Holography for $\mathcal{N} = 1^*$ on S^4

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with Henriette Elvang, Daniel Freedman, Silviu Pufu
Uri Kol, Tim Olson

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- Study $\mathcal{N} = 1$ theories hologrphically. Localization has not been successful for these theories.

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- $\mathcal{N} = 1^*$ SYM is a theory of an $\mathcal{N} = 1$ vector multiplet and 3 massive chiral multiplets in the adjoint of the gauge group. It is a massive deformation of $\mathcal{N} = 4$ SYM. There is a unique supersymmetric Lagrangian on S^4. [Pestun], [Festuccia-Seiberg]

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- The result from localization for $N, \lambda \gg 1$ for $\mathcal{N} = 2^*$ is [Russo-Zarembo]

$$F_{S^4}^{\mathcal{N}=2^*} = -\log \mathcal{Z}_{S^4}^{\mathcal{N}=2^*} = -\frac{N^2}{2}(1 + (mR)^2) \log \frac{\lambda(1 + (mR)^2) e^{2\gamma + \frac{1}{2}}}{16\pi^2},$$

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- Previous results for holography for $\mathcal{N}=1^*$ and $\mathcal{N}=2^*$ on $\mathbb{R}^4$. [FGPW], [GPPZ], [Pilch-Warner], [Buchel-Peet-Polchinski], [Evans-Johnson-Petrini], [Polchinski-Strassler], ... On S^4 the holographic construction is more involved.

Plan

- $\mathcal{N} = 1^*$ SYM theory on S^4
- The gravity dual of $\mathcal{N} = 2^*$ on S^4
- Holographic calculation of $F_{S^4}^{\mathcal{N}=2^*}$
- Holography for $\mathcal{N} = 1^*$ SYM theory on S^4
- Outlook

$\mathcal{N} = 1^*$ SYM theory on S^4

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$$A_\mu\,, \qquad X_{1,2,3,4,5,6}\,, \qquad \lambda_{1,2,3,4}\,.$$

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Organize this into an $\mathcal{N} = 1$ vector multiplet

$$A_\mu \,, \qquad \psi_1 \equiv \lambda_4 \,,$$

and 3 chiral multiplets

$$\chi_j = \lambda_j \,, \qquad Z_j = \frac{1}{\sqrt{2}}\big(X_j + iX_{j+3}\big), \qquad j = 1, 2, 3 \,.$$

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The $\mathcal{N} = 1^*$ theory is obtained by giving (independent) mass terms for the chiral multiplets.

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$$\begin{aligned}
\mathcal{L}^{S^4}_{\mathcal{N}=1^*} \;=\;& \mathcal{L}^{S^4}_{\mathcal{N}=4} \\
&+ \frac{2}{R^2}\,\mathrm{tr}\Big(Z_1\tilde{Z}_1 + Z_2\tilde{Z}_2 + Z_3\tilde{Z}_3\Big) \\
&+ \mathrm{tr}\left(m_1\tilde{m}_1 Z_1\tilde{Z}_1 + m_2\tilde{m}_2 Z_2\tilde{Z}_2 + m_3\tilde{m}_3 Z_3\tilde{Z}_3\right) \\
&- \frac{1}{2}\,\mathrm{tr}\left(m_1\chi_1\chi_1 + m_2\chi_2\chi_2 + m_3\chi_3\chi_3 + \tilde{m}_1\tilde{\chi}_1\tilde{\chi}_1 + \tilde{m}_2\tilde{\chi}_2\tilde{\chi}_2 + \tilde{m}_3\tilde{\chi}_3\tilde{\chi}_3\right) \\
&- \frac{1}{\sqrt{2}}\,\mathrm{tr}\left[m_i\epsilon^{ijk}Z_i\tilde{Z}_j\tilde{Z}_k + \tilde{m}_i\epsilon^{ijk}\tilde{Z}_i Z_j Z_k\right] \\
&+ \frac{i}{2R}\,\mathrm{tr}\left(m_1 Z_1^2 + m_2 Z_2^2 + m_3 Z_3^2 + \tilde{m}_1\tilde{Z}_1{}^2 + \tilde{m}_2\tilde{Z}_2{}^2 + \tilde{m}_3\tilde{Z}_3{}^2\right) .
\end{aligned}$$

15 (real) relevant couplings in the Lagrangian + 1 complex gaugino vev + 1 complexified gauge coupling. A 19 (real) parameter family of $\mathcal{N} = 1$ deformations of $\mathcal{N} = 4$ SYM. Only 18 of these parameters are visible as modes in IIB supergravity.

For $m_3 = \tilde{m}_3 = 0$, $m_1 = m_2 \equiv m$ and $\tilde{m}_1 = \tilde{m}_2 \equiv \tilde{m}$ we get the $\mathcal{N} = 2^*$ theory.

