Discretization of optimal control problems with partial differential equations

An introduction

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Partial differential equations, optimal design and numerics
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Plan of the talk

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Plan of the talk

1. Introduction to optimal control with elliptic PDEs
2. Discretization of optimal control problems (Neumann control)
3. Numerical example
4. Summary

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State equation

In optimal control problems one optimizes a state variable
with the help of a control variable:

- state variable $y \in Y$ satisfies a boundary value problem, e. g.

$$-\Delta y + y = f \quad \text{in } \Omega, \qquad y = g \quad \text{or} \quad \partial_n y = g \quad \text{on } \Gamma = \partial\Omega$$

  to be understood in weak sense if $Y = H^1(\Omega)$
  and in very weak sense if $Y = L^2(\Omega)$

- in my group were also investigated
  - PDEs with discontinuous coefficients
  - the Stokes equations
  - parabolic PDEs
  - semilinear elliptic problems

- interests for future work
  - PDEs of non-integer order
  - variational inequalities

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Control

- state variable $y \in Y$ satisfies a boundary value problem
- state equation contains a control variable $u \in U^{\mathrm{ad}} \subset U$
  - distributed control

$$-\Delta y + y = u \quad \text{in } \Omega, \qquad y = 0 \quad \text{on } \Gamma$$

  - Neumann boundary control

$$-\Delta y + y = 0 \quad \text{in } \Omega, \qquad \partial_n y = u \quad \text{on } \Gamma$$

  - Dirichlet boundary control

$$-\Delta y + y = 0 \quad \text{in } \Omega, \qquad y = u \quad \text{on } \Gamma$$

  defines a control-to-state mapping $S : U \to Y$, $u \mapsto y = Su$
- distinguish the function space U (e.g. L^2, energy, L^1, BV, ...) and a convex set of admissible controls $U^{\mathrm{ad}} \subset U$, e. g.

$$U^{\mathrm{ad}} := \{v \in U : a \leq v(x) \leq b \text{ a.e.}\}$$

Target functional

- state variable $y \in Y$ satisfies a boundary value problem
- control-to-state mapping $S : U \to Y$, $u \mapsto y = Su$
- control variable is used for optimization, here tracking type functional

Optimal control problem

$$\min_{(y,u)\in Y\times U^{\mathrm{ad}}} J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_U, \qquad \text{subject to} \quad y = Su \tag{1}$$

or shortly

$$\min_{u\in U^{\mathrm{ad}}} j(u) = J(Su,u)$$

- if S linear, then j convex, and the optimal control problem (1) has a unique optimal solution $(\bar{y},\bar{u})$
- if S is non-linear we loose uniqueness; we have to deal with local solutions

First order optimality conditions

- assume that control space U is Hilbert space
- optimal control $\bar{u}$ minimizes tracking type functional

$$j(u) = J(Su, u) := \frac{1}{2}(Su - y_d, Su - y_d)_{L^2(\Omega)} + \frac{\nu}{2}(u, u)_U,$$

- derivative of j in $\bar{u}$

$$\begin{aligned} j'(\bar{u})(v) &= (Sv, S\bar{u} - y_d)_{L^2(\Omega)} + \nu(v, \bar{u})_U \\ &= \langle v, S^*(S\bar{u} - y_d) + \nu N\bar{u}\rangle_{U,U^*} \\ &= \langle v, \bar{p} + \nu N\bar{u}\rangle_{U,U^*} \quad \text{with } \bar{p} = S^*(S\bar{u} - y_d) \end{aligned}$$

  with $N : U \to U^*$ such that $\langle v, Nu\rangle_{U,U^*} = (v, u)_U$ for all $u, v \in U$
- necessary and sufficient optimality condition:

$$\begin{aligned} &\bar{y} = S\bar{u}, \\ &\bar{p} = S^*(S\bar{u} - y_d) \\ &\langle u - \bar{u}, \bar{p} + \nu N\bar{u}\rangle_{U,U^*} \geq 0 \quad \forall u \in U^{\mathrm{ad}} \end{aligned}$$

- if S is non-linear, we need also a second order sufficient optimality condition

