Simulating lattice gauge field theories with Tensor Networks

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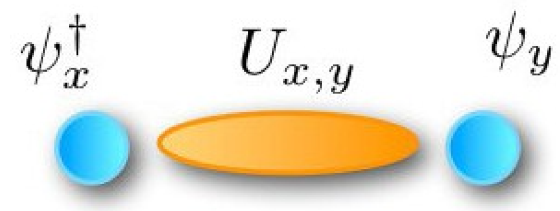
Abstract

A unified framework to describe lattice gauge theories by means of tensor networks is presented, based on the quantum link formulation. It is efficient as it exploits the high amount of local symmetry content native of these systems describing only the gauge invariant subspace. Compared to a standard tensor network description, the gauge invariant one allows to speed-up real and imaginary time evolution of a factor that is up to the square of the dimension of the link variable. Additionally, we present a cellular automata analysis which estimates the gauge invariant Hilbert space dimension as a function of the number of lattice sites, and that might guide the search for effective simplified models of complex theories.



A lattice gauge theory (LGT) has two types of local degrees of freedom:

- **Matter fields**: can be fermionic or bosonic matter, they live on the lattice <u>sites</u>
- **Gauge fields**: are bosonic, they live on the <u>links</u> of the lattice

QED

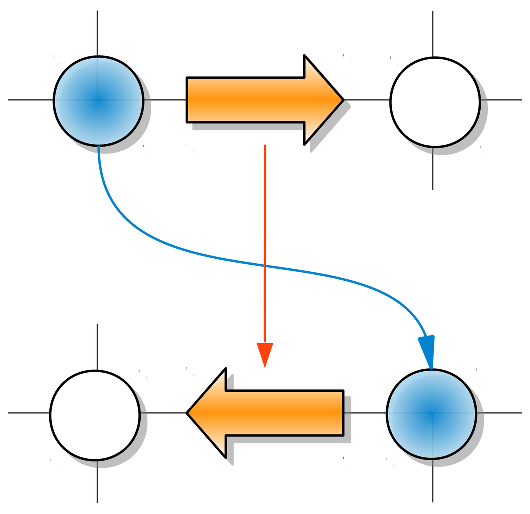
Abelian LGT interaction:

$$H_{\rm int}^{[A]} = \sum_{x,\vec{a}} \psi_x^\dagger \, U_{x,x+\vec{a}} \, \psi_{x+\vec{a}} + {\rm h.c.}$$

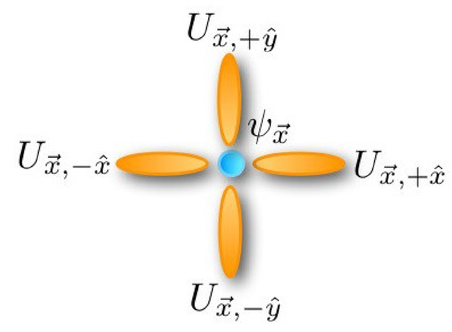
QCD

Non-abelian LGT interaction:

$$H_{\rm int}^{[NA]} = \sum_{x,\vec{a},\mu,\nu} \psi_x^{\mu\dagger} \, U_{x,x+\vec{a}}^{\mu,\nu} \, \psi_{x+\vec{a}}^{\nu} + {\rm h.c.}$$

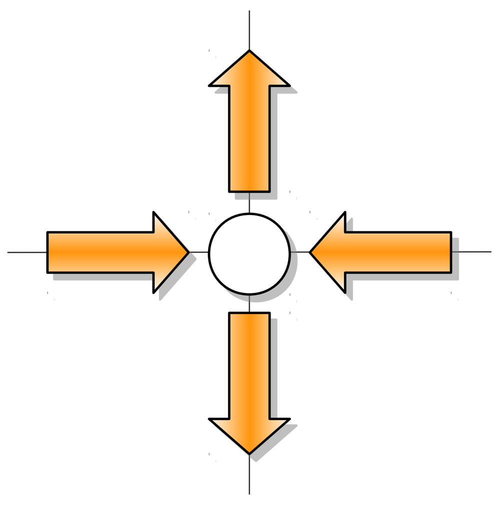


Gauge symmetry (generalized Gauss' Law)



The dynamics preserves an extensive number of symmetries, each one having support on a vertex (site and links connected to it). The "physical" quantum space is made out of states that belong to a specific irreducible representation for each of these gauge symmetries (e.g. invariant irrep). Such gauge constraint is also known as generalized Gauss' Law.