Results from localization for $\mathcal{N} = 2^*$

The $SU(N)$ gauge symmetry is generally broken to $U(1)^{N-1}$ by a vev for Z_3.

$$Z_3 = \text{diag}(a_1, \ldots, a_N) \ .$$

After supersymmetric localization the path integral for the theory on S^4 reduces to a finite dimensional integral over the Coulomb moduli a_i. [Pestun]

$$\mathcal{Z} = \int \prod_{i=1}^{N} da_i \, \delta\left(\sum_{i=1}^{N} a_i\right) \prod_{i<j} (a_i - a_j)^2 \mathcal{Z}_{\text{1-loop}} |\mathcal{Z}_{\text{inst}}|^2 e^{-S_{\text{cl}}} \ ,$$

where

$$S_{\text{cl}} = \frac{8\pi^2 N}{\lambda} \sum_{i=1}^{N} a_i^2 \ , \qquad \mathcal{Z}_{\text{1-loop}} = \prod_{i=1}^{N} \frac{H^2(a_i - a_j)}{H(a_i - a_j + mR) H(a_i - a_j - mR)} \ ,$$

$$H(x) = \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2}\right)^n e^{-x^2/n} \ .$$

The function $\mathcal{Z}_{\text{inst}}$ is Nekrasov's partition function with parameters $\varepsilon_1 = \varepsilon_2 = 1/R$.

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This answer depends on the regularization scheme. The scheme independent quantity is

$$\boxed{\frac{d^3 F_{S^4}}{d(mR)^3} = -2N^2 \frac{mR((mR)^2 + 3)}{((mR)^2 + 1)^2}}$$

This is the unambiguous result one can aim to compute holographically.

The gravity dual

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The result is that only 3 real scalars, 2 Abelian gauge fields and the metric survive in the bosonic sector. The 3 scalars, $\{\phi, \psi, \chi\}$, are dual to 3 relevant operators

$$\mathcal{O}_\phi\,, \quad \Delta_{\mathcal{O}_\phi} = 2\,; \qquad \mathcal{O}_\psi\,, \quad \Delta_{\mathcal{O}_\psi} = 3\,; \qquad \mathcal{O}_\chi\,, \quad \Delta_{\mathcal{O}_\chi} = 2\,.$$

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It is convenient to use

$$\eta = e^{\phi/\sqrt{6}}\,, \qquad z=\frac{1}{\sqrt{2}}\big(\chi+i\psi\big)\,, \qquad \tilde{z}=\frac{1}{\sqrt{2}}\big(\chi-i\psi\big)\,.$$

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For $\chi = 0$ recover the truncation for $\mathcal{N}=2^*$ on $\mathbb{R}^4$. [Pilch-Warner]

Supergravity setup

The Euclidean Lagrangian is

$$\mathcal{L} = \frac{1}{2\kappa^2}\left[-\mathcal{R} + 12\frac{\partial_\mu \eta \partial^\mu \eta}{\eta^2} + 4\frac{\partial_\mu z \partial^\mu \tilde{z}}{(1-z\tilde{z})^2} + \mathcal{V}\right] ,$$

$$\mathcal{V} \equiv -\frac{4}{L^2}\left(\frac{1}{\eta^4} + 2\eta^2\frac{1+z\tilde{z}}{1-z\tilde{z}} + \frac{\eta^8}{4}\frac{(z-\tilde{z})^2}{(1-z\tilde{z})^2}\right) .$$

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To preserve the isometries of S^4 take the "domain-wall" Ansatz

$$ds^2 = L^2 e^{2A(r)} ds^2_{S^4} + dr^2\,,$$
$$\eta = \eta(r)\,, \qquad z = z(r)\,, \qquad \tilde{z} = \tilde{z}(r)\,.$$

The masses of the scalars around the AdS_5 vacuum are

$$m_\phi^2 L^2 = m_\chi^2 L^2 = -4\,, \qquad m_\psi^2 L^2 = -3\,.$$

The BPS equations

Plug the Ansatz in the supersymmetry variations of the 5d $\mathcal{N} = 8$ theory and use the "conformal Killing spinors" on S^4

$$\hat{\nabla}_\mu \zeta = \frac{1}{2} \gamma_5 \gamma_\mu \zeta \ ,$$

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to derive the BPS equations

$$z' = \frac{3\eta'(z\tilde{z}-1)\left[2(z+\tilde{z})+\eta^6(z-\tilde{z})\right]}{2\eta\left[\eta^6\left(\tilde{z}^2-1\right)+\tilde{z}^2+1\right]} \ ,$$

$$\tilde{z}' = \frac{3\eta'(z\tilde{z}-1)\left[2(z+\tilde{z})-\eta^6(z-\tilde{z})\right]}{2\eta\left[\eta^6\left(z^2-1\right)+z^2+1\right]} \ ,$$

$$(\eta')^2 = \frac{\left[\eta^6\left(z^2-1\right)+z^2+1\right]\left[\eta^6\left(\tilde{z}^2-1\right)+\tilde{z}^2+1\right]}{9L^2\eta^2(z\tilde{z}-1)^2} \ ,$$

$$e^{2A} = \frac{(z\tilde{z}-1)^2\left[\eta^6\left(z^2-1\right)+z^2+1\right]\left[\eta^6\left(\tilde{z}^2-1\right)+\tilde{z}^2+1\right]}{\eta^8\left(z^2-\tilde{z}^2\right)^2} \ .$$

