appear))

Problem $\mathcal{P}^{\Lambda,\epsilon}$ has a unique solution $f^{\epsilon,v} \in C_V$. Moreover, $f^{\epsilon,v}$ is even with respect to x.

- Existence by a minimizing sequence
- Uniqueness by strict convexity
- Symmetry thanks to the symmetry of $R_{Michell}$ through $x \mapsto -x$.

Remark: also valid if $\Lambda = \infty$ with $R_{Michell}$ instead of R^{Λ}.

Continuity of the optimum with respect to v

Theorem (Dambrine, P. & Rousseaux (to appear))

Let $\bar{v} > 0$. Then $f^{\epsilon,v}$ converges strongly in H to $f^{\epsilon,\bar{v}}$ as $v \to \bar{v}$.

idea of proof

- $N^{\Lambda,\epsilon}(v,\cdot)$ Γ-converges to $N^{\Lambda,\epsilon}(\bar{v},\cdot)$ for the weak topology in H, thanks to $\Lambda < \infty$.
- strong convergence thanks to the convergence of the H^1-norm

Remark: result also valid if $\epsilon > 0$ depends continuously on v.

Regularity of the solution

Theorem (Dambrine, P. & Rousseaux (to appear))

The solution $f^{\epsilon,v}$ of problem $\mathcal{P}^{\Lambda,\epsilon}$ belongs to $W^{2,p}(\Omega)$ for all $1 \leq p < \infty$. In particular, $f^{\epsilon,v} \in C^1(\overline{\Omega})$.

Sketch of proof (regularity)

The problem is a perturbation of an obstacle-type problem for the Dirichlet energy

- The Euler-Lagrange equation gives a variational inequality for an obstacle-type problem
- By a standard result, the regularity of the obstacle problem is given by the regularity of unconstrained problem
- The unconstrained problem reads $-\Delta f^{\epsilon,v} = w$ with $w \in L^\infty(\Omega)$, and homogeneous Dirichlet BC on 3 sides + no-flux BC on 1 side of the rectangle, hence $f^{\epsilon,v} \in W^{2,p}(\Omega)$ for all $1 \leq p < \infty$.

Numerical methods

- Q^1 finite element discretization of the space H
- the integrals

$$J(\lambda) = \int_{-L/2}^{L/2} \int_0^T \frac{\partial f(x,z)}{\partial x} \exp\left(-\frac{\lambda^2 gz}{U^2}\right) \sin\left(\frac{\lambda gx}{U^2}\right) dxdz. \quad (12)$$

  are computed exactly on the basis functions
- the antisymmetric contribution $I(\lambda)$ is dropped (since the minimizer is even with respect to x).
- for the last integral $R_{Michell}$, we use a midpoint formula which preserves nonnegativity of the quadratic form + Tarafder's trick to improve accuracy
- Uzawa algorithm for the resolution

Numerical test

- $\rho = 1000\,\mathrm{kg}\cdot\mathrm{m}^{-3}$, $g = 9.81\,\mathrm{m}\cdot\mathrm{s}^{-2}$, $L = 2\,\mathrm{m}$, $T = 20\,\mathrm{cm}$, $V = 0.03\,\mathrm{m}^3$.
- $N_x = 100$ and $N_z = 20$
- $\epsilon = \frac{1}{2}\rho C_{vd} U^2$ with $C_{vd} = 0.01$
- $\boxed{Fr = U/\sqrt{gL}}$

Scaling

Let $T = \alpha \bar{T}$ / $L = \alpha \bar{L}$ / $f = \alpha \bar{f}$ / $x = \alpha \bar{x}$ / $z = \alpha \bar{z}$
The wave resistance reads

