Quantum Field Tomography

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We introduce the concept of quantum field tomography, the efficient and reliable reconstruction of unknown quantum fields based on data of correlation functions. At the basis of the analysis is the concept of continuous matrix product states, a complete set of variational states grasping states in quantum field theory. We innovate a practical method, making use of and developing tools in estimation theory applied in the context of compressed sensing such as Prony methods and matrix pencils, allowing us to faithfully reconstruct quantum field states based on low-order correlation functions. In the absence of a phase reference, we highlight how specific higher order correlation functions can still be predicted. We exemplify the functioning of the approach by reconstructing randomised continuous matrix product states from their correlation data and study the robustness of the reconstruction for different noise models. Furthermore, we apply the method to data generated by simulations based on continuous matrix product states and using the time-dependent variational principle. The presented approach is expected to open up a new window into experimentally studying continuous quantum systems, such as encountered in experiments with ultra-cold atoms on top of atom chips. By virtue of the analogy with the input-output formalism in quantum optics, it also allows for studying open quantum systems.

Motivation

- Full quantum state tomography is highly inefficient: Exponentially many parameters.
- Continuous system: In principle infinitely many degrees of freedom.
- Ansatz: Identify the right model in which to represent the states with appropriate sparsity structure—physical corner of Hilbert space [1, 2].
- Using tensor networks, tomography of quantum fields becomes feasible.

Translationally invariant cMPS [3]

- Variational class of 1D quantum field states.
- $|\Psi_{Q,R}\rangle = \mathrm{tr}_{\mathrm{aux}}\left(\mathscr{P}\, e^{\int_0^L dx \left(Q\otimes\hat{1}+R\otimes\hat{\psi}^\dagger(x)\right)}\right)|\Omega\rangle$
  - Vacuum state $|\Omega\rangle$, field operators $\hat{\psi}^\dagger(x)$ with $\left[\hat{\psi}(y),\hat{\psi}^\dagger(x)\right]=\delta(y-x)$
  - Length of the system L
  - Variational parameters $Q, R \in \mathbb{C}^{d\times d}$—in finite-dimensional auxiliary system
  - Bond dimension d
- The larger d, the better the "physical corner of Hilbert space" is covered.

Correlation Functions in cMPS formalism

Example: density-like 3-point function.

$$\begin{aligned} &C^{(3)}(\tau_1,\tau_2)\\ &= \left\langle \Psi_{Q,R} \right| \hat{\psi}^\dagger(\tau_1+\tau_2)\hat{\psi}^\dagger(\tau_1)\hat{\psi}^\dagger(0)\hat{\psi}(0)\hat{\psi}(\tau_1)\hat{\psi}(\tau_1+\tau_2)\left|\Psi_{Q,R}\right\rangle\\ &= \mathrm{tr}\left(\left(\overline{R}\otimes R\right)e^{T\tau_1}\left(\overline{R}\otimes R\right)e^{T\tau_2}\left(\overline{R}\otimes R\right)e^{T(L-\tau_1-\tau_2)}\right)\\ &\xrightarrow{L\to\infty} \sum_{k_1,k_2=1}^{d^2} \underbrace{M_{1,k_1}M_{k_1,k_2}M_{k_2,1}}_{:=\rho_{k_1,k_2}} e^{\lambda_{k_1}\tau_1}e^{\lambda_{k_2}\tau_2} \end{aligned}$$