Plan of the talk

Finite element method: basic idea

- Neumann boundary value problem

$$-\Delta y + y = f \quad \text{in } \Omega, \quad \partial_n y = u \quad \text{on } \Gamma$$

  in weak form

$$y \in V : a(y, v) = (f, v)_{L^2(\Omega)} + (u, v)_{L^2(\Gamma)} \quad \forall v \in V$$

  where $V = H^1(\Omega)$ and

$$a(y, v) = (\nabla y, \nabla v)_{L^2(\Omega)} + (y, v)_{L^2(\Omega)}$$

- Galerkin: choose finite dimensional subspace $V_h \subset V$ and solve

$$y_h \in V_h : a(y_h, v_h) = (f, v_h)_{L^2(\Omega)} + (u, v_h)_{L^2(\Gamma)} \quad \forall v_h \in V_h$$

- FEM: piecewise polynomial functions on mesh $\mathcal{T}_h$

Finite element error estimates

Let $\mathcal{T}_h$ be quasi-uniform and $V_h = \{y_h \in C(\bar{\Omega}) : y_h|_T \in \mathcal{P}_1 \ \forall T \in \mathcal{T}_h\}$, then

Finite element error estimates

$$\|y - y_h\|_{L^2(\Omega)} \le Ch^2 \|y\|_{H^2(\Omega)}$$
$$\|y - y_h\|_{L^2(\Gamma)} \le Ch^2 |\ln h| \, \|y\|_{W^{2,\infty}(\Omega)}$$

- Note that $y \in H^2(\Omega)$ only if $\omega < \pi$, and $y \in W^{2,\infty}(\Omega)$ only if $\omega < \pi/2$ where ω is the maximal internal angle of the domain

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- $L^2(\Gamma)$ error estimate holds also for little less regular functions ($\omega < 2\pi/3$) (see [Pfefferer 2014])

J. Pfefferer: Numerical analysis for elliptic Neumann boundary control problems on polygonal domains. PhD thesis, Universität der Bundeswehr München, 2014

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Finite element error estimates

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- $L^2(\Gamma)$ error estimate holds also for little less regular functions ($\omega < 2\pi/3$) (see [Pfefferer 2014])
- $y \in W^{2,p}(\Omega)$: standard strategies give $L^2(\Gamma)$ estimates of order $2 - 1/p$ only

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Finite element error estimates

Let $\mathcal{T}_h$ be quasi-uniform and $V_h = \{y_h \in C(\bar{\Omega}) : y_h|_T \in \mathcal{P}_1 \ \forall T \in \mathcal{T}_h\}$, then

Finite element error estimates

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- Note that $y \in H^2(\Omega)$ only if $\omega < \pi$, and $y \in W^{2,\infty}(\Omega)$ only if $\omega < \pi/2$ where ω is the maximal internal angle of the domain
- $L^2(\Gamma)$ error estimate holds also for little less regular functions ($\omega < 2\pi/3$) (see [Pfefferer 2014])
- $y \in W^{2,p}(\Omega)$: standard strategies give $L^2(\Gamma)$ estimates of order $2 - 1/p$ only
- lower convergence order for less regular solutions: let $\lambda = \pi/\omega$ then

$$\|y - y_h\|_{L^2(\Omega)} \le Ch^{2\min\{1,\lambda-\varepsilon\}}$$
$$\|y - y_h\|_{L^2(\Gamma)} \le Ch^{\min\{2,1/2+\lambda-\varepsilon\}} |\ln h|^{3/2} \quad \text{[Pfefferer 2014]}$$

J. Pfefferer: Numerical analysis for elliptic Neumann boundary control problems on polygonal domains. PhD thesis, Universität der Bundeswehr München, 2014

Neumann boundary control problem

Optimal control problem

$$\min J(y,u) = \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)}, \quad y_d \in C^{0,\sigma}(\Omega)$$
$$-\Delta y + y = 0 \quad \text{in } \Omega, \quad \partial_n y = u \quad \text{on } \Gamma \quad \text{in weak sense}$$
$$a \le u(x) \le b \quad \text{for a.a. } x \in \Gamma$$

$$V = H^1(\Omega)$$
$$U^{ad} = \{u \in L^2(\Gamma) : a \le u(x) \le b \text{ a.e. on } \Gamma\}$$

Optimality system

$$\bar y \in V: \quad a(\bar y, v) = (\bar u, v)_{L^2(\Gamma)} \quad \forall v \in V$$
$$\bar p \in V: \quad a(v, \bar p) = (\bar y - y_d, v)_{L^2(\Omega)} \quad \forall v \in V$$
$$\bar u \in U^{ad}: \quad (\bar p_{|\Gamma} + \nu \bar u, u - \bar u)_{L^2(\Gamma)} \ge 0 \quad \forall u \in U^{ad}$$

Short notation

$$\bar y = S\bar u$$
$$\bar p = P(\bar y - y_d)$$
$$\bar u = \Pi_{[a,b]}(-\tfrac{1}{\nu}\bar p_{|\Gamma})$$

with $\left(\Pi_{[a,b]} v\right)(x) = \min\{\max\{v(x), a\}, b\}$. Note that $S^* v = (Pv)_{|\Gamma}$.

