Multi-scale entanglement renormalization ansatz (MERA)
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Why this talk, here?

MERAs ↔ AdS/CFT

Swingle, 2009

“Entanglement renormalization for quantum fields”
Haegeman, Osborne, Verschelde, Verstraete, 2011

“Holographic Geometry of Entanglement Renormalization in Quantum Field Theories”
Nozaki, Ryu, Takayanagi, 2012

“Time Evolution of Entanglement Entropy from Black Hole Interiors”
Hartman, Maldacena, 2013

“Exact holographic mapping and emergent space-time geometry”
Xiaoliang Qi, 2013

“Holographic quantum error-correcting codes: Toy models for the bulk/boundary correspondence”
Pastawski, Yoshida, Harlow, Preskill, 2015

“Integral Geometry and Holography”
Czech, Lamprou, McCandlish, Sully, 2015

MERA (2005)
What can MERA do, for sure?

1D quantum / 2D classical Hamiltonian
- on the lattice
- at a critical point

Numerical determination of conformal data:
- central charge $c$
- scaling dimensions $\Delta_\alpha \equiv h_\alpha + \bar{h}_\alpha$
  and conformal spins $s_\alpha \equiv h_\alpha - \bar{h}_\alpha$
- OPE coefficients $C_{\alpha\beta\gamma}$

Numerical results (approx. an hour on your laptop)

- $\Delta_\sigma \approx 0.124997$
- $\Delta_\epsilon \approx 0.99993$
- $\Delta_\mu \approx 0.125002$
- $\Delta_\psi \approx 0.500001$
- $\Delta_{\bar{\psi}} \approx 0.500001$

- $C_{\epsilon\sigma\sigma} = \frac{1}{2}$
- $C_{\epsilon\mu\mu} = -\frac{1}{2}$
- $C_{\epsilon\psi\bar{\psi}} = i$
- $C_{\epsilon\bar{\psi}\psi} = -i$
- $C_{\psi\mu\sigma} = \frac{e^{-i\pi}}{\sqrt{2}}$
- $C_{\bar{\psi}\mu\sigma} = \frac{e^{i\pi}}{\sqrt{2}}$

$\Delta_\Pi = 0$
Part 1: (old stuff)

Tensor networks + Renormalization group = Multi-scale entanglement renormalization ansatz (MERA)

- quantum circuit
- RG transformation

Part 2: (recent developments)

Tensor network renormalization (TNR)

- RG flow in the space of tensors
- Local scale transformations on the lattice
  - plane to cylinder
  - hyperbolic plane (MERA)
  - thermal states / black holes
Many-body wave-function of $N$ spins

$$|\Psi\rangle = \sum_{i_1, i_2, \ldots, i_N} \Psi_{i_1 i_2 \ldots i_N} |i_1 i_2 \ldots i_N\rangle$$

$$T_{ij} = \sum_k R_{ik} S_{kj}$$

$$a = \hat{y}^\dagger \cdot M \cdot \hat{x}$$

$$t_r(ABCD)$$

Why bother?

$$\sum_{ijklmnop} A_{ijk} B_{jlm} C_{nk} D_{kmr} x_i y_l z_n v_r$$
Many-body wave-function of $N$ spins

$$|\Psi\rangle = \sum_{i_1, i_2, \ldots, i_N} \Psi_{i_1 i_2 \cdots i_N} |i_1 i_2 \cdots i_N\rangle$$

Tensor network states

$$\Psi = \begin{array}{c}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_N \\
\end{array}$$

Generic state

$\mathcal{H}(N)$

Tensor network

$\alpha = 1, 2, \ldots, \chi$

Ground states of local Hamiltonians (area law for EE)

$\mathcal{H}(N)$

Generic state (volume law for EE)

$\mathcal{H}(N)$
Multi-scale entanglement renormalization ansatz (MERA)

- Variational class of states for 1d systems, which extends in space and scale
- Variational parameters for different length scales stored in different tensors
- It is secretly a quantum circuit and an RG transformation
MERA as a quantum circuit

\[ |0\rangle \]

*disentangler*

two-body unitary gate

*isometry*

also a two-body unitary gate
MERA as a quantum circuit

\[ |0\rangle = \text{also a two-body unitary gate} \]