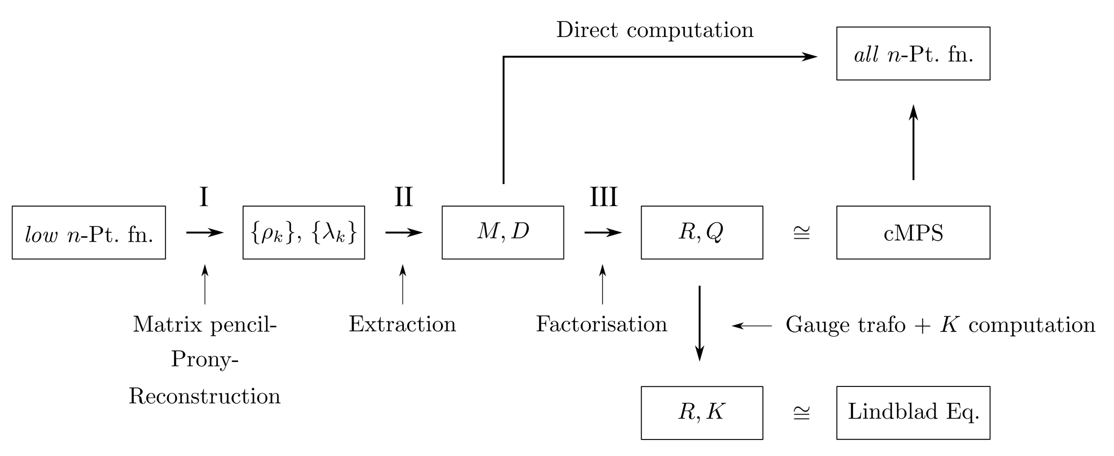
- Transfer matrix $T=\overline{Q}\otimes\mathbb{1}_d+\mathbb{1}_d\otimes Q+\overline{R}\otimes R\in\mathbb{C}^{d^2\times d^2}$ consisting of the variational parameter matrices Q and R.
- M is $\overline{R}\otimes R$ in the diagonal basis of transfer matrix T.
- $\lambda_k\in\mathbb{C}$, eigenvalues of T—"poles".
- $\rho_{k_1,k_2}\in\mathbb{C}$—"residues".

Goal

- Given: n-point function

$$C^{(n)}(\tau_1,\dots,\tau_{n-1})=\sum_{k_1,\dots,k_{n-1}=1}^{d^2} M_{1,k_1}M_{k_1,k_2}\dots M_{k_{n-1},1}e^{\lambda_{k_1}\tau_1}\dots e^{\lambda_{k_{n-1}}\tau_{n-1}}$$

  of quantum field state $|\Psi_{Q,R}\rangle=\mathrm{tr}_{\mathrm{aux}}\left(\mathscr{P}\,e^{\int_0^L dx\left(Q\otimes\hat{1}+R\otimes\hat{\psi}^\dagger(x)\right)}\right)|\Omega\rangle$.
- Task: Reconstruction of parameter matrices Q and R and predict correlation functions of arbitrary order.



Reconstruction of the poles and residues (I)

- Given:

$$C^{(n)}(\tau_1,\dots,\tau_{n-1})=\sum_{k_1,\dots,k_{n-1}=1}^{d^2}\rho_{k_1,\dots,k_{n-1}}e^{\lambda_{k_1}\tau_1}\dots e^{\lambda_{k_{n-1}}\tau_{n-1}}$$

- Wanted: $\{\lambda_k\},\{\rho_{k_1,\dots,k_n}\}$.
- Nonlinear problem – hard to solve!

Reconstruction of the poles (Ia): Matrix Pencil Method [4]

- Determine poles $\{\lambda_k\}$ independently. from the residues $\{\rho_{k,\dots}\}$
- Input: Discretized 2-point function $C_j=\sum_{k=1}^{d^2}\rho_k e^{\lambda_k\Delta\tau\cdot j}$, $j=0,\dots,N-1$, sampling interval $\Delta\tau$, generalization to n-point function possible.
- Build two Hankel matrices $C^{[1]}=\begin{pmatrix} C_0 & C_1 & \dots C_{N/2-1}\\ C_1 & C_2 & \dots\ C_{N/2}\\ \vdots & \vdots & \vdots\\ C_{N/2-1} & C_{N/2}\dots & C_{N-2}\end{pmatrix}$ and

$$C^{[2]}=\begin{pmatrix} C_1 & C_2 & \dots\ C_{N/2}\\ C_2 & C_3 & \dots C_{N/2+1}\\ \vdots & \vdots & \vdots\\ C_{N/2} & C_{N/2+1}\dots & C_{N-1}\end{pmatrix}.$$

