Optimal persistent currents for interacting bosons on a ring with gauge field

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OUTLINE

• Introduction & connection to experiments
• Definition of the problem
• Analytical treatment
• Numerical approach (MPS & co.)
• Conclusions & open problems

Aharanov-Bohm effect & persistent current

Introduction

- U(1) gauge potential $\implies$ geometric phase
  \[ \nabla \times \vec{A} = \vec{B} \quad \varphi = \int \vec{A} \cdot d\vec{x} \]
- threaded flux $\Phi = \oint \vec{A} \cdot d\vec{l} = \oint C \vec{B} \cdot d\vec{S}$
- flux quantum $\Phi_0 = h/e$
Aharanov-Bohm effect & persistent current

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- minimal coupling \((\vec{p} - e\vec{A})^2 / 2m\) can be related to rotation as well

- U(1) gauge potential \(\Rightarrow\) geometric phase
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- threaded flux \(\Phi = \oint \vec{A} \cdot d\vec{l} = \int_\circ \vec{B} \cdot d\vec{S}\)

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- **minimal coupling**
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  \[\varphi = \int \mathbf{A} \cdot d\mathbf{x}\]

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- **flux quantum** \(\Phi_0 = h/e\)

- **quantum fluid + multiply connected geometry**
  \[I(\Omega) = -\frac{1}{2\pi\hbar} \frac{\partial E(\Omega)}{\partial \Omega}\]

- **physics periodic in terms of** \(\Omega = 2\pi \Phi / \Phi_0\)
  (inequivalent sectors … winding number...)
Persistent currents in condensed matter

Introduction

Persistent Current $\iff$ macroscopic many-body quantum coherence

- bulk superconductors
  
  B. S. Deaver and W. M. Fairbank, PRL 7, 43 (1961)
  N. Byers and C. N. Yang, PRL 7, 46 (1961)
  L. Onsager, PRL 7, 50 (1961)

- normal metallic rings
  
  L. P. Levy, et al., PRL 64, 2074 (1990)
  H. Bluhm et al., PRL 102, 136802 (2009)

- SQUID = superconducting quantum interference device

![Diagram of a SQUID](image)
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Persistent Current $\leftrightarrow$ macroscopic many-body quantum coherence

? Effects of interactions & barrier/impurities & statistics ?

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Ultragold atoms: a quantum engineering platform

Introduction

- isolated neutral quantum systems (long coherence times)
- high tunability of microscopic parameters (also interactions!)
- possibility of inducing artificial gauge potentials
- access to many microscopic observables

\[ H = -J \sum_{\langle i,j \rangle} b_i^\dagger b_j + \sum_i \epsilon_i \hat{n}_i + \frac{1}{2} U \sum_i \hat{n}_i (\hat{n}_i - 1) \]

J. Dalibard, F. Gerbier, G. Juzeliunas, and P. Öhberg, RMP 83, 1523 (2011)
Cold atoms in ring traps

Introduction

- persistent currents flowing for up to 40s!
- applications for:
  - quantum info [atomic qubit]
  - high-precision measurements [interferometry]
  - fundamental questions :)

Ramanathan et al., PRL 106, 130401 (2011); Wright et al., PRL 110, 025302 (2013); Moulder et al., PRA 86, 013629 (2012); Beattie, et al., PRL 110, 025301 (2013);

Ryu, et al., PRL 111, 205301 (2013),…

D. W. Hallwood, et al., PRA 82, 063623 (2010)
D. Solenov, D. Mozyrsky, PRA 82, 061601 (2010)
A. Nunnennkamp, et al, PRA 84, 053604 (2011)
C. Schenke et al., PRA 85, 053627 (2012)
Richness & oddness of a 1D scenario

Introduction

- obtained by strong transverse confinement and/or optical lattice

\[ \hbar \omega_\perp \gg k_B T, \mu \]

\[ \Psi_B(r_1, \ldots, r_N) = \psi^{1D}_B(x_1, \ldots, x_N) \prod_{i=1}^{N} \phi_0(r_{\perp i}) \]

Greiner et al., PRL 87, 160405 (2001)
Moritz et al., PRL 91, 250402 (2003)

๏ Quantum fluctuations are crucial

๏ Interaction growth with diluteness!

\[ \gamma = \frac{E_{\text{int}}}{E_{\text{kin}}} = \frac{gn}{\hbar^2 n^2 / m} = \frac{gm}{\hbar^2 n} \]

\[ n = \frac{N}{L} \]

๏ Fermionization of hard-core bosons

Paredes, et al., Nature 429, 6989 (2004);
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\[ \langle \hat{\Psi}^\dagger(x) \hat{\Psi}(x') \rangle \sim \frac{1}{|x - x'|^{1/2K}} \]
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- first proposals & proof of principles for 1D rings available!