Variational discretization

$$V_h = \{y_h \in C(\bar\Omega) : y_h|_T \in \mathcal{P}_1\ \forall T \in \mathcal{T}_h\}$$
$$U^{ad} = \{u \in L^2(\Gamma) : a \le u(x) \le b \text{ a.e. on } \Gamma\}$$

Optimality system

$$\bar y_h^s \in V_h : \quad a(\bar y_h^s, v_h) = (\bar u_h^s, v_h)_{L^2(\Gamma)} \quad \forall v_h \in V_h$$
$$\bar p_h^s \in V_h : \quad a(v_h, \bar p_h^s) = (\bar y_h^s - y_d, v_h)_{L^2(\Omega)} \quad \forall v_h \in V_h$$
$$\bar u_h^s \in U^{ad} : \quad (\bar p_{h|\Gamma}^s + \nu \bar u_h^s, u - \bar u_h^s)_{L^2(\Gamma)} \ge 0 \quad \forall u \in U^{ad}$$

Short notation

$$\bar y_h^s = S_h \bar u_h^s$$
$$\bar p_h^s = P_h(\bar y_h^s - y_d)$$
$$\bar u_h^s = \Pi_{[a,b]}\left(-\tfrac{1}{\nu}\bar p_{h|\Gamma}^s\right)$$

Note that $S_h^* v = (P_h v)_{|\Gamma}$. Note further that $\bar u_h^s \notin V_{h|\Gamma}$.

M. Hinze: A variational discretization concept in control constrained optimization: The linear-quadratic case
Comput. Optim. Appl. 30(2005), 45–61

E. Casas, M. Mateos: Error estimates for the numerical approximation of Neumann control problems.
Comput. Optim. Appl., 39(2008), 265–295

M. Hinze, U. Matthes: A note on variational discretization of elliptic Neumann boundary control.
Control & Cybernetics 38(2009), 577–591

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Full discretization

$$V_h = \{y_h \in C(\bar\Omega) : y_h|_T \in \mathcal{P}_1 \ \forall T \in \mathcal{T}_h\}$$
$$U_h^{ad} = \{u_h \in L^\infty(\Gamma) : u_h|_G \in \mathcal{P}_0 \ \forall G \in \mathcal{G}_h\} \cap U^{ad}$$

Optimality system

$$\begin{aligned}
\bar y_h \in V_h : &\quad a(\bar y_h, v_h) = (\bar u_h, v_h)_{L^2(\Gamma)} && \forall v_h \in V_h \\
\bar p_h \in V_h : &\quad a(v_h, \bar p_h) = (\bar y_h - y_d, v_h)_{L^2(\Omega)} && \forall v_h \in V_h \\
\bar u_h \in U_h^{ad} : &\quad (\bar p_{h|\Gamma} + \nu \bar u_h, u_h - \bar u_h)_{L^2(\Gamma)} \ge 0 && \forall u_h \in U_h^{ad}
\end{aligned}$$

Short notation

$$\bar y_h = S_h \bar u_h$$
$$\bar p_h = P_h(\bar y_h - y_d)$$
$$\bar u_h = \Pi_{[a,b]}(-\tfrac{1}{\nu} R_h \bar p_{h|\Gamma})$$

R_h is the interpolation operator on U_h (midpoints). – Note that $S_h^* v = (P_h v)_{|\Gamma}$.

disadvantage: approximation of control less accurate than state and co-state

E. Casas, M. Mateos, F. Tröltzsch: Error estimates for the numerical approximation of boundary semilinear elliptic control problems. Comput. Optim. Appl. 31(2005), 193–219

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Full discretization and postprocessing

$$V_h = \{y_h \in C(\bar\Omega) : y_h|_T \in \mathcal{P}_1 \ \forall T \in \mathcal{T}_h\}$$
$$U_h^{ad} = \{u_h \in L^\infty(\Gamma) : u_h|_G \in \mathcal{P}_0 \ \forall G \in \mathcal{G}_h\} \cap U^{ad}$$

