TAE 2014
Lattice exercises

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1 Gauge transformation for QED

In continuum QED, a gauge transformation has the following effect on the fields

$$A_\mu(x) \to A_\mu(x) - \partial_\mu\alpha(x), \tag{1}$$
$$\psi(x) \to e^{-ig\alpha(x)}\psi(x), \tag{2}$$
$$\bar{\psi}(x) \to \bar{\psi}(x)e^{ig\alpha(x)}, \tag{3}$$

and as we can readily check, the lagrangian remains invariant

$$\bar{\psi}(x)\left(\partial_\mu - igA_\mu(x)\right)\psi(x) \to \bar{\psi}(x)e^{ig\alpha(x)}\left(\partial_\mu - igA_\mu(x) + ig\partial_\mu\alpha(x)\right)e^{-ig\alpha(x)}\psi(x) =$$
$$\bar{\psi}(x)e^{ig\alpha(x)}\left(\partial_\mu\left(e^{-ig\alpha(x)}\psi(x)\right) - e^{-ig\alpha(x)}igA_\mu(x)\psi + e^{-ig\alpha(x)}\partial_\mu\alpha(x)\psi(x)\right) =$$
$$\bar{\psi}(x)e^{ig\alpha(x)}\left(e^{-ig\alpha(x)}\partial_\mu\psi(x) - ig\partial_\mu\alpha(x)\psi(x) - e^{-ig\alpha(x)}iA_\mu(x)\psi(x) + e^{-ig\alpha(x)}\partial_\mu\alpha(x)\psi(x)\right) =$$
$$\bar{\psi}(x)\left(\partial_\mu - igA_\mu(x)\right)\psi(x).$$

On the lattice, if we call the gauge transformation $G(x)$, the variables transform as

$$U_\mu(x) \to G(x)U_\mu(x)G^\dagger(x + a\mu), \tag{4}$$
$$\psi(x) \to G(x)\psi(x), \tag{5}$$
$$\bar{\psi}(x) \to \bar{\psi}(x)G^\dagger(x). \tag{6}$$

The gauge transformation for $U(1)$ is the same as in the continuum, $G(x) = e^{-ig\alpha(x)}$, where $\alpha(x)$ takes values on the points of the lattice. Therefore, the transformation applied to the spinor fields trivially give the same result as in the continuum. The question is, what happens to the gauge field $A_\mu(x)$?

So we apply the gauge transformation to the link

$$U_\mu(x) = e^{-igaA_\mu(x)}, \tag{7}$$

and calculate the way the gauge field transform:

$$G(x)U_\mu(x)G^\dagger(x+a\mu) = e^{-ig\alpha(x)}e^{-igaA_\mu(x)}e^{ig\alpha(x+a\mu)} = e^{-iga(A_\mu(x)-a\partial_\mu\alpha(x))}, \tag{8}$$

where we have used the expression of the discrete derivative

$$\partial_\mu \alpha\left(x\right) = \frac{\alpha\left(x+a\mu\right)-\alpha\left(x\right)}{a}. \tag{9}$$

2 Recovering the continuum gauge action from the lattice

2.1 Abelian case: QED

The link connecting the points x and $x+a\mu$ is given by

$$U_\mu\left(x\right) = e^{-igaA_\mu(x)}. \tag{10}$$

And the corresponding plaquette is given by the smallest closed product of links, that is

$$P_{\mu\nu} = U_\mu\left(x\right)U_\nu\left(x+a\mu\right)U_\mu^\dagger\left(x+a\nu\right)U_\nu^\dagger\left(x\right). \tag{11}$$

Since our group is abelian (more exactly, $U(1)$ for QED), we can solve directly the product of links by substituting every plaquette in (12) by (11),

$$P_{\mu\nu} = e^{-iga[A_\mu(x)+A_\nu(x+a\mu)-A_\mu(x+a\nu)-A_\nu(x)]}. \tag{12}$$

Now we recall the definition of the discrete derivative on the lattice,

$$\partial_\mu A_\nu\left(x\right) = \frac{A_\nu\left(x+a\mu\right)-A_\nu\left(x\right)}{a}, \tag{13}$$

and we identify potential derivatives on the $A_\mu\left(x\right)$ field of equation (13),

$$P_{\mu\nu} = \exp\left\{-iga\left[\underbrace{A_\nu\left(x+a\mu\right)-A_\nu\left(x\right)}_{a\partial_\mu A_\nu(x)} - \left[\underbrace{A_\mu\left(x+a\nu\right)-A_\mu\left(x\right)}_{a\partial_\nu A_\mu(x)}\right]\right]\right\}. \tag{14}$$

If we just substitute the differences by the derivative, the expression we get is

$$P_{\mu\nu} = e^{-iga[a\partial_\mu A_\nu(x)-a\partial_\nu A_\mu(x)]} = e^{-iga^2F_{\mu\nu}(x)}, \tag{15}$$

where $F_{\mu\nu}$ is the field strength tensor

$$F_{\mu\nu}\left(x\right) = \partial_\mu A_\nu\left(x\right)-\partial_\nu A_\mu\left(x\right). \tag{16}$$