- Decomposition: $C^{[1]}=V_1\cdot A\cdot V_2$, $C^{[2]}=V_1\cdot A\cdot V_0\cdot V_2$.
- Diagonal matrices $A=\mathrm{diag}(\rho_1,\dots,\rho_{d^2})$, $V_0=\mathrm{diag}\left(e^{\lambda_1\Delta\tau},\dots,e^{\lambda_{d^2}\Delta\tau}\right)$ and Vandermonde matrices

$$V_1=V_2^T=\begin{pmatrix} 1 & 1 & \dots & 1\\ e^{\lambda_1\Delta\tau} & e^{\lambda_2\Delta\tau} & & e^{\lambda_{d^2}\Delta\tau}\\ \vdots & \vdots & & \vdots\\ \left(e^{\lambda_1\Delta\tau}\right)^{N/2-1} & \left(e^{\lambda_2\Delta\tau}\right)^{N/2-1} & \dots & \left(e^{\lambda_{d^2}\Delta\tau}\right)^{N/2-1}\end{pmatrix}$$

- V_0, V_1, V_2 contain the poles, A contains the residues.
- Consider the matrix pencil $C^{[2]}-\gamma C^{[1]}=V_1A(V_0-\gamma\mathbb{1}_{d^2})V_2$.
- $\gamma=e^{\lambda_k\Delta\tau}$ for $k=1,\dots,d^2 \iff \mathrm{rank}\left(C^{[2]}-\gamma C^{[1]}\right)=d^2-1$.
- We obtain λ_k from solving the GEVP $C^{[2]}v=\gamma C^{[1]}v$.
- Several refinements for better noise performance.

Reconstruction of the residues (Ib)

- Example: $C^{(2)}(\tau_1)=\sum_{k=1}^{d^2}\rho_k e^{\lambda_k\tau_1}$, n-point function analogously.
- Build Vandermonde matrix out of the determined λ_j.
- Solve

$$\begin{pmatrix} 1 & 1 & \dots & 1\\ e^{\lambda_1\Delta\tau} & e^{\lambda_2\Delta\tau} & & e^{\lambda_{d^2}\Delta\tau}\\ \vdots & \vdots & & \vdots\\ \left(e^{\lambda_1\Delta\tau}\right)^{N-1} & \left(e^{\lambda_2\Delta\tau}\right)^{N-1} & \dots & \left(e^{\lambda_{d^2}\Delta\tau}\right)^{N-1}\end{pmatrix}\begin{pmatrix}\rho_1\\ \rho_2\\ \vdots\\ \rho_{d^2}\end{pmatrix}=\begin{pmatrix}C_0\\ C_1\\ \vdots\\ C_{N-1}\end{pmatrix}$$

  in the least square sense.

Reconstruction of M (II)

- Given: Poles and residues of 3-point function

$$C^{(3)}(\tau_1,\tau_2)=\sum_{k_1,k_2=1}^{d^2}\rho_{k_1,k_2}e^{\lambda_{k_1}\tau_1}e^{\lambda_{k_2}\tau_2}$$

- Wanted: Decomposition of $\rho_{k_1,k_2}=M_{1,k_1}M_{k_1,k_2}M_{k_2,1}$.
- $M\in\mathbb{C}^{d^2\times d^2}$ can be transformed without changing $C^{(n)}$ such that $M_{1,k}=1$ for $k=1,\dots,d^2$.
- $\Rightarrow \frac{\rho_{k,j}}{\rho_{k,1}}=\frac{1\cdot M_{j,k}M_{k,1}}{1\cdot 1\cdot M_{k,1}}=M_{j,k}$
- Better noise properties: Averaging over all combinations of residues that lead to $M_{j,k}$.
- Having determined M and poles $\{\lambda_j\}$, density-like correlation functions $C^{(n)}$ of arbitrary order n can be constructed

Reconstruction of R (IIIa)

- Given: $D=\mathrm{diag}(\lambda_1,\dots,\lambda_{d^2})=X^{-1}TX$ and $M=X^{-1}\left(\overline{R}\otimes R\right)X$
- Indeterminate: change-of-basis matrix X.
- Gauge invariance: (Q,R) and $(Z^{-1}QZ, Z^{-1}RZ)$ for arbitrary invertible Z represent the same state.
- We need to determine Q and R only up to conjugation with an invertible matrix.
- Strategy: Choose R diagonal and determine Q accordingly.
- Diagonalize $M\mapsto M_{\mathrm{diag}}$, $\sigma\left(M_{\mathrm{diag}}\right)=\sigma(M)=\sigma\left(\overline{R}\otimes R\right)$.
- $\Rightarrow$ The entries of M_{diag} can be permuted such that the diagonal matrix has tensor-product form: $M_{\mathrm{diag}}\mapsto\overline{R_{\mathrm{rec}}}\otimes R_{\mathrm{rec}}$.
- Diagonal matrix R_{rec} is *similar* to R:
  It exists a gauge matrix W with $R_{\mathrm{rec}}=W^{-1}RW$.

