# Anyons & Topological Phases II

Marc D. SCHULZ

University of Minnesota

Numerical and analytical methods
for strongly correlated systems

29/07/2014

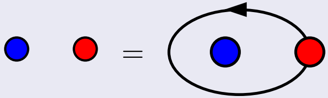
I don't really read a lot.
Maybe I should.
Syd Barrett

Based i.a. on very fruitful discussions with:
Sébastien Dusuel, Kai P. Schmidt, Julien Vidal, Michael Kamfor, Fiona Burnell, Roman Orus, Oliver Buerschaper, Kris Coester, Steven Simon, Eddy Ardonne, ...

# Summary part I

## What happened so far:

- point-like particles in $2D$: anyons



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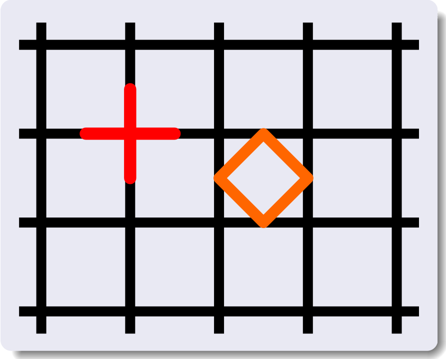

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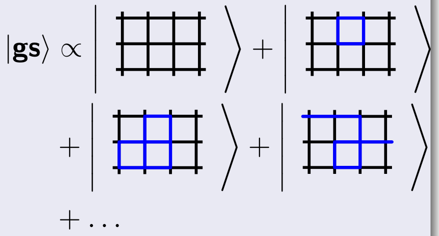

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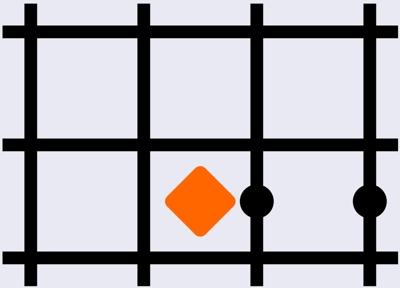
- excitations: endpoints of string operators



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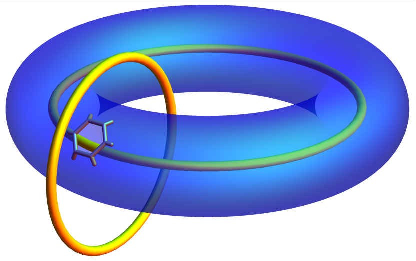

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- point-like particles in $2D$: anyons
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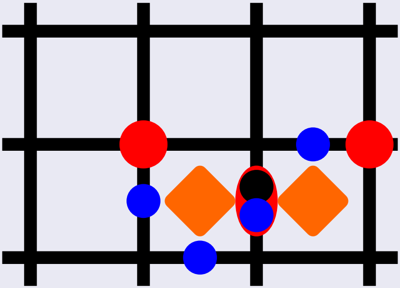
- excitations: endpoints of string operators
- non-contractible strings $\Leftrightarrow$ ground-state degeneracy
- Aharanov-Bohm effect measures excitations



# Outline

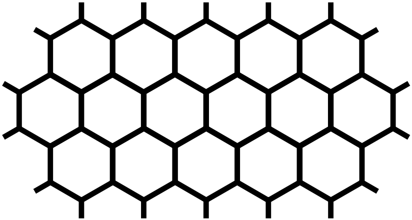
1. Review of $\mathbb{Z}_2$-string net model
2. Particle picture
3. Non-Abelian anyons

Be aware of oversimplifications!

