# TAE 2014
# Quantum Field Theory exercises

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September 24, 2014

## 1 Massive spin-1 field

### 1.1 Exercise 1: Equations of motion

Given the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{1}$$

and remembering the equations of motion

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu A_\nu\right)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0, \tag{2}$$

show that

$$\left[g^{\mu\nu}\left(\Box + m^2\right) - \partial^\mu\partial^\nu\right] A_\nu = 0, \tag{3}$$
$$\partial_\mu A^\mu = 0. \tag{4}$$

#### 1.1.1 Solution

In order to solve this problem correctly, one needs to remember the rules of the tensor derivative

$$\frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha_\beta. \tag{5}$$

With this, we write the equations of motion (2) for the Lagrangian (1) to get an identity. We evaluate (2) piece by piece:

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = \frac{\partial\left(\frac{m^2}{2}A_\mu A^\mu\right)}{\partial A_\nu} = \frac{\partial\left(\frac{m^2}{2}A_\mu g^{\mu\rho}A_\rho\right)}{\partial A_\nu} =$$
$$\frac{m^2}{2}\left[\delta^\nu_\mu g^{\mu\rho}A_\rho + A_\mu g^{\mu\rho}\delta^\nu_\rho\right] = m^2 A^\nu,$$

and from here one discovers that the tensor derivative rule is quite analogue to the normal derivative rule, although one should always proceed with care. For the other term in (2)

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu A_\nu\right)} = -\frac{1}{4}\partial_\mu \frac{\partial\left[\left(\partial_\sigma A_\rho - \partial_\rho A_\sigma\right)\left(\partial^\sigma A^\rho - \partial^\rho A^\sigma\right)\right]}{\partial\left(\partial_\mu A_\nu\right)} =$$

$$-\frac{1}{4}\partial_\mu\left[\left(g_{\sigma\alpha}g_{\rho\beta}\left(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha\right) + \delta^\mu_\sigma\delta^\nu_\rho - \delta^\mu_\rho\delta^\nu_\sigma\right)\left(\partial^\sigma A^\rho - \partial^\rho A^\sigma\right)\right] =$$

$$-\partial_\mu\left[\partial^\mu A^\nu - \partial^\nu A^\mu\right]$$

So the final equations of motion are

$$\partial_\mu\partial^\mu A^\nu - \partial_\mu\partial^\nu A^\mu + m^2 A^\nu = 0. \tag{6}$$

Equation (6) will help us prove the two relations defined in (3) and (4). Let's start with the first one

$$\left[g^{\mu\nu}\left(\Box + m^2\right) - \partial^\mu\partial^\nu\right]A_\nu = \Box A^\mu + m^2 A^\mu - \partial^\mu\partial^\nu A_\nu$$

Substituting $m^2 A^\mu$ by the difference of derivatives, as (6) indicates,

$$\Box A^\mu - \partial_\nu\partial^\nu A^\mu + \partial_\nu\partial^\mu A^\nu - \partial^\mu\partial^\nu A_\nu = 0,$$

for by definition, the Lambertian is

$$\Box A^\mu = \partial_\rho\partial^\rho A^\mu.$$

The relation (4) is also easy to obtain. We just take a derivative in the first one

$$\partial_\mu\left[g^{\mu\nu}\left(\Box + m^2\right) - \partial^\mu\partial^\nu\right]A_\nu = \partial_\mu\partial_\rho\partial^\rho A^\mu + m^2\partial_\mu A^\mu - \partial_\mu\partial^\mu\partial^\nu A_\nu = m^2\partial_\mu A^\mu$$

but we know that this expression is equal to zero due to (3), so its derivative will also be zero. Therefore

$$m^2\partial_\mu A^\mu = 0,$$

and we have proved (4), provided that $m \neq 0$.

### 1.2 Exercise 2: Plane wave solutions

A plane-wave solution has the form

$$A^\nu = \epsilon^\nu e^{\pm ipx} \tag{7}$$

Taking into account the Lagrangian of the last exercise,

1. What are the conditions on the polarization vector?
2. Given $p^\mu = (E, 0, 0, p)$ with $p = |\vec{p}|$, construct a 3-orthogonal $\epsilon^\nu$ with $\epsilon^2 = -1$
3. Show that $\epsilon^\nu_3 = \epsilon^\nu_L \sim \frac{p^\nu}{m}$ for $p >> m$

### 1.2.1 Solution

The conditions on the polarization vector can be obtained from (4). Since we derived all the relations from the equations of motion, one doesn't expect to get new information from the other relations. Substituting (7) in (4)

$$\partial_\mu A^\mu = \pm p_\mu \epsilon^\mu e^{\pm ipx} = 0 \qquad \Longrightarrow \qquad p_\mu \epsilon^\mu = 0, \tag{8}$$

thus the polarization vector $\epsilon^\mu$ and the momentum $p_\mu$ must be orthogonal. This doesn't mean in principle that only transversal polarizations are allowed, because the $p^0$ component will always be non-zero.