$$R(v, f) = \alpha^3 \bar{R}(\alpha v, \bar{f}),$$

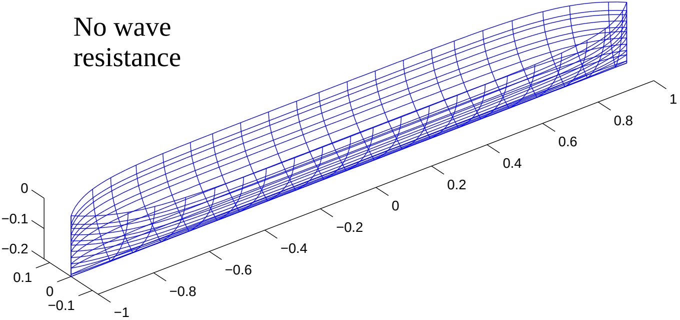
where $v = g/U^2$. It is natural to set $\bar{v} = \alpha v$, i.e. $U = \sqrt{\alpha}\bar{U}$, and with this choice,

$$Fr = U/\sqrt{gL} = \bar{Fr} = \bar{U}/\sqrt{g\bar{L}}.$$

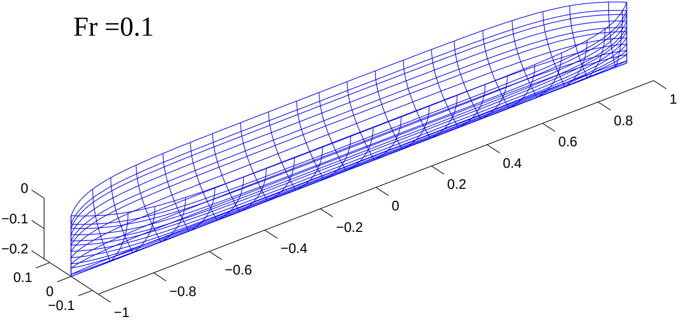
The viscous drag reads

$$\frac{1}{2}\rho U^2 C_{vd} \int_{\Omega} |\nabla f(x,z)|^2 dxdz = \alpha^3 \frac{1}{2}\rho \bar{U}^2 C_{vd} \int_{\bar{\Omega}} |\nabla \bar{f}(\bar{x},\bar{z})|^2 d\bar{x}d\bar{z}.$$

