Existence of weak solutions of doubly nonlinear parabolic equations

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Special cases of the equation

- Prototype for the equations under consideration ($E_T := E \times (0, T), E \subset \mathbb{R}^n, n \geq 2, p > \frac{2n}{n+2}, m > 1$):

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$$p = 2: \quad \partial_t u - \Delta u^m = f \qquad \text{(porous medium equation)}$$

$$m = 1: \quad \partial_t u - \Delta_p u = f \qquad \text{(p-Laplace equation)}$$

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- Classification:

	$0 < m < 1$	$m > 1$
$1 < p < 2$	doubly singular	singular-degenerate
$p > 2$	degenerate-singular	doubly degenerate

Cauchy-Dirichlet problem

- Model equation ($p > \frac{2n}{n+2}$, $m > 1$):

$$\partial_t u - \operatorname{div}\left(|u|^{m-1}|Du|^{p-2}Du\right) = f \quad \text{in } E_T,$$

where $f \geq 0$, $f \in L^\gamma(E_T)$; γ as small as possible?

- General Cauchy-Dirichlet problem in operator notation:

$$\text{(CP)} \quad \begin{cases} \partial_t u - \operatorname{div}\left(\mathbf{A}(x,t,u,Du)\right) = f & \text{in } E_T, \\ u = 0 & \text{on } \partial_{\text{par}} E_T, \end{cases}$$

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  - certain monotonicity and Lipschitz conditions for $\mathbf{A}$

Some previous results for DNPE

- Hölder regularity for bounded weak solutions (Ivanov, Porzio/Vespri)
- Harnack type inequalites for bounded weak solutions (Kinnunen/Kuusi, Vespri)
- Results regarding the asymptotic behavior of weak solutions (Manfredi/Vespri, Savaré/Vespri, Tedeev/Vespri)
- Local boundedness of the gradient for locally bounded, strictly positive weak solutions (Siljander)
- Uniqueness of bounded weak sol. "having some appr. scheme" (Ivanov)
- Existence of bounded weak solutions with $f \in L^{\gamma}(E_T), \gamma = \infty$ (Ivanov)

Definition of a weak solution

Definition

A non-negative function $u\colon E_T \to \mathbb{R}$ satisfying $u = 0$ on $\partial_{\text{par}} E_T$, $u \in C^0([0,T]; L^2(E))$ and $u^{\alpha+1} \in L^p((0,T); W^{1,p}(E))$ is termed a weak solution of (CP) if and only if the identity

$$\iint_{E_T} \left[-u\partial_t\varphi + \mathbf{A}(x,t,u,Du)\cdot D\varphi \right] dz = \iint_{E_T} f\varphi\, dz$$

holds true for any testing function $\varphi \in C_0^1(E_T)$.

- $\alpha := \frac{m-1}{p}$ is one of two common exponents in the definition of a solution in the context of the porous medium equation (alternative: $\beta := \frac{m-1}{p-1}$). The usage of α in our definition admits a smaller value for γ!

Existence of a weak solution: The statement

Theorem

Let $f \in L^\gamma(E_T, \mathbb{R}_{\geq 0})$ for $\gamma := 1 + \frac{n}{n(p+m-2)+2p}$ and assume that the previous structure conditions for $\mathbf{A}$ hold. Then, there exists at least one weak solution of (CP).

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- Our exponent is natural in the sense that it coincides for $p = 2$ with an earlier result for the porous medium equation by Bögelein, Duzaar, Gianazza: $\gamma = 1 + \frac{n}{nm+4}$.

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- Our exponent is natural in the sense that it coincides for $p = 2$ with an earlier result for the porous medium equation by Bögelein, Duzaar, Gianazza: $\gamma = 1 + \frac{n}{nm+4}$.
- For $m = 1$ (i. e. the case of the p-Laplacian equation), it coincides with the Hölder conjugate of $p\frac{n+2}{n}$.