The construction of an orthogonal polarization vector should be straightforward. The components of $\epsilon$ verify two constrains

$$\epsilon_0 E - \epsilon_3 p = 0, \tag{9}$$

$$(\epsilon_0)^2 - (\epsilon_1)^2 - (\epsilon_2)^2 - (\epsilon_3)^2 = -1. \tag{10}$$

From the first we get

$$\epsilon_0 = \epsilon_3 \frac{p}{E}. \tag{11}$$

We can introduce an angle $\theta$ representing how longitudinal our polarization vector is, so we split the norm in (10) in two:

$$(\epsilon_3)^2 - (\epsilon_0)^2 = \cos^2\theta,$$
$$(\epsilon_1)^2 + (\epsilon_2)^2 = \sin^2\theta.$$

For the first case we get

$$(\epsilon_3)^2 \left(1 - \frac{p^2}{E^2}\right) = \cos^2\theta \qquad \Longrightarrow \qquad \epsilon_3 = \frac{\cos\theta}{\sqrt{1 - \frac{p^2}{E^2}}}.$$

And for the second we need another angle $\phi$,

$$\epsilon_1 = \cos\phi\,\sin\theta, \qquad \epsilon_2 = \sin\phi\,\sin\theta,$$

So our most general polarization vector verifying all the constraints is

$$\epsilon_\mu = \left(\frac{p}{E}\frac{\cos\theta}{\sqrt{1 - \frac{p^2}{E^2}}}, \cos\phi\,\sin\theta, \sin\phi\,\sin\theta, \frac{\cos\theta}{\sqrt{1 - \frac{p^2}{E^2}}}\right). \tag{12}$$

If we add as a constraint that our polarization vector must be longitudinal (3-orthogonal vector), then $\theta = 0$ and

$$\epsilon_\mu = \left(\frac{\frac{p}{E}}{\sqrt{1 - \frac{p^2}{E^2}}}, 0, 0, \frac{1}{\sqrt{1 - \frac{p^2}{E^2}}}\right). \tag{13}$$

If the momentum is much larger than the mass in (13), we can operate with the third component of the polarization

$$\epsilon_3 = \frac{1}{\sqrt{1-\frac{p^2}{E^2}}} = \frac{1}{\sqrt{\frac{E^2-p^2}{E^2}}} = \frac{E}{m} = \frac{p_0}{m},$$

$$\epsilon_0 = \frac{p}{E}\epsilon_3 = \frac{p}{m},$$

where we have used (11). If we compare now $\epsilon^\mu$ and $p^\mu$, we will realize that

$$\epsilon^\mu = \frac{p^\mu}{m}. \tag{14}$$

## 1.3 Exercise 3: Propagator

Verify that

$$D_{\rho\nu}(k) = \left[-g_{\rho\nu} + \frac{k_\nu k_\rho}{m^2}\right]\frac{1}{k^2 - m^2 + i\epsilon}, \tag{15}$$

is the solution of the Green equation

$$\left[\left(-k^2+m^2\right)g^{\mu\rho} + k^\mu k^\rho\right] D_{\rho\nu}(k) = g^\mu_\nu, \tag{16}$$

### 1.3.1 Solution

We try simple substitution of (15) in (16) and check whether (16) is fulfilled:

$$\left[\left(-k^2+m^2\right)g^{\mu\rho} + k^\mu k^\rho\right]\left[-g_{\rho\nu} + \frac{k_\nu k_\rho}{m^2}\right]\frac{1}{k^2-m^2+i\epsilon} =$$

$$\left[\left(k^2-m^2\right)\left(g^\mu_\nu - \frac{k^\mu k_\nu}{m^2}\right) - k^\mu k_\nu + \frac{k^2 k^\mu k_\nu}{m^2}\right]\frac{1}{k^2-m^2+i\epsilon} =$$

$$\left[\left(k^2-m^2\right)\left(g^\mu_\nu - \frac{k^\mu k_\nu}{m^2}\right) - \frac{k^\mu k_\nu}{m^2}\left(m^2-k^2\right)\right]\frac{1}{k^2-m^2+i\epsilon} =$$

$$\left[g^\mu_\nu - \frac{k^\mu k_\nu}{m^2} + \frac{k^\mu k_\nu}{m^2}\right] = g^\mu_\nu$$