# $\mathbb{Z}_2$-string net model: Lattice

## Lattice

defined on honeycomb lattice



M. Levin, X.-G. Wen, PRB **71** 045110 (2005)

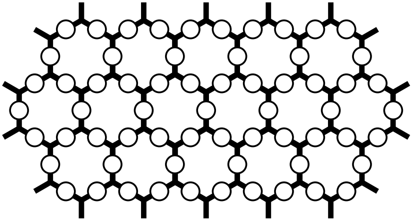
# $\mathbb{Z}_2$-string net model: Lattice

## Lattice

defined on honeycomb lattice

## Microscopic degrees of freedom

- located on the links
- can take values $\mathbf{1}$ and $(-\mathbf{1})$



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# $\mathbb{Z}_2$-string net model: Lattice

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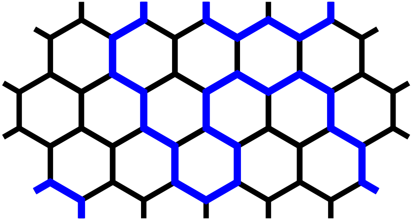

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# Model: Hilbert space $\mathcal{H}$

## Constraints for $\mathbb{Z}_2$-theory

Fusion rules of $\mathbb{Z}_2$ lead to vertex constraints

$$\mathbf{1} \otimes \mathbf{1} = \mathbf{1}$$

$$\mathbf{1} \otimes (-\mathbf{1}) = (-\mathbf{1}) \otimes \mathbf{1} = (-\mathbf{1}) \quad \longrightarrow$$

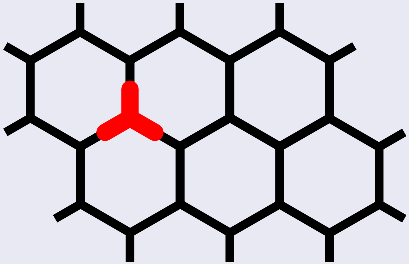
$$(-\mathbf{1}) \otimes (-\mathbf{1}) = \mathbf{1}$$

# Model: Hamiltonian

## The Hamiltonian

$$H_{\mathbb{Z}_2} = -\frac{J_s}{2} \sum_{\text{vertices } s} (\mathbb{1} + A_s) - \frac{J_p}{2} \sum_{\text{plaquettes } p} (\mathbb{1} + B_p)$$

- $A_s = \prod_{i \in s} \sigma_i^z$
- eigenvalues are $\pm 1$
  - $+1$: even number of $|\downarrow\rangle$
  - $-1$: odd number of $|\downarrow\rangle$
  
  $\Rightarrow$ charge at vertex $s$



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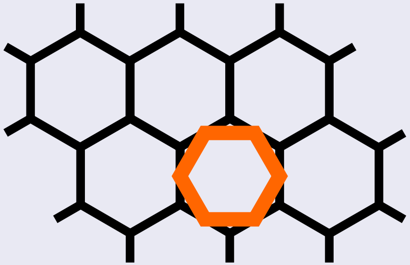
- $B_p = \prod_{i \in p} \sigma_i^x$
- eigenvalues are $\pm 1$
  - $+1$: e.g. $| \quad \rangle + | \quad \rangle$
  - $-1$: e.g. $| \quad \rangle - | \quad \rangle$
  
  $\Rightarrow$ flux at plaquette $p$



# Model: Hamiltonian

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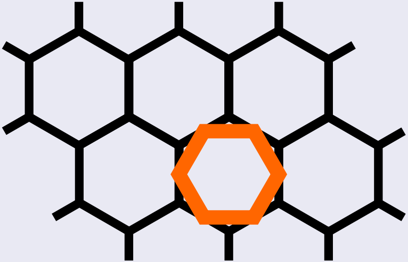
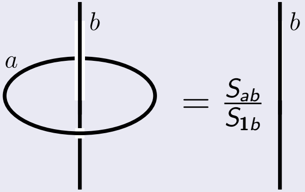


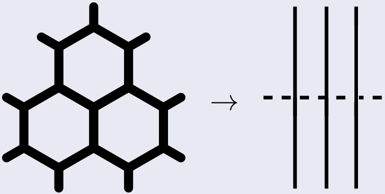
## From now on:

$J_s \to \infty$: only charge-free states.