The system Hamiltonian

\[ \mathcal{H} = \sum_{j=1}^{N} \left[ \frac{\hbar^2}{2M} \left( -i \frac{\partial}{\partial x_j} - \frac{2\pi \Omega}{L} \right)^2 + U \delta(x_j) + g \sum_{l<j}^{N} \delta(x_l - x_j) \right] \]

- ultracold bosons (T=0)
- mesoscopic sizes (no TL, for now)
- adimensional couplings

\[ \gamma = \frac{gm}{\hbar^2 n} \quad \lambda = \frac{mUL}{\pi\hbar^2} \]

TARGET: Persistent current \( I(\Omega) = -\frac{1}{2\pi\hbar} \frac{\partial E(\Omega)}{\partial \Omega} \) in all regimes of \( \gamma \) & \( \lambda \)

for interacting fermions ... Loss, PRL 69, 343 (1992); Mueller-Gröeling et al., EPL 22, 193 (1993) and BEC-BCS crossover ...

Presence of a barrier/defect

Setup

\[ \lambda = 0 \quad \text{Rotational Invariance} \]

\[ \text{total ang. mom. } \mathcal{L} \text{ preserved} \]

“internal” energy independent from \( \Omega \)

Current = perfect sawtooth \( \forall \gamma \)

of amplitude \( I_0 = \frac{2\pi \hbar \Omega}{mL^2} \)
Presence of a barrier/defect

\[ \lambda > 0 \] \quad U(1) \text{ Symmetry Breaking}

- single particle levels affected differently
- "internal" energy dependent from \( \Omega \)
- \( \text{Current depends on } \gamma \)!

- flux qubit around \( \Omega = 0.5 \)!
- Lieb-Liniger integrability destroyed!
- (here) adiabatic raising of barrier & focus on stationary regime...
Single-particle regimes: non-interacting

✓ eigenfunctions are plane waves
  + twisted boundary conditions
  + cusp at barrier position

\[ k_n = \pm \lambda \frac{\pi}{L} \frac{\sin(k_n L)}{\cos(2\pi \Omega) \mp \cos(k_n L)} \]

\[ \varepsilon_n = \hbar^2 k_n^2 / 2m \]

✓ ideal bosons scenario:
  condensation in the ground

\[ E = N\varepsilon_0 \]
Single-particle regimes: hard-core

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✓ ideal “fermionized” scenario
  => Friedel oscillations

\[ E = \sum_{n=0}^{N-1} \varepsilon_n \]
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Weakly interacting regime

✓ mean-field (~ classical) approach
\[ \langle \hat{\psi}(\theta) \rangle = |\Psi(\theta)| e^{i\phi(\theta)} \]

• non-linear Schrödinger equation
  (Gross-Pitaevski equation)
\[ \left[ \left( -i \frac{\partial}{\partial \theta} - \Omega \right)^2 + \lambda \delta(\theta) + \bar{g}|\Psi(\theta)|^2 \right] \Psi(\theta) = \mu \Psi(\theta) \]


✓ soliton pinned by barrier
\[ \xi = \hbar/\sqrt{2mn_0g} \]

Kanamoto et al., PRL 100, 060401 (2008)
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Analytic

MF artifact !?

\[ \gamma \]
\[ \lambda = 1.9 \]
\[ \lambda = 95.5 \]
\[ g = 10 \]
\[ g = 1.0 \]
Strongly interacting regime

- Effective field theory: Luttinger liquid
  \[
  \psi(x) = \sqrt{\rho(x)} e^{i\phi(x)}
  \]
  \[
  \rho(x) \simeq (n_0 + \partial_x \theta(x)/\pi) \sum_{l \in \mathbb{Z}} e^{2i(l\theta(x) + \pi n_0 x)}
  \]
  \[
  \omega(k) \simeq \hbar v_s |k|
  \]
  \[
  n_0 = N/L
  \]
  \[
  [\partial_x \theta(x), \phi(x')] = i\pi \delta(x-x')
  \]
  \[
  \langle \hat{\Psi}^\dagger(x)\hat{\Psi}(x') \rangle \simeq \frac{1}{|x-x'|^{1/2K}}
  \]

- Presence of gauge field ~ shift in the phase field
  \[
  \mathcal{H}_{LL} = \frac{\hbar v_s}{2\pi} \int_0^L dx \left[ K \left( \partial_x \phi(x) - \frac{2\pi}{L} \Omega \right)^2 + \frac{1}{K} (\partial_x \theta(x))^2 \right]
  \]
\textbf{Strongly interacting regime}

\begin{itemize}
  \item \textbf{effective field theory: Luttinger liquid}

  \[ \psi(x) = \sqrt{\rho(x)} e^{i \phi(x)} \]

  \[ \rho(x) \simeq (n_0 + \partial_x \theta(x)/\pi) \sum_{l \in \mathbb{Z}} e^{2il(\theta(x) + \pi n_0 x)} \]

  \[ [\partial_x \theta(x), \phi(x')] = i\pi \delta(x - x') \]