# 2 Scalar QED

## 2.1 Exercise 1: Lagrangian

Construct a Lagrangian $\mathcal{L}$ for scalar QED, with a scalar field $\phi(x) \neq \phi^\star(x)$ and a photon field $A_\mu(x)$

### 2.1.1 Solution

The first thing we might try is to adapt the Lagrangian of a free scalar field to QED by the principle of minimum substitution. The free scalar Lagrangian is

$$\mathcal{L}^R_{Free} = \partial^\mu\phi\partial_\mu\phi - m^2\phi^2. \tag{17}$$

But in order to introduce QED we need the Lagrangian to be invariant under $U(1)$ transformations. Obviously a real field is not enough, and we need to make our field complex

$$\mathcal{L}_{Free} = (\partial^\mu \phi)^\star \partial_\mu \phi - m^2 |\phi|^2 . \tag{18}$$

Now we can use the principle of minimum substitution

$$\partial_\mu \quad \rightarrow \quad D_\mu = \partial_\mu - ieA_\mu,$$

so the Lagrangian becomes

$$\mathcal{L} = (\partial^\mu \phi^\star + ieA^\mu \phi^\star)(\partial_\mu \phi - ieA_\mu \phi) - m^2 |\phi|^2 . \tag{19}$$

The new field $A_\mu$ becomes dynamic with the addition of the Yang-Mills kinetic term, and the final Lagrangian for scalar QED becomes

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + (\partial^\mu \phi^\star + ieA^\mu \phi^\star)(\partial_\mu \phi - ieA_\mu \phi) - m^2 |\phi|^2 . \tag{20}$$

## 2.2 Exercise 2: Interaction

Identify the interaction terms and derive the Feynman rules.

### 2.2.1 Solution

First we expand the kinetic term of the scalar field in (20)

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \partial^\mu \phi^\star \partial_\mu \phi + ieA^\mu \phi^\star \partial_\mu \phi - ieA_\mu \phi \partial^\mu \phi^\star + e^2 A_\mu^2 |\phi|^2 - m^2 |\phi|^2 . \tag{21}$$

Here we can easily identify a current

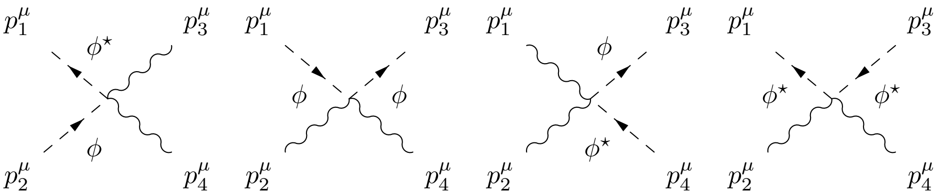
$$J_\mu = i \left( \phi^\star \partial_\mu \phi - \phi \partial_\mu^\star \phi \right), \tag{22}$$

that is nothing else than the Nöther current associated to charge conservation. One can obtain the expression in (22) by applying the Nöther theorem to the Lagrangian for global $U(1)$ transformations. The current couples to the electromagnetic field as $e\, A_\mu\, J^\mu$, so we can write

$$\mathcal{L} = \underbrace{-\frac{1}{4} F^{\mu\nu} F_{\mu\nu}}_{\text{Free gauge field}} + \underbrace{\partial^\mu \phi^\star \partial_\mu \phi - m^2 |\phi|^2}_{\text{Free massive scalar field}} + \underbrace{ieA^\mu \left( \phi^\star \partial_\mu \phi - \phi \partial_\mu \phi^\star \right) + e^2 A_\mu^2 |\phi|^2}_{\text{Interaction term}} . \tag{23}$$

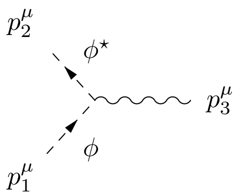
Therefore we have two kinds of vertex coming from the interaction Lagrangian: a four vertex with two scalar field lines and two photons, and the photon absorption/emission and and pair annihilation/creation vertices.