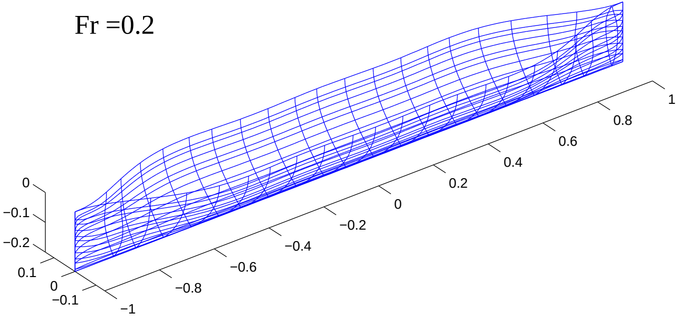
No wave resistance



Fr =0.1

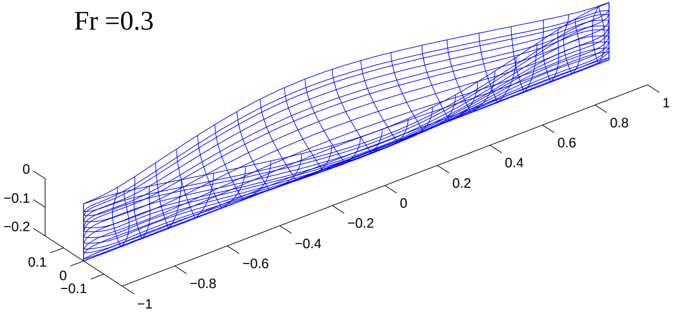


Fr =0.2

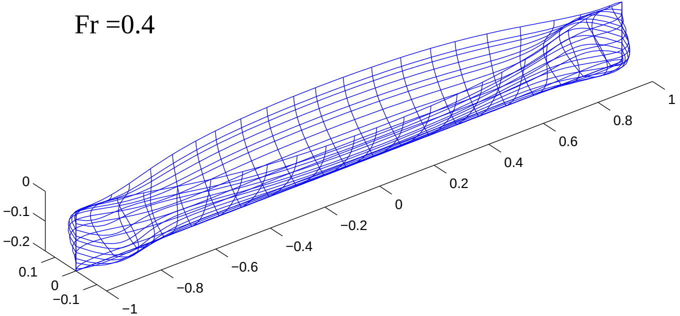


0
−0.1
−0.2
0.1
0
−0.1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

Fr =0.3



0
−0.1
−0.2
0.1
0
−0.1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


Fr =0.4
0
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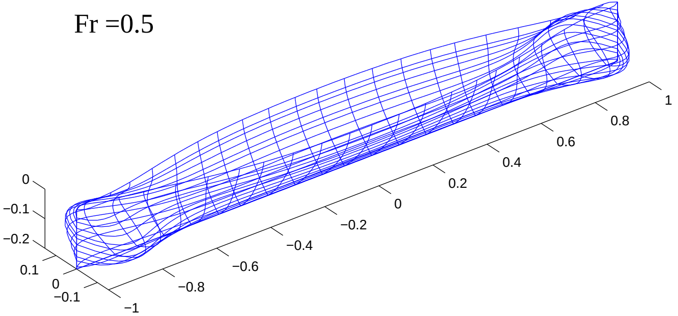
−0.2
0.1
0
−0.1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


Fr =0.5
0
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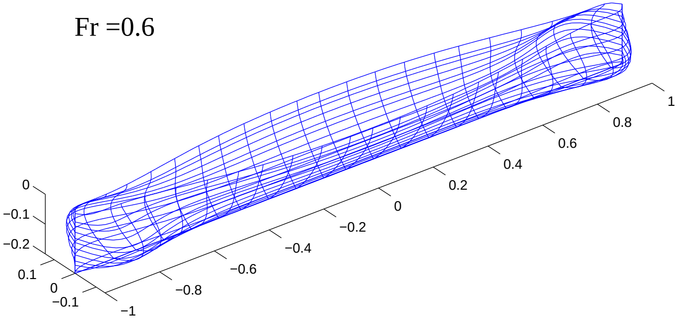
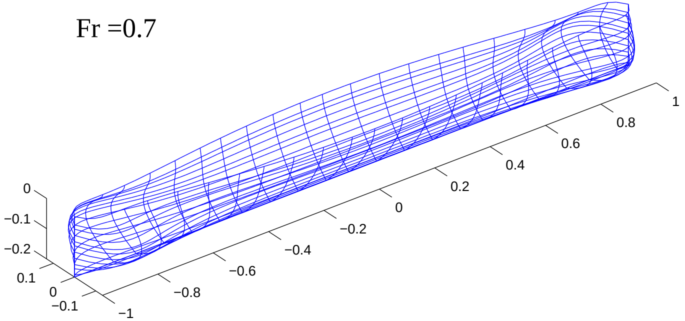
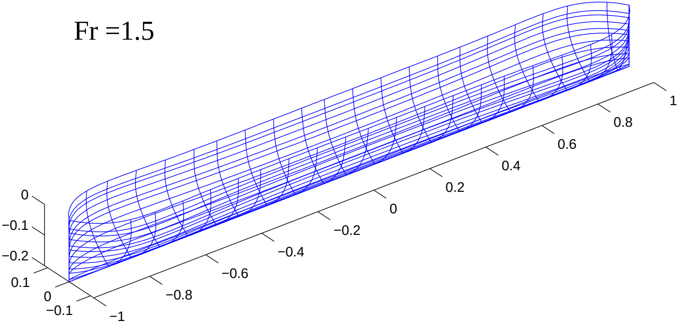
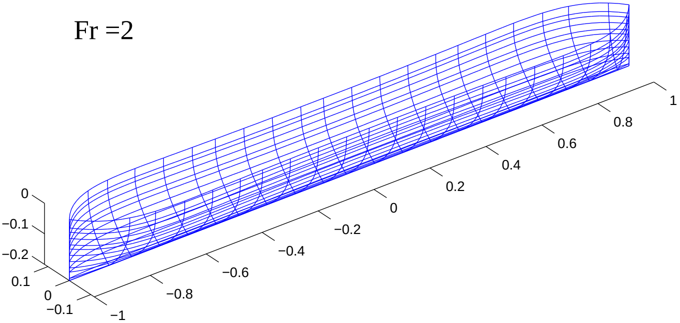
−0.2
0.1
0
−0.1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