**disentangler**

two-body unitary gate

**isometry**

two-body unitary gate
MERA as a quantum circuit

\[
|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle
\]

disentangler
two-body unitary gate

isometry
also a two-body unitary gate

|0\rangle
MERA as a quantum circuit

ground state ansatz \( |\Psi\rangle = U |0\rangle \otimes^N \)

Entanglement introduced by gates at different “times” (= length scales)
MERA = tensor network + isometric/unitary constraints

~ hyperbolic plane?  
(Swingle 2009)

~ de Sitter space?  
(Beny 2011, Czech 2015)

Causal structure

essential for many MERA properties and computational efficiency
Causal cone

past causal cone of region $A$
(boundary)

region $A$
(boundary)
Causal cone

past causal cone of region $A$ (boundary)

region $A$ (boundary)
MERA as RG Transformation

blocking variational optimization

Entanglement renormalization (2005)
MERA as RG Transformation


blocking variational optimization

Entanglement renormalization (2005)

failure to remove some short-range entanglement!

removal of all short-range entanglement
MERA as a sequence of ground state wave-functions

\[ |\Psi\rangle \rightarrow |\Psi'\rangle \rightarrow |\Psi''\rangle \rightarrow \ldots \]
MERA as a sequence of ground state wave-functions

\[ |\Psi'\rangle \]

\[ |\Psi\rangle \rightarrow |\Psi'\rangle \rightarrow |\Psi''\rangle \rightarrow \cdots \]
MERA as a sequence of ground state wave-functions

\[ |\Psi''\rangle \]

\[ |\Psi\rangle \rightarrow |\Psi'\rangle \rightarrow |\Psi''\rangle \rightarrow \cdots \]
MERERA defines an RG flow in the space of wave-functions

\[ |\Psi\rangle \rightarrow |\Psi'\rangle \rightarrow |\Psi''\rangle \rightarrow \ldots \]

\[ H \rightarrow H' \rightarrow H'' \rightarrow \ldots \]

... and in the space of Hamiltonians

\[ L \rightarrow L' \rightarrow L'' \rightarrow \ldots \]

local operators are mapped into local operators!
Entanglement entropy and correlations

- Entanglement entropy
  \[ S_L \approx \log (L) \]
  Computation of density matrix requires tracing out \( \sim \log(L) \) indices

- Two-point correlations
  \[ C(L) \approx L^{-2\Delta} \]
  Geodesic distance \( D \approx \log(L) \)
  \[ C(L) \approx e^{-D} = e^{-2\Delta \log(L)} = L^{-2\Delta} \]
Summary so far

- Variational parameters for different length scales
- It is secretly a **quantum circuit**

and an **RG transformation**

| | \( |\Psi\rangle \rightarrow |\Psi'\rangle \) | \( H \rightarrow H' \) |
|---|---|---|
| “removes short-range entanglement” | “entanglement at different length scales” | “preservation of locality” |

- Entanglement entropy and correlations as in 1+1 critical ground states
  
  \[ S_L \approx \log (L) \quad C(L) \approx L^{-2\Delta} \]

**blah, blah, blah... However, does it work?**

[Given lattice Hamiltonian \( H \), optimize variational parameters by energy minimization]
1D quantum Hamiltonian
- on the lattice
- at a critical point

Numerical determination of conformal data:
- central charge $c$
- scaling dimensions $\Delta_\alpha \equiv \frac{\hbar}{\alpha} + \frac{\hbar}{\alpha}$
- and conformal spins $s_\alpha \equiv \frac{\hbar}{\alpha} - \frac{\hbar}{\alpha}$
- OPE coefficients $C_{\epsilon\alpha\beta\gamma}$

Also:
1+1 critical systems with
- impurities/defects
- boundaries
- interfaces

2+1 gapped phases with
- topological order
- frustrated spins
- interacting fermions

(tensor networks have no sign problem)

Also:
- $\Delta_\epsilon \approx 0.99993$
- $\Delta_\mu \approx 0.125002$
- $\Delta_\psi \approx 0.500001$
- $\Delta_{\bar{\psi}} \approx 0.500001$