Optimality system

$$\begin{aligned}
\bar y_h \in V_h : &\quad a(\bar y_h, v_h) = (\bar u_h, v_h)_{L^2(\Gamma)} && \forall v_h \in V_h \\
\bar p_h \in V_h : &\quad a(v_h, \bar p_h) = (\bar y_h - y_d, v_h)_{L^2(\Omega)} && \forall v_h \in V_h \\
\bar u_h \in U_h^{ad} : &\quad (\bar p_{h|\Gamma} + \nu \bar u_h, u_h - \bar u_h)_{L^2(\Gamma)} \ge 0 && \forall u_h \in U_h^{ad}
\end{aligned}$$

Short notation

$$\begin{aligned}
\bar y_h &= S_h \bar u_h \\
\bar p_h &= P_h(\bar y_h - y_d) \\
\bar u_h &= \Pi_{[a,b]}(-\tfrac{1}{\nu} R_h \bar p_{h|\Gamma})
\end{aligned}$$

R_h is the interpolation operator on U_h (midpoints). – Note that $S_h^* v = (P_h v)_{|\Gamma}$.

Postprocessing

$$\tilde u_h = \Pi_{[a,b]}(-\tfrac{1}{\nu} \bar p_{h|\Gamma})$$

Note that $\tilde u_h \notin U_h^{ad}$ and $\tilde u_h \notin V_{h|\Gamma}$.

C. Meyer, A. Rösch. Superconvergence properties of optimal control problems. SIAM J. Control Optim. 43(2004), 970–985

M. Mateos, A. Rösch. On saturation effects in the Neumann boundary control of elliptic optimal control problems. Comput. Optim. Appl., 49(2011), 359–378

Error estimate for variational discretization

Theorem [Hinze/Matthes 09], [Mateos/Rösch 11]

On quasi-uniform meshes and for $\omega < \pi/2$ the estimate

$$\|\bar u - \bar u_h^s\|_{L^2(\Gamma)} + \|\bar y - \bar y_h^s\|_{L^2(\Omega)} + \|\bar p - \bar p_h^s\|_{L^2(\Gamma)} \lesssim h^{2-\varepsilon}$$

is valid.

Proof:

$$\nu\|\bar u - \bar u_h^s\|_{L^2(\Gamma)} \lesssim \|(S^* - S_h^*)(S\bar u - y_d)\|_{L^2(\Gamma)} + \|S_h^*(S - S_h)\bar u\|_{L^2(\Gamma)}$$

$$\|\bar y - \bar y_h^s\|_{L^2(\Omega)} \lesssim \|(S - S_h)\bar u\|_{L^2(\Omega)} + \|S_h(\bar u - \bar u_h^s)\|_{L^2(\Omega)}$$

$$\|\bar p - \bar p_h^s\|_{L^2(\Gamma)} \lesssim \|(S^* - S_h^*)(S\bar u - y_d)\|_{L^2(\Gamma)} + \|S_h^*(\bar y - \bar y_h^s)\|_{L^2(\Gamma)} \qquad \square$$

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Error estimate for variational discretization

Theorem [Hinze/Matthes 09], [Mateos/Rösch 11]

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$$\|\bar{u} - \bar{u}_h^s\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h^s\|_{L^2(\Omega)} + \|\bar{p} - \bar{p}_h^s\|_{L^2(\Gamma)} \lesssim h^{2-\varepsilon}$$

is valid.

Proof:

$$\nu\|\bar{u} - \bar{u}_h^s\|_{L^2(\Gamma)} \lesssim \|(S^* - S_h^*)(S\bar{u} - y_d)\|_{L^2(\Gamma)} + \|S_h^*(S - S_h)\bar{u}\|_{L^2(\Gamma)}$$

$$\|\bar{y} - \bar{y}_h^s\|_{L^2(\Omega)} \lesssim \|(S - S_h)\bar{u}\|_{L^2(\Omega)} + \|S_h(\bar{u} - \bar{u}_h^s)\|_{L^2(\Omega)}$$

$$\|\bar{p} - \bar{p}_h^s\|_{L^2(\Gamma)} \lesssim \|(S^* - S_h^*)(S\bar{u} - y_d)\|_{L^2(\Gamma)} + \|S_h^*(\bar{y} - \bar{y}_h^s)\|_{L^2(\Gamma)} \qquad \square$$

Order is $\min\{2, \frac{1}{2} + \lambda\} - \varepsilon$ in the general case [Pfefferer 2014], where $\lambda = \pi/\omega$. Note that this implies that the order $2 - \varepsilon$ is valid for $\omega < 2\pi/3$.