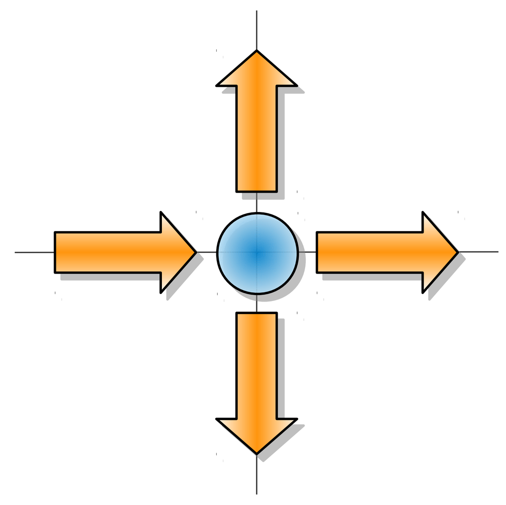
$$\rho - \vec{\nabla}\cdot\vec{E} = 0 \quad \Longrightarrow \quad G_x|\varphi_{\rm phys}\rangle = \left(\psi_x^\dagger\,\psi_x - \tfrac{1}{2}\textstyle\sum_{\vec{a}}\sigma^z_{x,x+\vec{a}}\right)|\varphi_{\rm phys}\rangle = 0$$



$$\left[H_{\rm int}^{[QED]}, G_x\right] = 0 \quad \forall x$$

The gauge constraint reduces the number of possible local configurations.

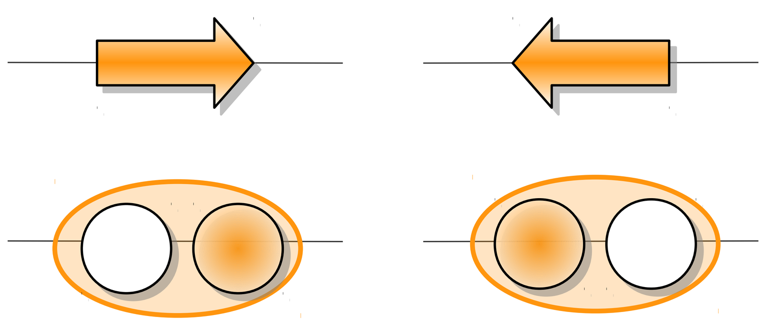
Here abelian fermionic lattice QED with two-level electric field: configs. reduced from 32 to d=10.



Quantum Link Prescription

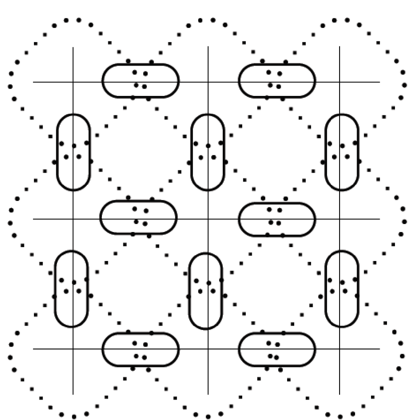
Once a finite-dimensional representation of the gauge group has been selected for the gauge boson, the bosonic operator can be recast as a bilinear operator. Effectively, the gauge boson is *split* into a pair of degrees of freedom, called "rishons".

$$U_{x,x+\vec{a}} \longrightarrow c_{x,\vec{a}}\, c^\dagger_{x+\vec{a},-\vec{a}}$$



An artificial, abelian symmetry arises: the total number of rishon per link is a conserved quantity. This "link symmetry" is also local, and commutes with the gauge symmetry. The two together form the full gauge group of the quantum link model.

$$G_x = \psi_x^\dagger\psi_x + \sum_{\vec{a}} c^\dagger_{x,\vec{a}} c_{x,\vec{a}} \qquad L_{x,x+\vec{a}} = c^\dagger_{x,\vec{a}}c_{x,\vec{a}} + c^\dagger_{x+\vec{a},-\vec{a}}c_{x+\vec{a},-\vec{a}}$$

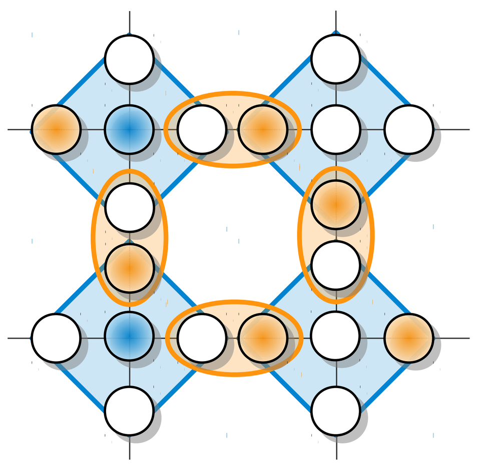


$$[H_{\rm int}^{[A]}, G_x] = [H_{\rm int}^{[A]}, L_{x,x+\vec{a}}] = [G_x, L_{y,y+\vec{a}}] = 0$$

