TAE 2014
LHC Exotics Exercises

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1 Naturalness and chance

A non-physics problem, that anyway might give us a hard time thinking about what is causal and what is casual in nature. The question is, why the following quotient of numbers

$$\frac{987654321}{123456789} = 8.0000000729 \tag{1}$$

is so extremely close to an integer? Solution in the next page, so you can think a bit about it

2 Solution

Although I'm not a mathematician and I might not known base concepts that would simplify this explanation, I hope it is clear enough for you. The matter is related to the fact that our numeration system is a base 10 system. The way we construct these two numbers, 987654321 and 123456789 depends only on this. For instance, if we used an octal system (base 8), we would write 1234567 and 7654321 as our numbers.

Of course with a different base, the numbers mean something different as well, in the sense that the positions of the digits carry a different weight. The most well known example is the binary system used in computers, which is base 2. You can check what the different numbers mean in different base in the following table:

Number	Base 2	Base 8	Base 10	Base 16
0	0	0	0	0
1	1	1	1	1
10	$2 = 2^1$	$8 = 8^1$	$10 = 10^1$	$16 = 16^1$
11	$3 = 2^1 + 1$	$9 = 8^1 + 1$	$11 = 10^1 + 1$	$17 = 16^1 + 1$
100	$4 = 2^2$	$64 = 8^2$	$100 = 10^2$	$256 = 16^2$
1000	$8 = 2^3$	$512 = 8^3$	$1000 = 10^3$	$4096 = 16^3$

In the first column I write a number combining the digits 0 and 1. This numbers has different interpretations according to the base we use, as I try to show in the following columns. For instance, 10 written in base 2 equals 2 in base 10. In the end the equivalence is very simple: for base N each digit ranges from 0 to $N-1$, and each time we advance one digit position, we add a factor N to the weight. The position 0 always has weight $1 = N^0$, but as we move the digits to the left, the weight increases potentially. This way, the position 1 has weight $N^1 = N$, the position 2 has weight N^2 and the position k has weight N^k. Let's analyze the number 3572 this way, and see what 3572 means in the different bases:

Position of digit $\Longrightarrow$	3	2	1	0	= Total in base 10
Weight base 10	10^3	10^2	10^1	10^0	
Weight base 8	8^3	8^2	8^1	8^0	
Weight base 12	12^3	12^2	12^1	12^0	
Weight base 16	16^3	16^2	16^1	16^0	
Number base 10	3×10^3	5×10^2	7×10^1	2×10^0	$= 3572$
Number base 8	3×8^3	5×8^2	7×8^1	2×8^0	$= 1914$
Number base 12	3×12^3	5×12^2	7×12^1	2×12^0	$= 5990$
Number base 16	3×16^3	5×16^2	7×16^1	2×16^0	$= 13682$

This way we can construct any number in any base. How would I write the equivalent of quotient (1) for base N? It would be for sure a different quotient, with base-N numbers in numerator and denominator, with $N-1$ digits each one of these numbers, like

$$\frac{(N-1)(N-2)(N-3)(\ldots)(2)(1)}{(1)(2)(\ldots)(N-3)(N-2)(N-1)}, \tag{2}$$

where the parenthesis mean that the number within is a digit belonging to the bigger number, so if I wanted to write 1564 in this notation, I would write (1)(5)(6)(4). The fraction (2) can be expressed in general as the sum

$$\frac{\sum_{i=1}^{N-1} iN^{i-1}}{\sum_{i=1}^{N-1} (N-i)\, N^{i-1}}. \tag{3}$$

Solving this quotient we can find out the mystery of this apparent fine-tuning of nature. In order to carry out these sums one can use the following trick

$$\sum_{i=1}^{N-1} iN^{i-1} = \sum_{i=1}^{N-1} \frac{d}{dN} N^i = \frac{d}{dN} \sum_{i=1}^{N'-1} N^i \text{ with } N' = N, \tag{4}$$

where the N' was introduced so we don't derivate it. The sum inside the derivative is a geometrical sum, and we know how to solve it

$$\sum_{i=1}^{N-1} N^i = \frac{N^N - 1}{N-1}, \tag{5}$$

and its derivative

$$\frac{d}{dN} \frac{N^N - 1}{N-1} = \frac{N \times N^{N-1}}{N-1} - \frac{N^N - 1}{(N-1)^2} = \frac{N^{N+1} - 2N^N + 1}{(N-1)^2}, \tag{6}$$

so we solved the numerator. The denominator is just

$$\sum_{i=1}^{N-1} (N-i)\, N^{i-1} = \sum_{i=1}^{N-1} N^i - \sum_{i=1}^{N-1} iN^{i-1} = \frac{N^N - 1}{N-1} - \frac{N^{N+1} - 2N^N + 1}{(N-1)^2} =$$

$$\frac{N^{N+1} - N^N - N + 1 - N^{N+1} + 2N^N - 1}{(N-1)^2} = \frac{N^N - N}{(N-2)^2}, \tag{7}$$

that is, the difference between (5) and (4), so we have all the ingredients to solve the quotient in an arbitrary base N. Let's do it

$$\frac{\sum_{i=1}^{N-1} iN^{i-1}}{\sum_{i=1}^{N-1} (N-i)\, N^{i-1}} = \frac{\frac{N^{N+1}-2N^N+1}{(N-1)^2}}{\frac{N^N-N}{(N-2)^2}} =$$

$$\frac{N^{N+1} - 2N^N + 1}{N^N - N} = \frac{N - 2 + \frac{1}{N^N}}{1 - \frac{1}{N^{N-1}}} \approx N - 2 + O\left(N^{2-N}\right)$$

The result is approximately an integer, and the correction to the result decrease exponentially as the base increases, so in the end there is no magic at all in the result.