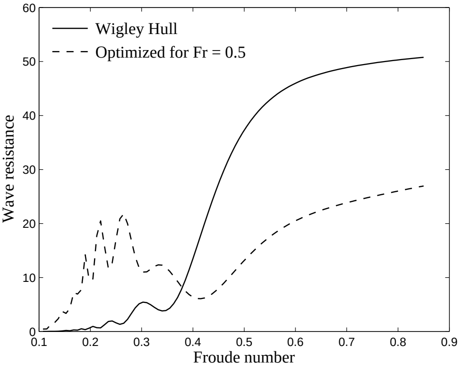

Fr =0.6
0
−0.1
−0.2
0.1
0
−0.1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


Fr =0.7
0
−0.1
−0.2
0.1
0
−0.1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


Fr =1.5
0
−0.1
−0.2
0.1
0
−0.1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


Fr =2
0
−0.1
−0.2
0.1
0
−0.1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

The bulbous bow of "Harmony of the Seas" (AFP / G. Gobet photo)
Speed : 20 knots / Length : 362m / Fr=0.17 (/T=9.1m / B=47m)
ITTC 1957 gives $C_{vd} = 0.0013$



Comparison with a Wigley hull

About the case $\epsilon = 0$

In this section, we assume

$$R^{\Lambda}(v, f) = \frac{4\rho g v}{\pi} \int_1^{\Lambda} |T_f(v, \lambda)|^2 \frac{\lambda^2}{\sqrt{\lambda^2 - 1}} d\lambda,$$

with $1 < \Lambda \leq \infty$ (i.e. "true" Michell wave resistance, or truncated Michell wave resistance).

Proposition (Krein'52)

Let $v > 0$. For all $f \in C_V$, $R^{\Lambda}(v, f) > 0$. More precisely,

$$\inf_{f \in C_V} R^{\Lambda}(v, f) > 0.$$

$\Rightarrow$ There is no ship with wave resistance equal to 0.

However, this is possible if $\boxed{L = \infty}$ (endless caravan of ships).
Indeed, choose

$$f(x,z) = g(x)h(z), \quad g(x) = \frac{\sin^2(ax)}{ax^2}$$

and h arbitrary. Then for $v < a$, $T_f(v,\lambda) = 0$ for all $\lambda \geq 1$ and so $R^\Lambda(v,f) = 0$.
Moreover, if $\boxed{L < \infty}$, for any $h \in C_c^\infty(\Omega)$, by setting $f = \partial_x^2 h + v\partial_z h$, we have by integration by parts:

$$T_f(v,\lambda) = i\lambda v \int_{-L/2}^{L/2} \int_0^T f(x,z) e^{-\lambda^2 vz} e^{-i\lambda vx} dxdz = 0,$$

and so

$$R^\Lambda(v,f) = 0.$$

(but in this case, f changes sign !)

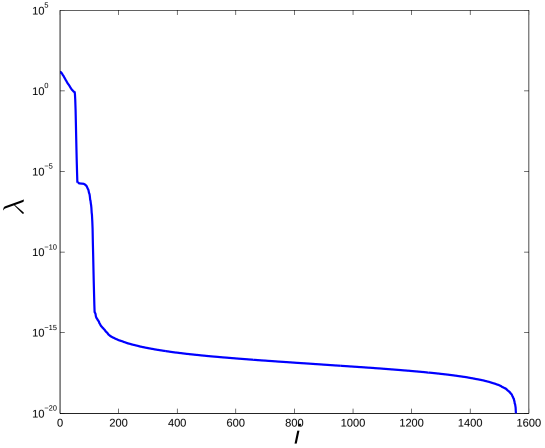


Figure: Eigenvalues of $M_w \approx R^{\Lambda}$ for a 100×30 grid

Letting $\epsilon \to 0$

Proposition (Dambrine, P. & Rousseaux (to appear))

The minimum value $N^{\Lambda,\epsilon}(v, f^{\epsilon,v})$ *tends to*

$$m^{\Lambda,v} := \inf_{f \in C_V} R^{\Lambda}(v, f)$$

as ϵ *tends to* 0.

Remark: Up to a subsequence, $f^{\epsilon,v}$ tends to a finite nonnegative measure with support in $\overline{\Omega}$, weakly-$\star$ in $(C(\overline{\Omega}))'$.