The first kind of vertex can be represented as

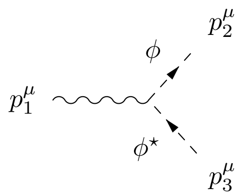



which is a four boson interaction without a counterpart in standard QED involving two photon interaction and also scalar-photon scattering. The weight associated with this vertex is $2ie^2 g_{\mu\nu}\delta\left(p_1^\mu + p_2^\mu - p_3^\mu - p_4^\mu\right)$, with the momenta $p^{1,2}$ coming from the left and $p^{3,4}$ from the right. The dirac-$\delta$ ensures momentum conservation, and the 2-factor comes from the symmetry in the electromagnetic field (since it appears squared, we have two different orderings for the photons involved in the vertex).

The second kind of vertex can be understood either as pair creation/annihilation

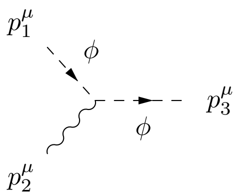



$-ie\left(p_1^\mu - p_2^\mu\right)\,\delta\left(p_1^\mu + p_2^\mu - p_3^\mu\right)$

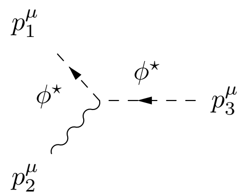



$-ie\left(p_2^\mu - p_3^\mu\right)\,\delta\left(p_1^\mu - p_2^\mu - p_3^\mu\right)$

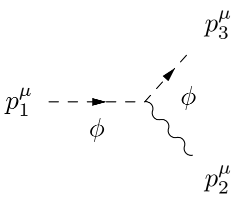
or as photon absorption/emission by a scalar electron



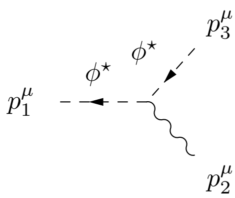

$-ie\left(p_1^\mu + p_3^\mu\right)\,\delta\left(p_1^\mu + p_2^\mu - p_3^\mu\right)$




$ie\left(p_1^\mu + p_3^\mu\right)\,\delta\left(p_1^\mu + p_2^\mu - p_3^\mu\right)$




$-ie\left(p_1^\mu + p_3^\mu\right)\,\delta\left(p_1^\mu - p_2^\mu - p_3^\mu\right)$




$ie\left(p_1^\mu + p_3^\mu\right)\,\delta\left(p_1^\mu - p_2^\mu - p_3^\mu\right)$

whose weights are written below each diagram. As you can immediately check, there are momentum factors appearing everywhere, due to the derivative that appears in the conserved current (22).

Finally, we need a rule for the internal scalar and photon lines, which will be given by the propagators. In the case of the photon, we already calculated it, and it is given by eq.(15) in the special case when $m = 0$. For the scalar field we are going to get exactly the same equation: being a spin-0 field, it behaves like a boson and follows the Klein-Gordon equation

$$\left(\Box + m^2\right)\phi = 0, \tag{24}$$

which is very similar to what we used for $A_\mu$ in (3). The difference between (3) and (24) arise because the photon is a spin-1 field, whereas our scalar has spin-0, so the solution for the propagator simplifies as

$$D\left(k\right) = \frac{1}{k^2 - m^2 + i\epsilon}, \tag{25}$$

with the following Green equation

$$\left(k^2 - m^2\right) D\left(k\right) = 1, \tag{26}$$

which is now trivial to solve. Therefore, the rules for the internal lines are

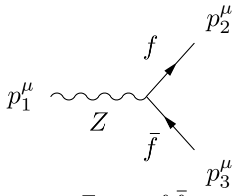

$\frac{i}{p^2-m^2+i\epsilon}$ $\qquad$ $-\frac{ig_{\mu\nu}}{p^2+i\epsilon}$

where the $i$ appears when we exponentiate the action.

# 3 Electroweak physics

## 3.1 Exercise 2: $Z-$boson decay width for $Z \longrightarrow f\bar{f}$

Calculate the partial decay width of a $Z$ boson into a pair fermion-antifermion of mass $m_f$. The fermion mass can be neglected in the calculation.