Selection rules (Gauge, Link):

$$G_x|\varphi_{\rm phys}\rangle = |\varphi_{\rm phys}\rangle \bar{N}_x$$

$$L_{x,x+\vec{a}}\,|\varphi_{\rm phys}\rangle = |\varphi_{\rm phys}\rangle \bar{M}_{x,x+\vec{a}}$$

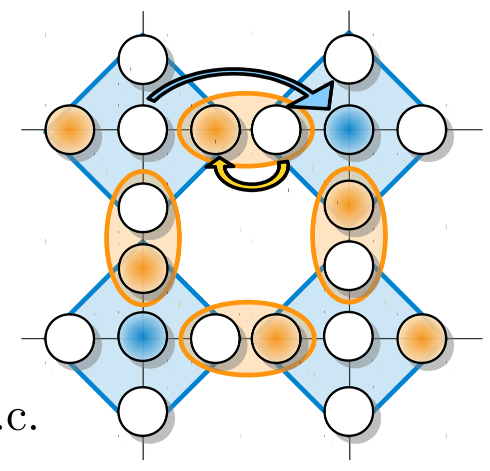


Gauge constraint: two particles, matter + rishons, in every vertex (blue square).

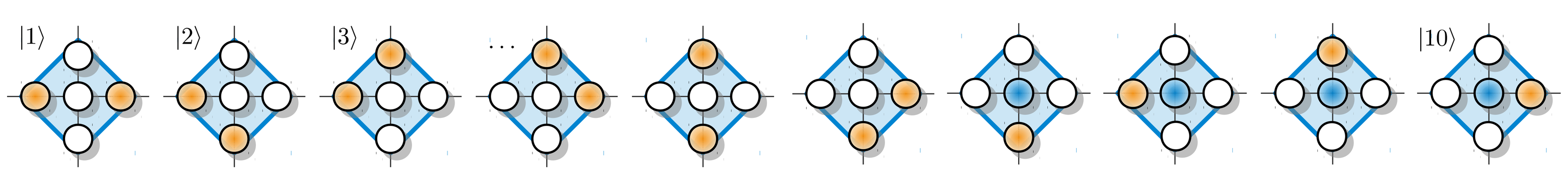
Link constraint: one particle on every link (orange bubble).

Preserved by:

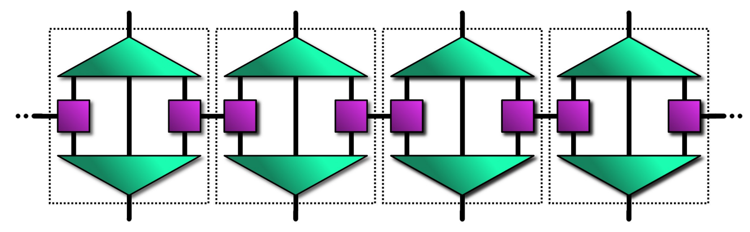
$$H_{\rm int}^{[A]} = \sum_{x,\vec{a}} \psi_x^\dagger \, c_{x,\vec{a}}\, c^\dagger_{x+\vec{a},-\vec{a}}\, \psi_{x+\vec{a}} + {\rm h.c.}$$



The gauge constraint defines the effective computational local basis (in this example: d = 10).



The link constraint imposes an abelian selection rule between nearest nerighbour computational states (example: |1, 2> forbidden, |1, 4> allowed. On two neighboring sites, 48 states survive out of 100).

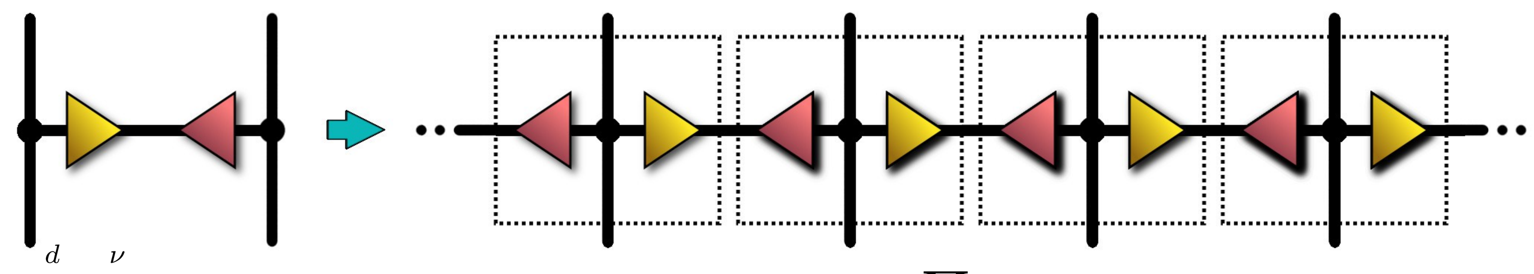


MPO / PEPO formulation of the combined link constraint

All the link constraints combined can be formulated as a many-body projector Q, which in 1D allows an exact Matrix Product Operator formulation (or a Projected Entangled Pair Operator in 2+D), with bondlink dimension bound by:

$$\nu = 1 + \bar{M}_{x,x+\vec{a}} \qquad \text{Where} \qquad L_{x,x+\vec{a}}\,|\varphi_{\rm phys}\rangle = |\varphi_{\rm phys}\rangle \bar{M}_{x,x+\vec{a}}$$

and can be made diagonal by choosing a suitable local basis (the one that simultaneously diagonalizes gauge and link constraints).

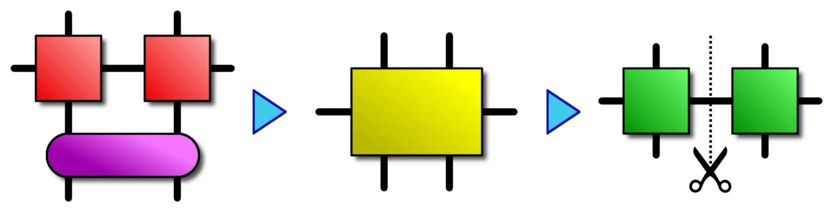


$$Q_{x,x+1} = \sum_{j_1,j_2}^{d}\sum_{\alpha}^{\nu} |j_1 j_2\rangle\langle j_1 j_2| Z_{j_1,\alpha} V_{\alpha,j_2} \qquad Q = \prod_x Q_{x,x+1}$$

Computational speed-up

Like for a global symmetry, upholding the local link constraint reduces the two-site computational space and allows one to perform numerical operations in a block-wise fashion. The advantage with respect to global symmetries is that we do not have to propagate the charges throughout the network, since the link symmetry is local: little bookkeeping!

Example: 1D gauge invariant time-evolution with a Matrix Product Density Operator (MPDO).



Step 1 - Contraction

Standard computational cost:

$$\sim d^2b^2m^3 + d^4b^2m^2$$

Reduced to:

$$\sim \chi b^2m^3 + \chi^2b^2m^2$$

Step 2 – Split (SVD-based)

Standard computational cost:

$$\sim d^3b^3m^3$$

Reduced to:

$$\sim d^3b^3m^3/\nu^2$$

With m being the original correlation bondlink dimension, b the bath bondlink dimension, χ the number of surviving states on two neighboring sites (in the example, speed-up of roughly four times).

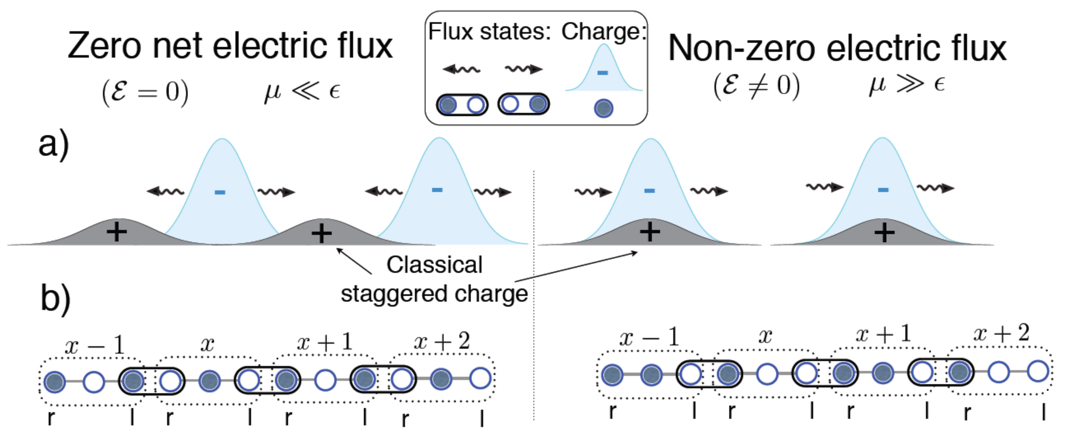
First results: 1D fermionic QED with two-level electric field

This scenario corresponds to a Schwinger model with d = 3 local states, and ν = 2 rishon charges.