Basic ideas of the proof

1) Regularization of (CP)

Let $(f_k)_{k\in\mathbb{N}} \subset L^\infty(E_T)$ be such that $f_k \nearrow f$ in $L^\gamma(E_T)$ and define the truncation operators $S_j^{(N)}(u) := \min\{\max\{u, \frac{1}{j}\}, N\}$ for $j, N \in \mathbb{N}$. Consider the following regularized Cauchy-Dirichlet problems

$$
\text{(RCP)} \quad \begin{cases} \partial_t u_{j,k}^{(N)} - \operatorname{div}\left(\mathbf{A}(x,t,S_j^{(N)}(u_{j,k}^{(N)}), Du_{j,k}^{(N)})\right) = f_k & \text{in } E_T, \\ u_{j,k}^{(N)} = \frac{1}{j} & \text{on } \partial_{\text{par}} E_T. \end{cases}
$$

Then, thanks to the truncation, the usual existence theorem for p-Laplacian equations from Lions (or Ladyženskaja/Solonnikov/Ural'ceva) is applicable and admits a solution $u_{j,k}^{(N)} \in C^0([0,T]; L^2(E)) \cap L^p((0,T); W^{1,p}(E))$.

Basic ideas of the proof

2) Properties of $u_{j,k}^{(N)}$: Part I

- $\sup_{E_T} u_{j,k}^{(N)} \leq C^*$, where $C^* \to \infty$ as $k \to \infty$ (via the Moser iteration method).
- $\inf_{E_T} u_{j,k}^{(N)} \geq \frac{1}{j}$ (via a comparison principle).

Hence, by choosing $N > C^*$, we can rewrite (RCP) in the following way:

$$
\text{(RCP)} \quad \begin{cases} \partial_t u_{j,k} - \operatorname{div}\left(\mathbf{A}(x,t,u_{j,k},Du_{j,k})\right) = f_k & \text{in } E_T, \\ u_{j,k} = \frac{1}{j} & \text{on } \partial_{\text{par}} E_T. \end{cases}
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Basic ideas of the proof

2) Properties of $u_{j,k}$: Part II (remember: $\alpha = \frac{m-1}{p}$)

- $u_{j,k}$ is in particular a weak solution since $|Du_{j,k}^{\alpha+1}| = cu_{j,k}^{\alpha}|Du_{j,k}| \leq c|Du_{j,k}| \in L^p(E_T)$.
- Energy estimate:

$$\sup_{t\in(0,T)} \int_{E\times\{t\}} u_{j,k}^2\,dx + \iint_{E_T} |Du_{j,k}^{\alpha+1}|^p\,dz \leq c,$$

uniformly in j, k. (Here, the Gagliardo-Nirenberg inequality determines the value of γ!)

Basic ideas of the proof

3) The limiting process

Due to the uniform energy estimate, there exist a (non-relabeled) subsequence $(u_{j,k})_{j,k\in\mathbb{N}}$ and functions $u \in L^2(E_T)$ and $v \in L^p(E_T, \mathbb{R}^n)$ such that

- $u_{j,k} \rightharpoonup u$ weakly in $L^2(E_T)$ as $(j,k) \to \infty$,
- $Du_{j,k}^{\alpha+1} \rightharpoonup v$ weakly in $L^p(E_T, \mathbb{R}^n)$ as $(j,k) \to \infty$.

It has to be elucidated that v is indeed the weak derivative of $u^{\alpha+1}$.

Basic ideas of the proof

4) Passage to the limit in the equation

Since $u_{j,k}$ is a weak solution of (RCP), we know

$$-\iint_{E_T} u_{j,k}\partial_t\varphi\,dz + \iint_{E_T} \mathbf{A}(x,t,u_{j,k},Du_{j,k})\cdot D\varphi\,dz = \iint_{E_T} f_k\varphi\,dz,$$

where the first and third integrals converge due to the convergence properties of $u_{j,k}$ and f_k. The convergence of the diffusion term follows from the convergence

$$Du_{j,k}^{\alpha+1} \to Du^{\alpha+1} \quad \text{a.e.} \quad \text{as } j,k\to\infty,$$

whose proof is quite intricate.

Next aim: Existence of solutions for measure data problem

- Previous Cauchy-Dirichlet problem:

$$(\mathrm{CP}) \quad \begin{cases} \partial_t u - \operatorname{div}\big(\mathbf{A}(x,t,u,Du)\big) = f & \text{in } E_T, \\ u = 0 & \text{on } \partial_{\mathrm{par}} E_T \end{cases}$$

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- Target: replace f by a non-negative Radon measure $\mu \in \mathcal{M}^+(E_T)$ and prove the existence of a very weak solution for the Cauchy-Dirichlet problem

$$(\mathrm{MDP}) \quad \begin{cases} \partial_t u - \operatorname{div}\big(\mathbf{A}(x,t,u,Du)\big) = \mu & \text{in } E_T, \\ u = 0 & \text{on } \partial_{\mathrm{par}} E_T. \end{cases}$$

Thank you for your attention!