With this calculation we can try to recover the continuum gauge action from the Wilson gauge action on the lattice. The Wilson gauge action is

$$S_G = \frac{2}{g^2}\sum_{x,\mu,\nu}\left(1-\mathrm{Re}P_{\mu\nu}(x)\right). \tag{17}$$

Substitute the plaquette for the expression we got in (16),

$$S_G = \frac{2}{g^2} \sum_{x,\mu,\nu} \left[1 - \mathrm{Re}\left(e^{-iga^2 F_{\mu\nu}(x)} \right) \right], \tag{18}$$

and finally expand the exponential

$$S_G = \frac{2}{g^2} \sum_x \left[1 - \mathrm{Re}\left(1 - iga^2 F_{\mu\nu}(x) - g^2 a^4 \left[F_{\mu\nu}(x) \right]^2 + O(a^6) \right) \right] =$$

$$\frac{2}{g^2} \sum_x \left[g^2 a^4 F_{\mu\nu}(x) F^{\mu\nu}(x) + O(a^6) \right].$$

We can observe that the discrete sum becomes an integral as $a \to 0$

$$a^4 \sum_x = \int_V d^4x, \tag{19}$$

giving rise to the following result

$$S_G = \int_V d^4x \left[F_{\mu\nu}(x) F^{\mu\nu}(x) + O(a^2) \right]. \tag{20}$$

and we recover (up to lattice artifacts) the continuum lattice action.

2.2 Non-abelian case: QCD

For the non-abelian case we can't solve the plaquette product in (12) we did before, and we have several alternatives. For instace, one can try to use the Baker-Campbell-Haussdorf formula

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\dots} \tag{21}$$

to get an analogue expression as in (13). After some algebra, one arrives to the following result

$$P_{\mu\nu} = \exp\{-iga^2 \left[\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \right] - \frac{1}{2} g^2 a^2 \left(\left[A_\mu(x), A_\nu(x+a\mu) \right] \right.$$
$$+ \left[A_\mu(x+a\nu), A_\nu(x) \right] - \left[A_\mu(x), A_\mu(x+a\nu) \right] - \left[A_\mu(x), A_\nu(x) \right]$$
$$\left. - \left[A_\nu(x+a\mu), A_\mu(x+a\nu) \right] - \left[A_\nu(x+a\mu), A_\nu(x) \right] \right)\}.$$

Here we are ommiting the color indices. They are irrelevant for the calculation, because at the end we will take the trace in color space. The first term in the exponent is the same as in the abelian case, containing the derivatives, whereas the second term, the sum of commutators, should give rise to the commutator in the definition of the field strength tensor for non-abelian fields

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + ig[A_\mu(x), A_\nu(x)], \tag{22}$$

As we very swiftly observe, the expression we derived has shifted fields with respect to the point x. In order to recover the commutator, we need to Taylor-expand those fields as

$$A_\mu\left(x+a\nu\right)=A_\mu\left(x\right)+a\partial_\nu A_\mu\left(x\right). \tag{23}$$

Since the derivatives appearing in our expression will carry an extra a factor, we drop them in our expansion, for we expect them to be small when close to the continuum limit $a\to 0$. On the other hand, the commutator terms now involving the same field at the same point will vanish

$$[A_\mu\left(x\right),A_\mu\left(x\right)]=0,$$

and the commutators with reversed fields, like

$$[A_\nu\left(x\right),A_\mu\left(x\right)]=-[A_\mu\left(x\right),A_\nu\left(x\right)],$$

can be easily put in the right order by changing sign. Therefore the final expression becomes

$$\begin{gathered}P_{\mu\nu}=\exp\{-iga^2\left[\partial_\mu A_\nu\left(x\right)-\partial_\nu A_\mu\left(x\right)\right]-\frac{1}{2}g^2a^2\left([A_\mu\left(x\right),A_\nu\left(x\right)]\right.\\ +[A_\mu\left(x\right),A_\nu\left(x\right)]-[A_\mu\left(x\right),A_\nu\left(x\right)]-[A_\nu\left(x\right),A_\mu\left(x\right)]=\\ \exp\{-iga\left[\partial_\mu A_\nu\left(x\right)-\partial_\nu A_\mu\left(x\right)\right]-\frac{1}{2}g^2a^2\left(2[A_\mu\left(x\right),A_\nu\left(x\right)]\right)\}.\end{gathered}$$

and by looking at (23), one can readily obtain the same result as in the abelian case

$$P_{\mu\nu}=e^{-ia^2gF_{\mu\nu}+O(a^3)}. \tag{24}$$

Substituting again in the expression of the action (note the trace in color space), we recover the continuum expression for the gauge action, exactly in the same way we did for abelian fields

$$S_G=\frac{2}{g^2}\sum_{x,\mu,\nu}\mathrm{ReTr}_{Color}\left(I-P_{\mu\nu}(x)\right)=\int_V d^4x\left[F_{\mu\nu}\left(x\right)F^{\mu\nu}\left(x\right)+O(a^2)\right], \tag{25}$$

with an implicit sum over color understood.

To match my notation with that of Elvira, notice that the N_c factor in $\beta=\frac{2N_c}{g^2}$ comes from the trace of the color matrices (including the identity!).