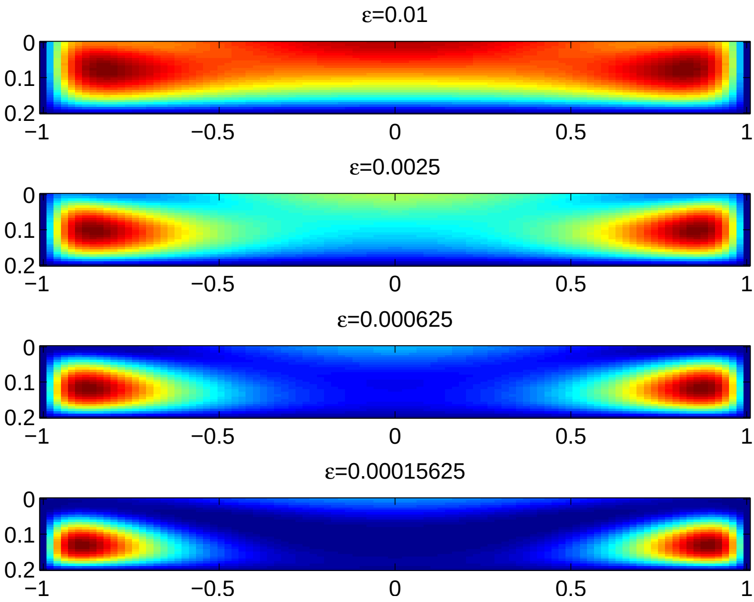


Figure: Color maps of the optimized hull function $f(x, z)$ for smaller and smaller values of ϵ.

The one dimensional case

For simplicity, we restrict the study to the functions $f(x, z) = f(x)$ with infinite draft T. Moreover, $f(\pm L/2) = 0$ and by symmetry, f is even. Then (for $\Lambda = \infty$),

$$R_{Michell} = \frac{4\rho g v}{\pi} \int_1^{\infty} S_f(v, \lambda)^2 \frac{1}{\sqrt{\lambda^2 - 1}} d\lambda$$

with

$$S_f(v, \lambda) = \int_{-L/2}^{L/2} f(x) \cos(\lambda v x) dx. \qquad (13)$$

We minimize $R_{Michell}$ in

$$C_V := \{f \in H_0^1(-L/2, L/2) \;:\; f \text{ even}, \int_{-L/2}^{L/2} f = V, \; f \geq 0 \text{ a.e.}\}.$$

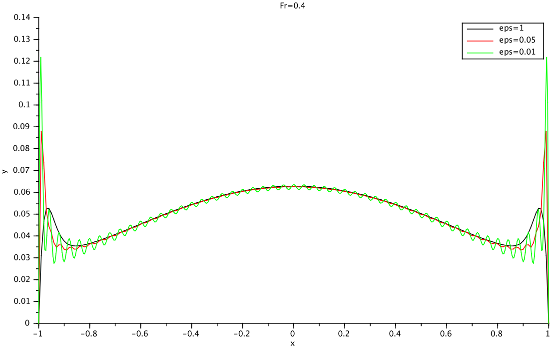
Proposition (1d case)

Any minimizing sequence (f_n) converges to the same finite nonnegative measure μ^v on $[-L/2, L/2]$. Moreover, μ^v belongs to $H^{-1/2}(-L/2, L/2)$.

Uniqueness: S_f is the Fourier transform of f, so by analycity, $R_{Michell}$ is a **norm** on $L^2(-L/2, L/2)$, which has a natural l.s.c. extension to a norm on $(C([-L/2, L/2])'$.