$Z \longrightarrow f\bar{f}$

### 3.1.1 Solution

The vertex in this decay carries a weight $ie\gamma_\mu(v_f - a_f\gamma_5)$, where

$$v_f = -\frac{s_W}{c_W}Q + \frac{T_I^3}{2c_W s_W}, \tag{27}$$

$$a_f = \frac{T_I^3}{2c_W s_W}. \tag{28}$$

It has such a complicated form because the $Z$ boson couples in a different way to left- and right-handed fermions. For left-handed fermions the coupling strength is given by $\frac{e}{c_W s_W}\left(T_I^3 - Qs_W^2\right)$, whereas the right-handed fermions don't have isospin and only couple to the electric charge $\frac{es_W}{c_W}Q$. The way we wrote the weight of the vertex ensures that left- and right- handed fermions are correctly accounted for, due to the $\gamma_5$ that introduces a different sign according to the chirality of the fermionic field. In the end the amplitude of the diagram is

$$\mathcal{M}_{if} = ie\bar{u}_{r1}\gamma_\mu(v_f - a_f\gamma_5)\epsilon^\mu v_{r2}, \tag{29}$$

being $r1$ and $r2$ the helicity of the fermions and $\epsilon^\mu$ the polarization of the $Z$ boson. The probability (and thence the decay width) is associated to the modulus squared of the amplitude, and since we don't care about the helicity of the final state, we must sum over helicities. On the other hand, we must average over the polarization of the $Z$ boson. Being this a spin-1 boson, averaging leads to a global factor $\frac{1}{3}$. First we write explicitly the vertex

$$\frac{1}{3}\sum_{r1,r2}|\mathcal{M}_{if}|^2 =$$

$$\frac{e^2}{3}\mathrm{Tr}\left\{\bar{u}\gamma_\mu\left(v_f - a_f\gamma_5\right)v\epsilon^\mu\left(\epsilon^\nu\right)^\star\bar{v}\gamma_0\left(v_f - a_f\gamma_5\right)\gamma_\nu^\dagger\gamma_0 u\right\},$$

where the second part is just the hermitian conjugated of the vertex. We can solve the product of polarizations by remembering that

$$\epsilon_\lambda^\mu\left(\epsilon_\lambda^\nu\right)^\star = -g^{\mu\nu} + \frac{p_1^\mu p_1^\nu}{M_Z^2}. \tag{30}$$

We substitute and start to move $\gamma$s around. Also the polarization product doesn't directly affect to the $\bar{v}$, so we can commute both terms, and since the trace is cyclic, we can also brin together the $u\bar{u}$ terms

$$= \frac{e^2}{3}\mathrm{Tr}\left\{u\bar{u}\left(v_f + a_f\gamma_5\right)\gamma_\mu v\bar{v}\left(-g^{\mu\nu} + \frac{p_1^\mu p_1^\nu}{M_Z^2}\right)\gamma_0\gamma_\nu^\dagger\gamma_0\left(v_f - a_f\gamma_5\right)\right\}$$

$$= \frac{e^2}{3}\mathrm{Tr}\left\{u\bar{u}\left(v_f + a_f\gamma_5\right)\gamma_\mu v\bar{v}\left(-g^{\mu\nu} + \frac{p_1^\mu p_1^\nu}{M_Z^2}\right)\gamma_\nu\left(v_f - a_f\gamma_5\right)\right\}.$$

Note that we applied $\gamma_0\left(\gamma_\mu\right)^\dagger\gamma_0 = \gamma_\mu$. Now we can use the completion rules for $f\bar{f}$

$$\sum_r u_r\bar{u}_r = \gamma_\mu p^\mu + m, \tag{31}$$

$$\sum_r v_r\bar{v}_r = \gamma_\mu p^\mu - m. \tag{32}$$

$$\tag{33}$$

The result is

$$= \frac{e^2}{3}\mathrm{Tr}\left\{\left(\gamma_\rho p_2^\rho + m_f\right)\left(v_f + a_f\gamma_5\right)\gamma_\mu\left(\gamma_\sigma p_3^\sigma - m_f\right)\left(-g^{\mu\nu} + \frac{p_1^\mu p_1^\nu}{M_Z^2}\right)\gamma_\nu\left(v_f - a_f\gamma_5\right)\right\}.$$

