Boundary controllability for a one-dimensional heat equation with two singular inverse-square potentials

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EXCELENCIA SEVERO OCHOA

Formulation of the problem

Let $T > 0$, $\mu_1, \mu_2 \leq 1/4$ and define $Q := (0,1) \times (0,T)$

$$\begin{cases} u_t - u_{xx} - \dfrac{\mu_1}{x^2} u - \dfrac{\mu_2}{(1-x)^2} u = 0 \quad (x,t) \in Q \\ \\ u(0,t) = f(t), \;\; u(1,t) = 0 \\ u(x,0) = u_0(x) \end{cases} \tag{1}$$

Theorem (Null-controllability)

For any time $T > 0$ and any initial datum $u_0 \in L^2(0,1)$ there exists a control function $f \in L^2(0,T)$ such that the solution of (1) satisfies $u(x,T) = 0$.

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State of the art

SINGULAR POTENTIALS

- J. Vancostenoble and E. Zuazua - Null controllability for the heat equation with singular inverse-square potentials (2008)
- S. Ervedoza - Control and stabilization properties for a singular heat equation with an inverse-square potential (2008)
- C. Cazacu - Controllability of the heat equation with an inverse-square potential localized on the boundary (2014)

DEGENERATE COEFFICIENTS $(u_t - (a(x)u_x)_x = 0)$

- P. Martinez and J. Vancostenoble - Carleman estimates for one-dimensional degenerate heat equations (2006)
- P. Cannarsa, P. Martinez and J. Vancostenoble - Carleman estimates for a class of degenerate parabolic operators (2008)
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Proposition

Let $\mu_1^*, \mu_2^* \in \mathbb{R}$ be such that $\mu_1^* + \mu_2^* \leq 1/4$. Then, for any $z \in H_0^1(0,1)$ it holds

$$\int_0^1 z_x^2 \, dx \geq \mu_1^* \int_0^1 \frac{z^2}{x^2} \, dx + \mu_2^* \int_0^1 \frac{z^2}{(1-x)^2} \, dx. \tag{2}$$

V. Felli and S. Terracini - Elliptic equations with multi-singular inverse-square potentials and critical nonlinearity

Proposition

There exists a constant $M > 0$ such that for any $z \in H_0^1(0,1)$ it holds

$$\int_0^1 z_x^2 \, dx + M \int_0^1 z^2 \, dx \geq \frac{1}{4} \int_0^1 \frac{z^2}{x^2} \, dx + \frac{1}{4} \int_0^1 \frac{z^2}{(1-x)^2} \, dx. \tag{3}$$

Sketch of the proof.

We rewrite $z = z_1 + z_2 + z_3$ with $z_i := z\phi_i, \quad i = 1, 2, 3$ and $(\phi_i)_{i=1,2,3}$ a partition of the unity such that

$$\mathrm{supp}(\phi_1) = (1/2, 1), \quad \mathrm{supp}(\phi_2) = (0, 1/2), \quad \phi_3 = 1 - \phi_1 - \phi_2$$

and we apply Hardy inequality. □

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Proposition

For all $\gamma < 2$ and $n > 0$ there exists a positive constant $C_0 = C_0(\gamma, n)$ such that, for any $z \in H^1_0(0,1)$ it holds

$$C_0 \int_0^1 z_x^2\, dx + \frac{2-\gamma}{2} \int_0^1 z^2\, dx \geq \frac{(1-\gamma)^2}{4} \int_0^1 \frac{z^2}{x^2}\, dx + n \int_0^1 \frac{z^2}{(1-x)^2}\, dx. \tag{4}$$

Sketch of the proof.

$$0 \leq \int_0^1 \left(x^{\frac{2-\gamma}{2}} z_x - \frac{\gamma-1}{2} \frac{z}{x^{\frac{\gamma}{2}}} + \frac{z}{1-x} \right)^2 dx.$$

We expand this expression, apply integration by parts and estimate using Hölder inequality, Cauchy-Schwarz inequality and Hardy-Poincaré inequalities. □