Output

E.g. critical Ising model (approx. an hour on your laptop)
Pfeifer, Evenbly, Vidal 08

$$\begin{align*}
C_{\epsilon\sigma\sigma} &= \frac{1}{2} \\
C_{\epsilon\mu\mu} &= -\frac{1}{2} \\
C_{\epsilon\psi\bar{\psi}} &= i \\
C_{\epsilon\bar{\psi}\psi} &= -i \\
C_{\psi\mu\sigma} &= \frac{e^{-\frac{i\pi}{4}}}{\sqrt{2}} \\
C_{\bar{\psi}\mu\sigma} &= \frac{i\pi}{e^{\frac{4}{4}}} \\
&= (\pm 6 \times 10^{-4})
\end{align*}$$
outline

Part 1: (old stuff)

- Tensor networks
- Renormalization group

+ quantum circuit
+ RG transformation

= Multi-scale entanglement renormalization ansatz (MERA)

Part 2: (recent developments)

- RG flow in the space of tensors
- Local scale transformations on the lattice
  - plane to cylinder
  - hyperbolic plane (MERA)
  - thermal states / black holes
Euclidean path integral

\[ Z(\lambda) = \text{tr} \ e^{-\beta H_{q}^{1d}} \]

Statistical partition function

\[ Z(T) = \sum_{\{s\}} e^{-\frac{1}{T} H_{\text{cl}}^{2d}} \]

as a tensor network

\[ Z = A_{\alpha \beta \gamma \delta} \]
Tensor Network Renormalization (TNR)

\[\text{[Evenbly, Vidal 14-15]}\]
Tensor Network Renormalization (TNR)

[Evenbly, Vidal 14-15]
TNR $\to$ proper RG flow
Example: 2D classical Ising

$A \to A' \to A'' \to \cdots \to A^{fp}$

below critical
$T = 0.9 \, T_c$

ordered (Z2) fixed point

critical
$T = T_c$

critical fixed point

above critical
$T = 1.1 \, T_c$

disordered (trivial) fixed point
local scale transformations

<table>
<thead>
<tr>
<th>Plane to cylinder</th>
<th>(radial quantization in CFT)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(z \equiv x + iy)</td>
<td>(z = 2^w)</td>
</tr>
<tr>
<td>(w \equiv s + i\theta)</td>
<td>(s \equiv \log_2 \left[ \sqrt{(x^2 + y^2)} \right])</td>
</tr>
</tbody>
</table>

- Extraction of scaling dimensions, OPE
local scale transformations

\[ |\Psi\rangle \sim e^{-\tau H} |\phi_0\rangle \]

Upper half plane to cylinder

TNR produces MERA
MERA = variational ansatz

MERA = by-product of TNR

energy optimization

energy minimization
- 1000s of iterations over scale
- local minima
- correct ground?

TNR \rightarrow MERA
- single iteration over scale
- rewrite tensor network for ground state
- certificate of accuracy
MERA for a thermal state (or black hole in holography)

\[ \rho_\beta \sim e^{-\beta H} \]
Multi-scale entanglement renormalization ansatz (MERA)

• Part I: variational ansatz for ground states of CFTs (on the lattice)

MERA \sim \text{CFT}

• Part II: by-product of coarse-graining the Euclidean path integral

TNR \to \text{MERA}

\text{MERA} \leftrightarrow \text{holography}

?
Tensor network for ground state/Hilbert space of CFT, organized in extra dimension corresponding to scale

**generic CFT** (no large $N$, strong interactions)

For smaller scale? $\rightarrow$ cMERA

Useful test bed

Generalized notion of *holographic* description?

<table>
<thead>
<tr>
<th>boundary</th>
<th>bulk</th>
</tr>
</thead>
<tbody>
<tr>
<td>state of CFT (or Hilbert space of CFT)</td>
<td>tensor network (?) (or unitary map, EHM)</td>
</tr>
<tr>
<td>scaling dimension $\Delta$</td>
<td>mass $\sim \Delta$</td>
</tr>
<tr>
<td>entanglement entropy</td>
<td>“minimal connecting surface”</td>
</tr>
<tr>
<td>global on-site symmetry (e.g. $Z_2$)</td>
<td>local/gauge symmetry (e.g. $Z_2$)</td>
</tr>
</tbody>
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THANK YOU!