At this point we are suggested to neglect fermionic masses. I'll keep them because they don't bother too much. Commuting the $\gamma_5$ along we can bring together the term $v_f \pm a_f\gamma_5$, and solve it in the trace

$$= \frac{e^2}{3}\mathrm{Tr}\left\{\left(\gamma_\rho p_2^\rho + m_f\right)\gamma_\mu\left(\gamma_\sigma p_3^\sigma - m_f\right)\left(-g^{\mu\nu} + \frac{p_1^\mu p_1^\nu}{M_Z^2}\right)\gamma_\nu\left(v_f - a_f\gamma_5\right)^2\right\}$$

$$= \frac{e^2}{3}\mathrm{Tr}\left\{\left(\gamma_\rho p_2^\rho + m_f\right)\gamma_\mu\left(\gamma_\sigma p_3^\sigma - m_f\right)\left(-g^{\mu\nu} + \frac{p_1^\mu p_1^\nu}{M_Z^2}\right)\gamma_\nu\left(v_f^2 + a_f^2 - 2v_f a_f\gamma_5\right)\right\}$$

$$= \frac{e^2}{3}\left(v_f^2 + a_f^2\right)\left(-g^{\mu\nu} + \frac{p_1^\mu p_1^\nu}{M_Z^2}\right)\mathrm{Tr}\left\{\left(\gamma_\rho p_2^\rho + m_f\right)\gamma_\mu\left(\gamma_\sigma p_3^\sigma - m_f\right)\gamma_\nu\right\},$$

where we have removed the terms proportional to $\gamma_5$ because they were bound to disappear when taking the trace: these terms result in a pure imaginary trace proportional to the fully antisymmetric tensor $\epsilon_{\mu\nu\sigma\rho}$, but we can't obtain a pure imaginary number from a modulus. Fortunately, these terms disappear once we contract the indices of the antisymmetric tensor with another symmetric tensor, like $g_{\mu\nu}$. The remaining terms of the trace can be easily evaluated, as long as we remember

$$\mathrm{Tr}\left\{\gamma_\mu p_1^\mu\gamma_\nu p_2^\nu\right\} = p_1 \cdot p_2, \tag{34}$$

$$\mathrm{Tr}\left\{\gamma_\mu\gamma_\nu\right\} = 4g_{\mu\nu}, \tag{35}$$

$$\mathrm{Tr}\left\{\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu\right\} = 4\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\sigma}g_{\mu\nu} + g_{\rho\nu}g_{\sigma\mu}\right), \tag{36}$$

and with this

$$= \frac{4e^2}{3}\left(v_f^2 + a_f^2\right)\left(-g^{\mu\nu} + \frac{p_1^\mu p_1^\nu}{M_Z^2}\right)\left[(p_2)_\mu (p_3)_\nu + (p_2)_\nu (p_3)_\mu - g_{\mu\nu}(p_2 \cdot p_3) - g_{\mu\nu} m_f^2\right]$$

$$= \frac{4e^2}{3}\left(v_f^2 + a_f^2\right)\left[\left(-2 + 4 - \frac{p_1^2}{M_Z^2}\right)(p_2 \cdot p_3) + 2\frac{(p_1 \cdot p_3)(p_2 \cdot p_3)}{M_Z^2} + 3m_f^2\right]$$

$$= \frac{4e^2}{3}\left(v_f^2 + a_f^2\right)\left[(p_2 \cdot p_3) + 2\frac{(p_1 \cdot p_3)(p_2 \cdot p_3)}{M_Z^2} + 3m_f^2\right]$$

In the CMS frame the $Z$ boson is at rest, and we can solve the scalar product of momenta very easily

$$p_1^\mu = (M_Z, \vec{0}), \tag{37}$$

$$p_2^\mu = (E, \vec{p}), \tag{38}$$

$$p_3^\mu = (E, -\vec{p}), \tag{39}$$

so the products

$$p_1 \cdot p_1 = M_Z^2$$

$$p_1 \cdot p_{2,3} = M_Z^2/2$$

$$p_2 \cdot p_3 = E^2 + p^2 = 2p^2 + m_f^2$$

and since $M_Z = 2E$ in our frame,

$$p^2 = \frac{M_Z^2}{4} - m_f^2 = \frac{M_Z^2}{4}\left(1 - \frac{4m_f^2}{M_Z^2}\right).$$

At this point we can really neglect the fermion masses, for we expect $m_f << M_Z$ and

$$1 - \frac{4m_f^2}{M_Z^2} \approx 1$$

Replacing the dot products,

$$= \frac{4e^2}{3}\left(v_f^2 + a_f^2\right)\left[(p_2 \cdot p_3) + 2\frac{(p_1 \cdot p_3)(p_2 \cdot p_3)}{M_Z^2}\right]$$

$$= \frac{4e^2 M_Z^2}{3}\left(v_f^2 + a_f^2\right).$$

Thence the final result for the amplitude is

$$\frac{1}{3}\sum_{r1,r2} |\mathcal{M}_{if}|^2 = \frac{e^2 M_Z^2}{3c_W^2 s_W^2}\left[\left(T_I^3 - 2Qs_W^2\right)^2 + \left(T_I^3\right)^2\right]. \tag{40}$$