UV expansion

The (constant curvature) metric on $\mathbb{H}^5$ is

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$$e^{2A} = \frac{e^{2r}}{4} + \frac{1}{6}(\mu^2 - 3) + \ldots \ ,$$

$$\eta = 1 + e^{-2r}\left[\frac{2\mu^2}{3} r + \frac{\mu(\mu + v)}{3}\right] + \ldots \ ,$$

$$\frac{1}{2}(z + \tilde{z}) = e^{-2r}\Big[2\mu\, r + v\Big] + \ldots \ ,$$

$$\frac{1}{2}(z - \tilde{z}) = \mu\, e^{-r} + e^{-3r}\left[\frac{4}{3}\mu\Big(\mu^2 - 3\Big)\, r + \frac{1}{3}\Big(2v(\mu^2 - 3) + \mu(4\mu^2 - 3)\Big)\right] + \ldots \ .$$

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Here μ and v are integration constants. Think of them as the "source" and "vev" for the operator $\mathcal{O}_\chi$. Compare to field theory to identify $\mu = imR$.

IR expansion

At $r = r_*$ the S^4 shrinks smoothly to zero size. Solve the BPS equations close to $r = r_*$, and require that the solution is smooth to find

IR expansion

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$$e^{2A} = (r - r_*)^2 + \frac{7{\eta_0}^{12} + 20}{81{\eta_0}^4}(r - r_*)^4 + \dots \; ,$$

$$\eta = \eta_0 - \left(\frac{{\eta_0}^{12} - 1}{27{\eta_0}^3}\right)(r - r_*)^2\left[1 - \left(\frac{85 + 131{\eta_0}^{12}}{810{\eta_0}^4}\right)(r - r_*)^2 + \dots\right],$$

$$\frac{1}{2}(z + \tilde{z}) = \sqrt{\frac{{\eta_0}^6 - 1}{{\eta_0}^6 + 1}}\left[\frac{{\eta_0}^6}{{\eta_0}^6 + 2} - \frac{2{\eta_0}^8(4{\eta_0}^6 + 5)}{15({\eta_0}^6 + 2)^2}(r - r_*)^2 + \dots\right],$$

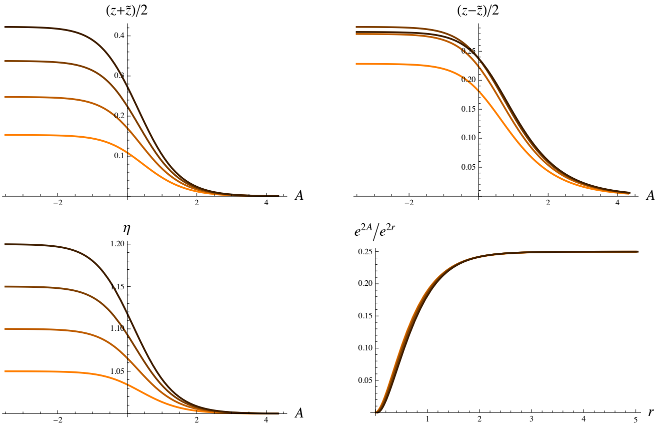
$$\frac{1}{2}(z - \tilde{z}) = \sqrt{\frac{{\eta_0}^6 - 1}{{\eta_0}^6 + 1}}\left[\frac{2}{{\eta_0}^6 + 2} + \frac{{\eta_0}^2(3{\eta_0}^{12} - 10{\eta_0}^6 - 20)}{15({\eta_0}^6 + 2)^2}(r - r_*)^2 + \dots\right].$$

Here, η_0 is the only free parameter since one can set $r_* = 0$ by the shift symmetry of the equations.