Braiding a charge around plaquette measures flux ...




... allows to convert plaquette picture in flux-line picture

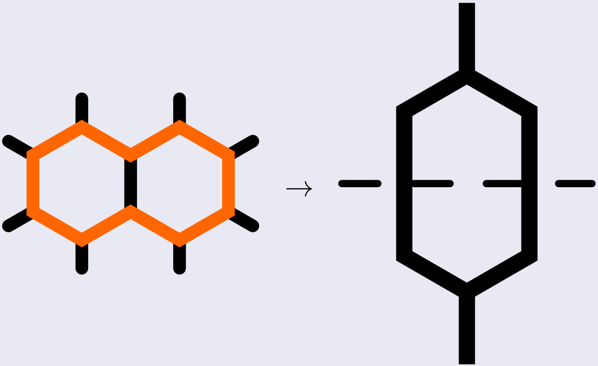


# Consider non-local fluxes

### simplest example for LRE-excitations

$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$:
singlet $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$ and triplet $|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$
locally indistinguishable

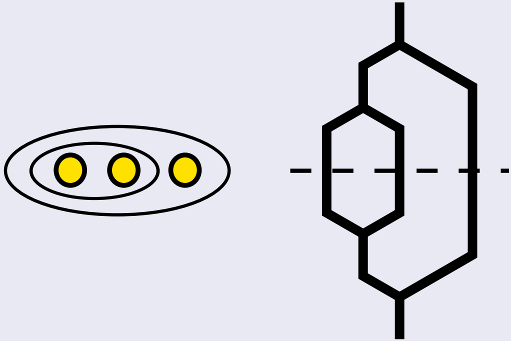
# Consider non-local fluxes



In this case: measure above & below the surface.

# Basis

- A possible basis is e.g. given by



- Order of fusion is arbitrary but to be fixed

# Example: adding 3 spin-$\frac{1}{2}$

## left first

$$
\begin{aligned}
&\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \\
=&\left(\frac{1}{2} \otimes \frac{1}{2}\right) \otimes \frac{1}{2} \\
=&(0 \oplus 1) \otimes \frac{1}{2} \\
=&\frac{1}{2}^{(0)} \oplus \frac{1}{2}^{(1)} \oplus \frac{3}{2}
\end{aligned}
$$

## right first

$$
\begin{aligned}
&\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \\
=&\frac{1}{2} \otimes \left(\frac{1}{2} \otimes \frac{1}{2}\right) \\
=&\frac{1}{2} \otimes (0 \oplus 1) \\
=&\frac{1}{2}^{(0)} \oplus \frac{1}{2}^{(1)} \oplus \frac{3}{2}
\end{aligned}
$$

but e.g. different singlets are not the same!

# Example: adding 3 spin-$\frac{1}{2}$

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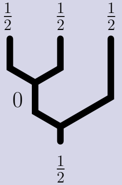
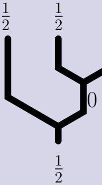


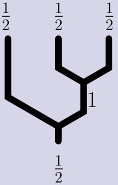

$=$

## right first

$-\frac{1}{2}$



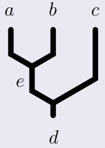

$-\ \frac{\sqrt{3}}{2}$



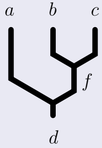
# F-moves

## F-moves

- tabulated for several theories
  e.g. E. Rowell et al. CMP **292**,343 (2009)
- not unique for given theory!
- consistent




$$= \sum_f F^{a,b,e}_{c,d,f}$$



# Particle picture

## flux-line description

- depends only on order of fusion
- does not involve other quantities
- yields eigenbasis of $H_{\mathbb{Z}_2}$

# Conclusion particle picture

- Particle picture describes topological eigenstates
- "gauge" independent
- facilitates handling of the states

$\Rightarrow$ now apply this to the non-Abelian case.