In this expression we have expanded the $v_f$ and the $a_f$ operators. Now we can add the phase space factors, neglecting again fermion masses,

$$\frac{d\Gamma\left(Z \to f\bar{f}\right)}{d\Omega} = \frac{N_c}{64\pi^2 M_Z}\sqrt{1 - \frac{4m_f^2}{M_Z^2}}\,|\mathcal{M}|^2 \approx \frac{N_c}{64\pi^2 M_Z}|\mathcal{M}|^2$$

where we took into account the number of colors for quarks. Substituting the amplitudes

$$\frac{d\Gamma\left(Z\to f\bar{f}\right)}{d\Omega}=\frac{N_c}{64\pi^2 M_Z}\left[\frac{1}{3}\sum_{r1,r2}\left|\mathcal{M}_{if}\right|^2\right]=$$

$$N_c\frac{e^2 M_Z}{192\pi^2 c_W^2 s_W^2}\left[\left(T_I^3-2Qs_W^2\right)^2+\left(T_I^3\right)^2\right]. \tag{41}$$

To get the whole decay width we must integrate the solid angle

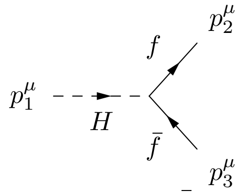
$$\Gamma\left(Z\to f\bar{f}\right)=N_c\frac{e^2 M_Z}{48\pi c_W^2 s_W^2}\left[\left(T_I^3-2Qs_W^2\right)^2+\left(T_I^3\right)^2\right], \tag{42}$$

where $N_c=3$ for quarks and $N_c=1$ for the rest of the fermions.

# 4 Higgs physics

## 4.1 Exercise 1: Decay width for $H \longrightarrow f\bar{f}$

Calculate the partial decay width of a Higgs boson into a pair fermion-antifermion of mass $m_f$




$H \longrightarrow f\bar{f}$

### 4.1.1 Solution

The strength of the vertex for the Higgs interaction (according to the Yukawa couplings arising in the Lagrangian) is $\frac{m_f}{v}$, therefore the amplitude of the diagram will be

$$\mathcal{M}_{if} = \frac{m_f}{v}\bar{u}_{r1}v_{r2}, \tag{43}$$

where $r1$ and $r2$ refer to the helicity. The square of the amplitude is the magnitude we are interested in

$$\sum_{r1,r2} |\mathcal{M}_{if}|^2 = \frac{m_f^2}{v^2}\mathrm{Tr}\left\{\left(\gamma_\mu p_2^\mu + m_f\right)\left(\gamma_\mu p_3^\mu - m_f\right)\right\} = \frac{4m_f^2}{v^2}\left(p_2 \cdot p_3 - m_f^2\right), \tag{44}$$

where we have used eq. (31), (32) and (34). Like in the $Z$ decay, in the CMS frame we have

$$p_1^\mu = (M_H, \vec{0}), \tag{45}$$
$$p_2^\mu = (E, \vec{p}), \tag{46}$$
$$p_3^\mu = (E, -\vec{p}), \tag{47}$$

therefore $p_2 \cdot p_3 = E^2 + p^2 = 2p^2 + m_f^2$ and since $M_H = 2E$ in our frame,

$$p^2 = \frac{M_H^2}{4} - m_f^2 = \frac{M_H^2}{4}\left(1 - \frac{4m_f^2}{M_H^2}\right).$$

Substituting in (44) one arrives to the final expression

$$\sum_{r1,r2} |\mathcal{M}_{if}|^2 = \frac{2m_f^2}{v^2}M_H^2\left(1 - \frac{4m_f^2}{M_H^2}\right). \tag{48}$$

For quarks one should take into account the number of colors, so we can add the $N_c$ factor in (48)

$$\frac{d\Gamma\left(H \to f\bar{f}\right)}{d\Omega} = \frac{N_c}{64\pi^2 M_H}\sqrt{1 - \frac{4m_f^2}{M_H^2}}\left[\sum_{r1,r2} |\mathcal{M}_{if}|\right] =$$

$$N_c \frac{m_f^2}{32\pi^2 v^2} M_H \left(1 - \frac{4m_f^2}{M_H^2}\right)^{\frac{3}{2}}. \tag{49}$$