Numerical solutions

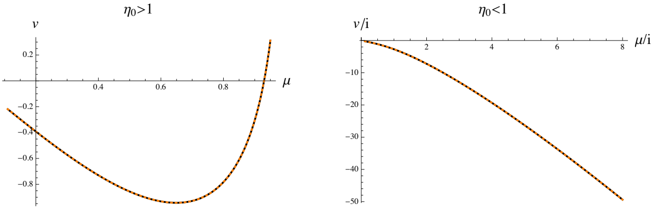
One can find numerical solutions by "shooting" from the IR to the UV. There is a one (complex) parameter family parametrized by η_0, so

$$v = v(\eta_0) \,, \qquad \text{and} \qquad \mu = \mu(\eta_0)\,.$$



Numerical solutions

For real η_0 one finds the following results for $v(\mu)$



From the numerical results one can extract the following dependence

$$\boxed{v(\mu) = -2\mu - \mu\,\log(1-\mu^2)}$$

Holographic calculation of $F_{S^4}^{\mathcal{N}=2^*}$

Calculating F from supergravity

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- Without knowing this finite counterterm we can only hope to match $\frac{d^5 F}{d\mu^5}$ with field theory.

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For both examples we found explicit supergravity truncations and (numerical) BPS solutions on S^4.

The hard part is to extract the third derivative of the free energy from the numerical solutions.

No results from localization to guide us.

LS on S^4

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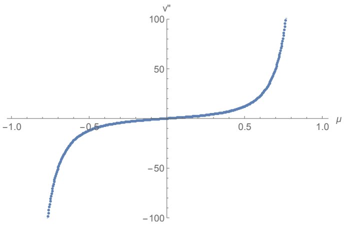
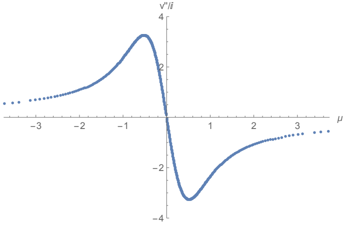
On the whole complex μ-plane $v''(\mu)$ vanishes linearly around $\mu \approx 0$. It is an odd function of μ and falls off for $|\mu| \gg 1$ as

$$|v''(\mu)| \sim \frac{2}{|\mu|}\,.$$





There is a pole at $\mu = \pm 1$ with

$$|v''(\mu)| \sim \frac{1}{(1 \pm \mu)^3}\,.$$

Summary

- We found a 5d supegravity dual of $\mathcal{N} = 2^*$ SYM on S^4.
- After careful holographic renormalization we computed the universal part of the free energy of this theory.
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- Extension of these results to $\mathcal{N} = 1^*$ SYM where our results can be viewed as supergravity "lessons" for the dynamics of the gauge theory.
- The results extend immediately to $\mathcal{N} = 2^*$ mass deformations of quiver gauge theories obtained by $\mathbb{Z}_k$ orbifolds of $\mathcal{N} = 4$ SYM.
  [Azeyanagi-Hanada-Honda-Matsuo-Shiba]

Outlook

- Uplift of the $\mathcal{N}=2^*$ solution to IIB supergravity. Will allow for a holographic calculation of Wilson or 't Hooft line vevs. In addition one can study probe D3-branes. [Chen-Lin-Zarembo]
- Holography for $\mathcal{N}=1^*$ on other 4-manifolds. [Cassani-Martelli]
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- Extensions to other dimensions.
- Can we see some of the large N phase transitions argued to exist by Russo-Zarembo in IIB string theory?
- Revisit supersymmetric localization for $\mathcal{N} = 1$ theories on S^4. Can one find the exact partition function (modulo ambiguities)? [Gerchkovitz-Gomis-Komargodski]
- For 4d $\mathcal{N} = 2$ conformal theories Z_{S^4} gives the Zamolodchikov metric. What is the "meaning" of Z_{S^4} for non-conformal theories?
- Broader lessons for holography from localization?

GRACIAS!

Comments

For a CFT on S^4 of radius R with cutoff $\epsilon \to 0$

$$F_{S^4} = \alpha_4 \left(\frac{R}{\epsilon}\right)^4 + \alpha_2 \left(\frac{R}{\epsilon}\right)^2 + \alpha_0 - a_{\text{anom}} \log\left(\frac{R}{\epsilon}\right) + \mathcal{O}(\epsilon/R) \ .$$

For supersymmetric theories with a supersymmetric regularization scheme $\alpha_4 = 0$.

For massive theories there is an extra scale, m, so $\alpha_2 = \alpha_2(m\epsilon)$ and $\alpha_0 = \alpha_0(m\epsilon)$. Expand this for small $m\epsilon$

$$\alpha_2 = \tilde{\alpha}_2 + m^2\epsilon^2\beta_2 + \mathcal{O}(m^4\epsilon^4) \ ,$$
$$\alpha_0 = \tilde{\alpha}_0 + \mathcal{O}(m^2\epsilon^2) \ .$$

The nonuniversal contribution to the free energy is

$$\tilde{\alpha}_2 \left(\frac{R}{\epsilon}\right)^2 + \tilde{\alpha}_0 + \beta_2 (mR)^2 \ .$$

Thus 3 derivatives w.r.t. mR eliminate the ambiguity.

For nonsupersymmetric theories the 5th derivative of F_{S^4} w.r.t. mR is universal.