M. Hinze, U. Matthes: A note on variational discretization of elliptic Neumann boundary control. Control & Cybernetics 38(2009), 577–591

J. Pfefferer: Numerical analysis for elliptic Neumann boundary control problems on polygonal domains. PhD thesis, Universität der Bundeswehr München, 2014

Error estimates for the postprocessing approach 1

Let R_h be the interpolation operator on U_h (midpoints).

Lemma [Mateos/Rösch 11]

On quasi-uniform meshes and for $\omega < \pi/2$ the estimates

$$\|S_h(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)} \lesssim h^2$$
$$\|R_h\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \lesssim h^{2-\varepsilon} \quad \text{(supercloseness)}$$

hold under the assumption that the control has only finitely many kinks.

Note that

$$\|\bar{u} - R_h\bar{u}\|_{L^2(\Gamma)} \lesssim h.$$

Error estimates for the postprocessing approach 1

Let R_h be the interpolation operator on U_h (midpoints).

Lemma [Mateos/Rösch 11]

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hold under the assumption that the control has only finitely many kinks.

Pfefferer [2014] showed for the general case ($\lambda = \pi/\omega$):

$$\|S_h(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)} \lesssim h^{\min\{2,1/2+\lambda-\varepsilon\}}$$
$$\|R_h\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \lesssim h^{\min\{2,1/2+\lambda-\varepsilon\}}|\ln h|^{3/2}$$

Note that this implies that the order $2 - \varepsilon$ is valid for $\omega < 2\pi/3$.

M. Mateos, A. Rösch. On saturation effects in the Neumann boundary control of elliptic optimal control problems. Comput. Optim. Appl., 49(2011), 359–378

J. Pfefferer: Numerical analysis for elliptic Neumann boundary control problems on polygonal domains. PhD thesis, Universität der Bundeswehr München, 2014

Error estimates for the postprocessing approach 2

Theorem [Mateos/Rösch 11]

On quasi-uniform meshes and for $\omega < \pi/2$ the estimates

$$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} + \|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma)} \lesssim h^{2-\varepsilon}$$

hold under the assumption that the control has only finitely many kinks.

Proof:

$$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \lesssim \|(S - S_h)\bar{u}\|_{L^2(\Omega)} + \|S_h(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)} + \|S_h(R_h\bar{u} - \bar{u}_h)\|_{L^2(\Omega)}$$

$$\|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} \lesssim \|(S^* - S_h^*)(S\bar{u} - y_d)\|_{L^2(\Gamma)} + \|S_h^*(\bar{y} - \bar{y}_h)\|_{L^2(\Gamma)}$$

$$\|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma)} \lesssim \|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)}$$

□

Error estimates for the postprocessing approach 2

Theorem [Mateos/Rösch 11]

On quasi-uniform meshes and for $\omega < \pi/2$ the estimates

$$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} + \|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} + \|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma)} \lesssim h^{2-\varepsilon}$$

hold under the assumption that the control has only finitely many kinks.

Proof:

$$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \lesssim \|(S - S_h)\bar{u}\|_{L^2(\Omega)} + \|S_h(\bar{u} - R_h\bar{u})\|_{L^2(\Omega)} + \|S_h(R_h\bar{u} - \bar{u}_h)\|_{L^2(\Omega)}$$

$$\|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)} \lesssim \|(S^* - S_h^*)(S\bar{u} - y_d)\|_{L^2(\Gamma)} + \|S_h^*(\bar{y} - \bar{y}_h)\|_{L^2(\Gamma)}$$

$$\|\bar{u} - \tilde{u}_h\|_{L^2(\Gamma)} \lesssim \|\bar{p} - \bar{p}_h\|_{L^2(\Gamma)}$$

□

- order is $\min\{2, \frac{1}{2} + \lambda\} - \varepsilon$ in the general case [Pfefferer 2014], $\lambda = \pi/\omega$.
- Winkler [2015] showed that one obtains without postprocessing for all ω

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \lesssim h$$

M. Winkler: Finite element error analysis for Neumann boundary control problems on polygonal and polyhedral domains. PhD thesis, Universität der Bundeswehr München, 2015