Reconstruction of Q (IIIb)

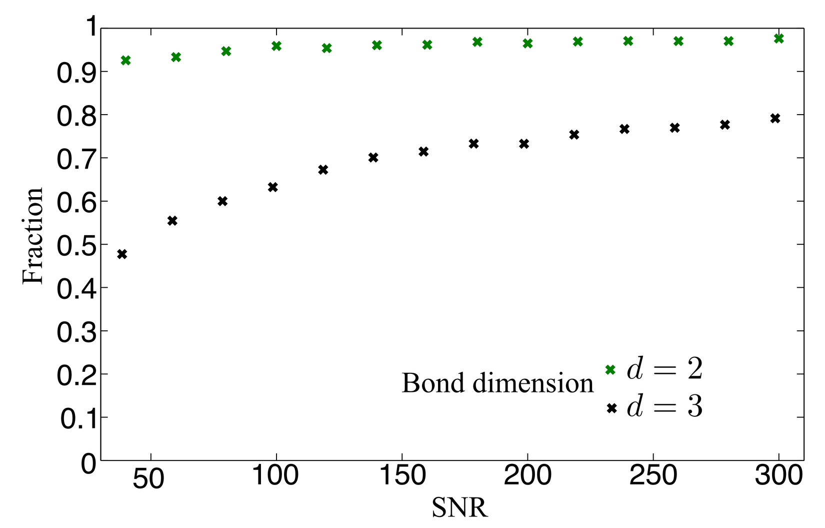
- Given: $D=\mathrm{diag}(\{\lambda_k\})$, $M=X^{-1}\left(\overline{R}\otimes R\right)X$.
- $D-M=X^{-1}\left(\overline{Q}\otimes\mathbb{1}_d+\mathbb{1}_d\otimes Q\right)X$.
- Q will in general not be diagonal in the same gauge where R is diagonal.
- Applying to $D-M$ the same change-of-basis matrix, that transforms M to $\overline{R_{\mathrm{rec}}}\otimes R_{\mathrm{rec}}$, yields

$$\overline{Q_{\mathrm{rec}}}\otimes\mathbb{1}_d+\mathbb{1}_d\otimes Q_{\mathrm{rec}}.$$

- One can show that Q_{rec} is similar to Q and $Q_{\mathrm{rec}}=W^{-1}QW$ with the same W as above.
- Q_{rec} and R_{rec} are gauged correctly: The variational parameter matrices of $|\Psi_{Q,R}\rangle$ are reconstructed.