Estimate: use Fatou's lemma and the standard definition of $H^{-1/2}(\mathbf{R})$ by Fourier transform. Indeed,

$$H^{-1/2}(\mathbf{R}) := \{g \in \mathcal{S}'(\mathbf{R}) \ : \ \int_{\mathbf{R}} (1+\lambda^2)^{-1/2} |\hat{g}(\lambda)|^2 d\lambda < \infty\}.$$



Solution for $\epsilon = 1$, $\epsilon = 0.05$ and $\epsilon = 0.01$ $(Fr = 0.4)$

$1d$ Resolution without positivity condition (Krein'52)

If we suppress the positivity condition, then the minimization problem is quadratic with linear constraint. The Euler-Lagrange equation reads: find $f : I \to \mathbf{R}$ s.t.

$$\int_I K_v(x-\xi) f(\xi) d\xi = cst, \quad \forall x \in I, \tag{14}$$

where $I = (-L/2, L/2)$ and

$$K_v(x-\xi) = \int_1^\infty \frac{\cos(\lambda v(x-\xi))}{\sqrt{\lambda^2 - 1}} d\lambda.$$

This is a *Fredholm integral equation of the first kind.* Well-known category of ill-posed problems !

We have

$$K_v(x) = c_v \ln(1/|x|) + g(x),$$

where g is continuously differentiable on $\bar{I}$ and twice continuously differentiable on $\bar{I} \setminus \{0\}$.

Keeping only the first term of K_v in (14), the solution is given by

$$f(x) = \frac{C}{\sqrt{(L/2)^2 - x^2}},$$

where C is a constant.

Singularity at $x = \pm L/2$. In particular, $f \notin H^1(I)$.

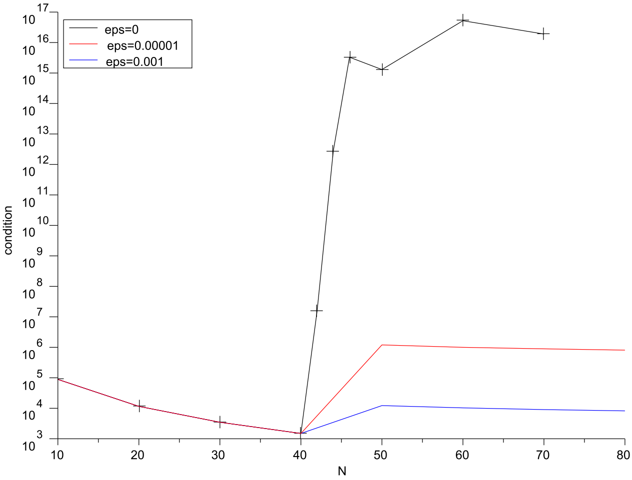
A numerical experiment (1d)

Discretization of the Euler-Lagrange equation (14) by P^1 finite element in $H^1(-L/2, L/2)$, and its ϵ-regularized version (Tykhonov regularization).

$Fr = 0.4$ ($L = 3$ / $V = 0.1$)

$N =$ number of degrees of freedom

$\kappa =$ condition number of the (augmented) linear system



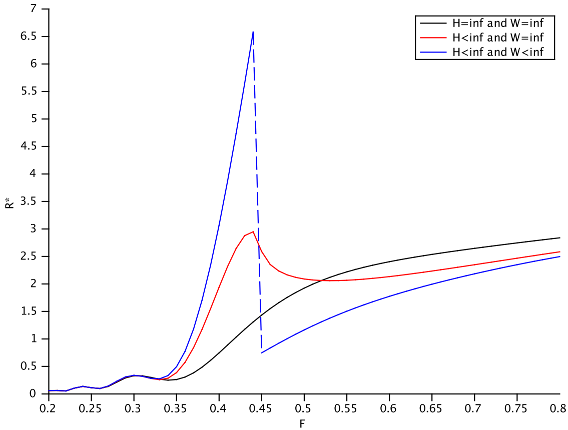
Condition number vs degrees of freedom (1d)

Conclusion and perspectives

Other formulas

Michell assumes an unbounded domain, i.e. depth $H = \infty$ and width $W = \infty$. There are also integral formulas for:

- $H = \infty$ and $W < \infty$ (**Sretensky'36**)
- $H < \infty$ and $W = \infty$ (**Sretensky'37**)
- $H < \infty$ and $W < \infty$ (**Sretensky'37** and **Keldish-Sedov'37**)
- Multilayers (dead-water effects) ...



Wave resistance of a Wigley hull for 3 different domains

- Fixed speed $U \Rightarrow$ range of speeds
- fixed domain of parameters $\Rightarrow$ varying domain (shape optimization)
- ...

Thank you for your attention !