Plan of the talk

Numerical example

Consider the intersection of a square with a circular sector of opening ω,

$$\Omega_\omega = \{(r\cos\phi, r\sin\phi) \in \mathbb{R}^2 : r \in (0, \sqrt{2}), \phi \in [0, \omega]\} \cap (-1, 1)^2,$$

and first order optimality conditions

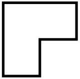
$$\begin{aligned} -\Delta y + y &= 0 && \text{in } \Omega, & \qquad -\Delta p + p &= y - y_d && \text{in } \Omega, \\ \partial_n y &= u + g_2 && \text{on } \Gamma, & \partial_n p &= g_1 && \text{on } \Gamma, \end{aligned}$$

and $u = \Pi_{[-0.5,0.5]}\left(-p_{|\Gamma}\right)$ on Γ.

We choose data $\lambda = \frac{\pi}{\omega}$, y_d, g_1, g_2, such that

$$\begin{aligned} \bar{y} &= 0 && \text{in } \Omega \\ \bar{p} &= r^\lambda \cos(\lambda\phi) && \text{in } \Omega \\ \bar{u} &= \Pi_{[-0.5,0.5]}(-\bar{p}) && \text{on } \Gamma \end{aligned}$$

$\omega = 2\pi/3$

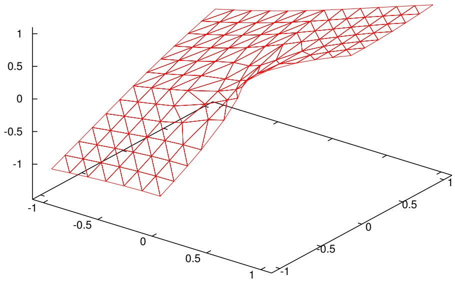


$\omega = 3\pi/2$

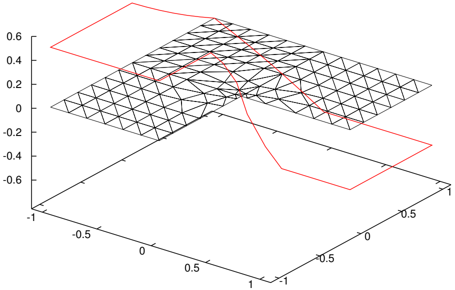
and consider the cases $\omega = 2\pi/3$ and $\omega = 3\pi/2$.

Plot of the approximate solution

$\bar{p}_h$



$\tilde{u}_h = \Pi_{[-0.5,0.5]}(-\bar{p}_h)$



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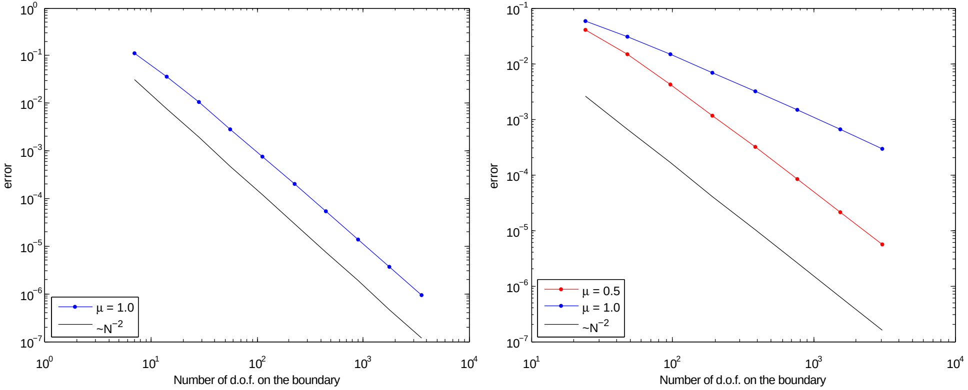
$L^2(\Gamma)$-error in the control (Postprocessing approach)



$\omega = 2\pi/3$ $\qquad$ $\omega = 3\pi/2$

T. Apel, J. Pfefferer, A. Rösch. Finite element error estimates on the boundary with application to optimal control. Math. Comp., 84:33–70, 2015.

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Plan of the talk

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Summary

- An introduction was given to optimal control problems with
  - linear elliptic state equation,
  - quadratic target functional and
  - control constraints.

  State constraints will be discussed later by W. Wollner.

- Discretization error estimates in L^2-norms were presented
  - for elliptic Neumann boundary control problems
  - with L^2 regularization
  - posed on a two-dimensional domain
  - discretized on quasi-uniform meshes

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    see talk by S. Rogovs.

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