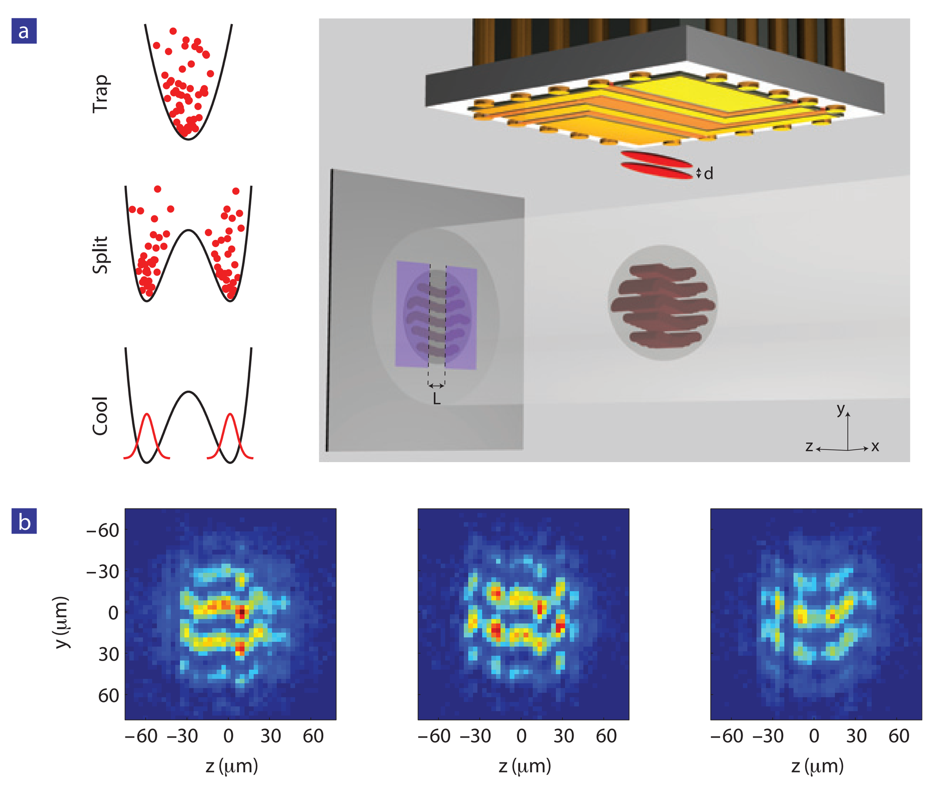
Numerical Simulations

- Randomly generated noisy cMPS correlation functions.
- Recovery rate of the poles using the Matrix Pencil Method:

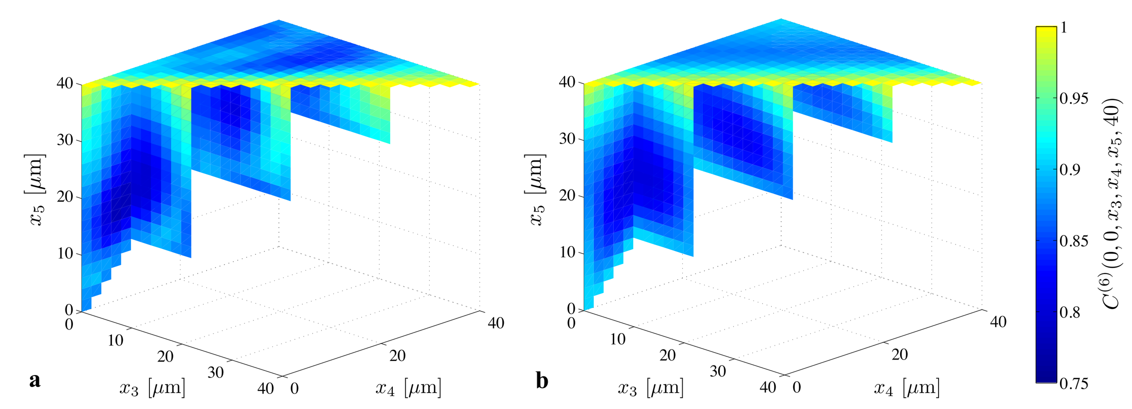


Experimental realisation [5, 6]

- Setup: Coherent split of ultracold atoms (1D BEC) in Atom Chip
- Correlation functions from ToF measurements.



- Successful reconstruction of 6-point function from 4-point function with a mean error of 2.1 %.



References

[1] Cramer, M., et al., "Efficient quantum state tomography," *Nature Comm.* **1** (2010), 149.

[2] Eisert, J., et al. Area laws for the entanglement entropy. *Rev. Mod. Phys.* **82** (2010), 277.

[3] Verstraete, F. and Cirac, J. I., "Continuous matrix product states for quantum fields," *Phys. Rev. Lett.* **104** (2010) 190405.

[4] Hua, Y., and Sarkar, T., "Matrix pencil method for estimating parameters of exponentially damped/undamped sinusoids in noise," *IEEE Transactions on Acoustics, Speech and Signal Processing* **38** (1990), 814.

[5] M. Gring, et al., "Relaxation and prethermalization in an isolated quantum system," *Science* **337** (2012) 1318.

[6] T. Langen, et al., "Local emergence of thermal correlations in an isolated quantum many-body system," *Nature Phys.* **9** (2013) 640.