There is no angular dependence, so we can just integrate the solid angle

$$\Gamma\left(H \to f\bar{f}\right) = N_c \frac{m_f^2}{8\pi v^2} M_H \left(1 - \frac{4m_f^2}{M_H^2}\right)^{\frac{3}{2}}, \tag{50}$$

where $N_c = 3$ for quarks and $N_c = 1$ for the rest of the fermions. We can remove $v^2$ by remembering

$$G_F = \frac{\sqrt{2} g_2^2}{8M_W^2}, \qquad v = \frac{2M_W}{g_2} \qquad \Longrightarrow \qquad \frac{1}{v^2} = \sqrt{2} G_F$$

so we get

$$\Gamma\left(H \to f\bar{f}\right) = N_c \frac{G_F m_f^2}{4\pi\sqrt{2}} M_H \left(1 - \frac{4m_f^2}{M_H^2}\right)^{\frac{3}{2}}, \tag{51}$$

### 4.1.2 Evaluation for particular cases

Let's evaluate the former branching ratio for $f = b, \tau, \mu$. We just need to substitute in (51) for the corresponding mass and take into account the number of colors. The three cases are shown in the following table

| | $N_c$ | $m_f$(GeV) | $\Gamma\left(H \to f\bar{f}\right)$MeV |
|---|---|---|---|
| $\mu$ | 1 | 0.1056 | $9.15 \times 10^{-4}$ |
| $\tau$ | 1 | 1.777 | 0.259 |
| $b$ | 3 | 4.5 | 4.94 |

As it can be seen in the table, the $\Gamma\left(H \to b\bar{b}\right)$ is the most important channel by far. The decay to $\tau^-\tau^+$ has also a non-negligible contribution, whereas the $\Gamma\left(H \to \mu^-\mu^+\right)$ is so low that most likely we won't observe this process.

## 4.2 Exercise 2: Finding the Landau pole

The Renormalization Group Equation (RGE) for the running of the Higgs self-coupling at 1-loop order is given by

$$\beta(\lambda) = Q^2 \frac{d\lambda}{dQ^2} = \frac{3}{4\pi^2} \lambda^2 \tag{52}$$

Find $\lambda\left(Q^2\right)$ using the boundary condition

$$\lambda\left(v^2\right) = \frac{M_H^2}{2v^2} \tag{53}$$

What is the scale $Q = \Lambda_c$ at which $\lambda\left(Q^2\right)$ diverges for $M_H = 125$ GeV?

### 4.2.1 Solution

We solve the RGE equation to obtain $\lambda$ as a function of $Q^2$

$$\frac{4\pi^2}{3}\frac{d\lambda}{\lambda^2} = \frac{dQ^2}{Q^2},$$

$$-\frac{4\pi^2}{3}\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right) = \ln\frac{Q^2}{Q_0^2},$$

and now we impose the boundary conditions in order to find $\lambda_0$ and $Q_0$

$$Q_0 = v, \qquad \lambda_0 = \frac{M_H^2}{2v^2} \implies -\frac{4\pi^2}{3}\left(\frac{1}{\lambda} - \frac{2v^2}{M_H^2}\right) = \ln\frac{Q^2}{v^2},$$

$$\frac{1}{\lambda} = -\frac{3}{4\pi^2}\ln\frac{Q^2}{v^2} + \frac{2v^2}{M_H^2}$$

$$\lambda\left(Q^2\right) = \frac{1}{\frac{2v^2}{M_H^2} - \frac{3}{4\pi^2}\ln\frac{Q^2}{v^2}}. \tag{54}$$

In order to have a pole the condition is

$$\frac{2v^2}{M_H^2} - \frac{3}{4\pi^2}\ln\frac{Q^2}{v^2} = 0,$$

therefore

$$Q = ve^{\frac{4\pi^2 v^2}{3M_H^2}}.$$

The vacuum expectation value of the Higgs is not a free parameter, and can be related to the mass of the $W^\pm$ boson. Its measured value is $v = 246$GeV. For a Higgs mass $M_H = 125$ GeV we get $Q \approx 3.3 \times 10^24$ GeV, further than the Planck scale $\Lambda_G \approx 1.22 \times 10^19$ GeV.