P. Martinez and J. Vancostenoble - Carleman estimates for one-dimensional degenerate heat equations (2006)

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Hilbert Uniqueness Method

Adjoint system

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Theorem (Observability inequality)

Let $T > 0$*. For any* $v_T \in L^2(0,1)$ *the solution of* (5) *satisfies*

$$\int_0^1 v(x,0)^2 \, dx \le C \int_0^T \left[x^{2\lambda_1} v_x^2 \right] \Big|_{x=0} dt, \tag{6}$$

with

$$\lambda_1 := \frac{1}{2} \left(1 - \sqrt{1 - 4\mu_1} \right). \tag{7}$$

Carleman estimate

Theorem

There exists a constant $R_0 > 0$ such that, for all $R \geq R_0$, every solution v of (5) *satisfies*

$$
\begin{aligned}
& R^3 C_1 \int_Q \theta^3 \left[x^{6\lambda_1}(1-x)^5\right] v^2 e^{-2R\sigma}\, dxdt + RC_2 \int_Q \theta \frac{v^2}{(1-x)^2} e^{-2R\sigma}\, dxdt \\
& + RC_3 \int_Q \theta \frac{v^2}{x^{1-2\lambda_1}} e^{-2R\sigma}\, dxdt + RC_4 \int_Q \theta \left[x^{2\lambda_1}(1-x)\right] v_x^2 e^{-2R\sigma}\, dxdt \\
& \leq RC_5 \int_0^T \theta \left[x^{2\lambda_1} v_x^2\right]\Big|_{x=0}\, dt,
\end{aligned}
\tag{8}
$$

where $C_i,\ i = 1, \dots, 5$ are positive constants and, for $\varpi, \beta > 0$, the weight function σ is defined as $\sigma(x,t) := \theta(t)p(x)$ with

$$
\theta(t) := \left(\frac{1}{t(T-t)}\right)^3,\ \ p(x) := \varpi + \frac{\beta x^{2\lambda_1+1}}{2\lambda_1+1}\left[1 - \frac{2\lambda_1+1}{\lambda_1+1}x + \frac{2\lambda_1+1}{2\lambda_1+3}x^2\right].
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$$+ RC_3 \int_Q \theta \frac{v^2}{x^{1-2\lambda_1}} e^{-2R\sigma}\, dxdt + RC_4 \int_Q \theta \left[x^{2\lambda_1}(1-x)\right] v_x^2 e^{-2R\sigma}\, dxdt$$
$$\leq RC_5 \int_0^T \theta \left[x^{2\lambda_1} v_x^2\right]\Big|_{x=0} dt, \tag{9}$$

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Proof of the observability inequality

- From (9) we have

$$\int_Q \theta \frac{v^2}{x^{1-2\lambda_1}} e^{-2R\sigma}\, dxdt \le \mathcal{C}_1 \int_0^T \theta \left[x^{2\lambda_1} v_x^2\right]\Big|_{x=0} dt;$$

- There exist two positive constants $\mathcal{P}_1$ and $\mathcal{P}_2$ such that

$$\frac{\theta e^{-2R\sigma}}{x^{1-2\lambda_1}} \ge \mathcal{P}_1 \ \text{in}\ (0,1)\times\left[\frac{T}{4},\frac{3T}{4}\right], \quad \theta e^{-2R\varpi\theta} \le \mathcal{P}_2 \ \text{in}\ (0,T);$$

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$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^1 v^2\, dxdt \le \mathcal{C}_2 \int_0^T \theta \left[x^{2\lambda_1} v_x^2\right]\Big|_{x=0} dt.$$

- Using (3)

$$\frac{d}{dt}\int_0^1 v^2\, dx \ge -M \int_0^1 v^2\, dx;$$

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$$\int_0^1 v(x,0)^2\, dxdt \le \frac{2}{T} e^{2MT} \mathcal{C}_2 \int_0^T \theta \left[x^{2\lambda_1} v_x^2\right]\Big|_{x=0} dt.$$

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THANKS FOR YOUR ATTENTION!