# Golden string-net model

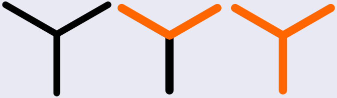
## Constraints for Fibonacci-theory

Fusion rules of Fibonacci-CFT lead to vertex constraints

$$\mathbf{1} \otimes \mathbf{1} = \mathbf{1}$$

$$\mathbf{1} \otimes \tau = \tau \otimes \mathbf{1} = \tau \qquad \longrightarrow$$

$$\tau \otimes \tau = \mathbf{1} + \tau$$



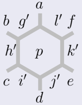
# Model: Hamiltonian

## The Hamiltonian

$$H_{\mathrm{Fib}} = - \; J_p \sum_{\text{plaquettes } p} B_p$$

- $B_p = \frac{1}{1+\varphi^2}\mathbb{1} + \frac{\varphi}{1+\varphi^2} B_p^{\tau}, \varphi = \frac{1+\sqrt{5}}{2}$
- $B_p^{\tau}$ $\Bigg|$ 

$\Bigg\rangle =$

$$\sum_{g',h',i',j',k',l'} F^{a\,g\,l}_{\tau l' g'} F^{b\,h\,g}_{\tau g' h'} F^{c\,i\,h}_{\tau h' i'} F^{d\,j\,i}_{\tau i' j'} F^{e\,k\,j}_{\tau j' k'} F^{f\,l\,k}_{\tau k' l'}$$ $\Bigg|$ 

$\Bigg\rangle$

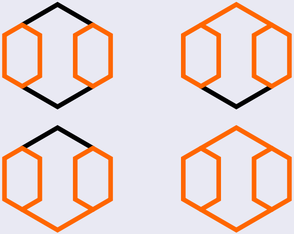
# Properties of $H_{\mathrm{Fib}}$

## Properties of $H_{\mathrm{Fib}}$

- $B_p$ projector
- eigenvalue 1: no flux through $p$
- eigenvalue 0: flux through $p$
- $\left[B_p, B_{p'}\right] = 0$
- ground state superposition of all possible string-net configuration

# Particle picture for $H_{\mathrm{Fib}}$

## Non-trivial degeneracy

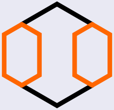


Hilbert-space grows as
$\left(1+\varphi^2\right)^{N_p} \approx 3.6^{N_p} > 2^{N_p}$

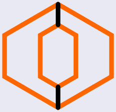
# Braiding

## Ordering in particle basis arbitrary

Exchange positions e.g. via:



$$= \sum_{a,b} F^{\tau\tau\mathbf{1}}_{\tau\tau\mathbf{a}} F^{\tau\tau\mathbf{1}}_{\tau\tau\mathbf{b}}$$




$$\xrightarrow{\text{braiding}} \sum_{a,b} F^{\tau\tau\mathbf{1}}_{\tau\tau\mathbf{a}} F^{\tau\tau\mathbf{1}}_{\tau\tau\mathbf{b}}$$

$$= \sum_{a,b} \underbrace{F^{\tau\tau\mathbf{1}}_{\tau\tau\mathbf{a}} F^{\tau\tau\mathbf{1}}_{\tau\tau\mathbf{b}} \left(R^{\tau\tau}_{a}\right)^{-1} R^{\tau\tau}_{b}}_{=:c_{a,b}}$$

R-moves allow deformation of fluxlines

## R-move


$$= R^{a\,b}_{c}$$


$$= 0.38$$

$$+(-.15 + 0.46\mathrm{i})$$

$$+(-.15 - 0.46\mathrm{i})$$

$$+0.62$$

# Summary & Conclusion

- Topological phases harbor exotic excitations
- Excitations characterize e.g. ground-state degeneracy on non-trivial topology
- Non-Abelian excitations show a specific degeneracy pattern in the excited states
- Non-trivial properties encoded in $F$- and $R$-matrices

# Notice:

Blackboard pictures missing!