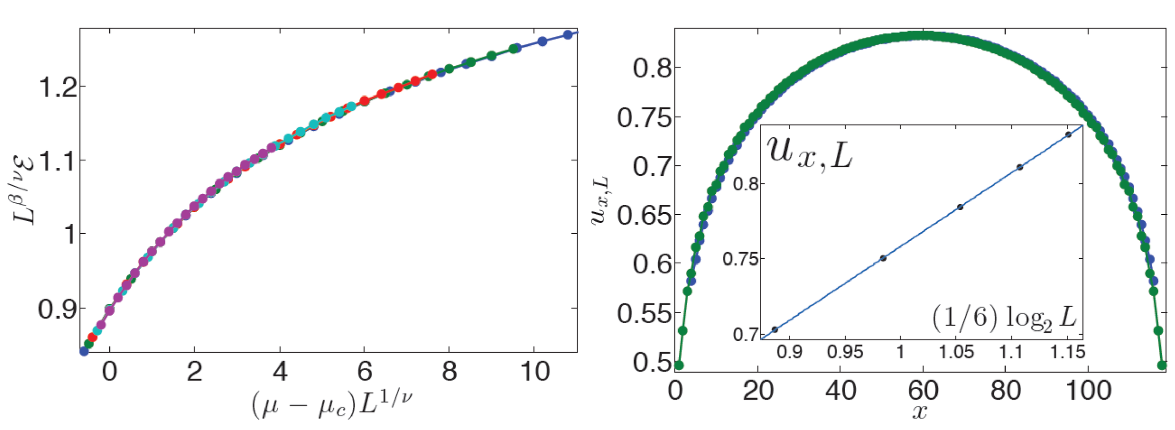
It exhibits a second-order quantum phase transition between two phases, driven by the staggered chemical potential:

<u>Disordered phase</u> - Staggered charges and electric flux direction.

<u>Ordered phase</u> - No charges and uniform electric flux direction.



The global symmetry being broken is the Charge-Parity symmetry, which is a Z_2 group. One expects to find the critical exponents of the 1+1D Ising model.



Critical point found via finite size scaling.

$$\mu_c = \epsilon(0.655 \pm 0.003)$$

Critical exponents compatible with quantum ising.

$$\beta = 1/8 \qquad \nu = 1 \qquad c = 1/2$$

Growth of Hilbert spaces dimension: the Cellular Automata

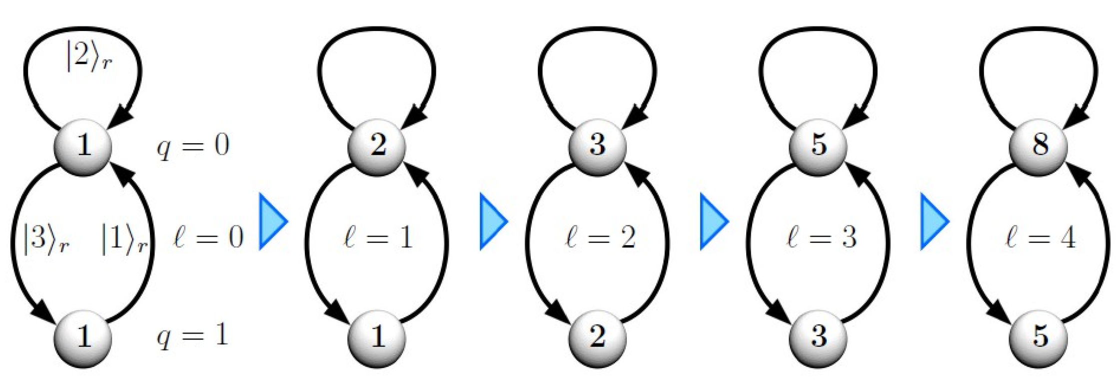
Due to the presence of the link constraint, the Hilbert dimension of a quantum link model on ℓ vertices is less than Dim(ℓ) = d^ℓ. In **1D** it is easy to calculate Dim(ℓ) recursively, with a Cellular Automata machinery.

1) Draw the automata (bubbles and arrows) according to the link constraint rules.

2) At zero lattice sites,start from 1 in every bubble.

3) for every lattice site we add, we propagate the numbers through the arrows.

4) Sum the numbers within the bubbles to obtain the total Hilbert space dimension.



Example above: for the fermionic QED with two-level electric field, one obtains the Fibonacci sequence.

$$Dim(\ell) = (\phi^{\ell+3} - (1-\phi)^{\ell+3})/\sqrt{5}$$

with ϕ being the golden ratio.

arXiv:1404.7439; Phys. Rev. Lett. 112, 201601 (2014).