  \item \textbf{weak barrier \sim backscattering term}

  \[ \mathcal{H}_b = \int dx \, U_0 \delta(x) \rho(x) \]

  \[ \mathcal{H}_{LL} = \frac{\hbar v_s}{2\pi} \int_0^L dx \left[ K \left( \partial_x \phi(x) - \frac{2\pi}{L} \Omega \right)^2 + \frac{1}{K} (\partial_x \theta(x))^2 \right] + 2U_0 n_0 \cos[2\theta(0)] \]

  \[ \mathcal{H}_J = E_0 (J - \Omega)^2 + n_0 U_{\text{eff}} \sum_J |J + 1\rangle \langle J| + \text{h.c.} \]

  \[ \lambda_{\text{eff}} = \lambda (d/L)^K \]
\end{itemize}
Strongly interacting regime

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\[ \rho(x) \simeq \left( n_0 + \partial_x \theta(x)/\pi \right) \sum_{l \in \mathbb{Z}} e^{2i(l(\theta(x) + \pi n_0 x))} \]

\[ [\partial_x \theta(x), \phi(x')] = i\pi \delta(x - x') \]

✓ weak barrier ~ backscattering term

less interactions ==> more number fluctuations ==> screen the barrier

\[ \delta E(0.5 + \delta \Omega) \propto \left( \delta \Omega^2 - \sqrt{\delta \Omega^2 + \lambda_{\text{eff}}^2} \right) \]

\[ \lambda_{\text{eff}} = \lambda (d/L)^K \]
Strongly interacting regime

- **effective field theory: Luttinger liquid**
  \[ \psi(x) = \sqrt{\rho(x)} e^{i\phi(x)} \]
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  \[ [\partial_x \theta(x), \phi(x')] = i\pi \delta(x - x') \]

- **strong barrier ~ weak link tunnelling**
  i.e. almost obc with \[ t = f(K) \lambda^{-K} \]
  \[ \mathcal{H}_{LL} = \frac{\hbar v_s}{2\pi} \int_0^L dx \left[ K \left( \partial_x \phi(x) - \frac{2\pi}{L} \Omega \right)^2 + \frac{1}{K} (\partial_x \theta(x))^2 \right] - 2 \, t \, n_0 \cos[\phi(L) - \phi(0) + 2\pi \Omega] \]
  \[ E(\Omega) = -2 \, t_{\text{eff}} \, n_0 \cos(2\pi \Omega) \]
  \[ t_{\text{eff}} = t(d/L)^{1/K} \]
Strongly interacting regime

✅ effective field theory: Luttinger liquid
\[ \psi(x) = \sqrt{\rho(x)} e^{i\phi(x)} \]
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✅ strong barrier ~ weak link tunnelling

i.e. almost obc with \[ t = f(K) \lambda^{-K} \]

more interactions ==> more phase fluctuations ==> weaker current

\[ E(\Omega) = -2 t_{\text{eff}} n_0 \cos(2\pi \Omega) \]

\[ t_{\text{eff}} = t(d/L)^{1/K} \]
Scanning through diverse regimes: MPS

MPS-PBC variational ansatz

\[ |\psi\rangle = \sum_{\{s\}} s_1^{s_L} \text{Tr}(A_{s_1} \ldots A_{s_L}) |s_1 s_2 \ldots s_L\rangle \]

\[ A[j] = s_j \alpha \beta \]

\[ s_j \in \{0, \ldots, n_j^{\text{max}}\} \]

\[ \alpha, \beta = 1 \ldots m \]

Verstraete, et al, PRL 93, 227205 (2004);
Schollwock, Ann. Phys. 326, 96 (2011);

lattice discretization (@ low filling): Bose-Hubbard + Peierls phase

\[ \mathcal{H}_{\text{lat}} = -t_{\text{BH}} \sum_{j=1}^{N_s} \left( e^{-\frac{2\pi i \Omega}{N_s}} b_j^\dagger b_{j+1} + \text{H.c.} \right) + \frac{U_{\text{BH}}}{2} \sum_{j=1}^{N_s} n_j(n_j-1) + \sum_j \left( \lambda_{\text{BH}} \delta_{j,1} n_j - \mu n_j \right) \]
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\[ A[j]_{s_j \alpha \beta} \]

\[ s_j \in \{0, \ldots, n_j^{\text{max}}\} \]

\[ \alpha, \beta = 1 \ldots m \]

\[ \lambda = 0.1 \]
\[ \lambda = 1.9 \]
\[ \lambda = 9.5 \]
\[ \lambda = 19.1 \]
\[ \lambda = 38.2 \]
\[ \lambda = 95.5 \]

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Schollwock, Ann. Phys. 326, 96 (2011);

... not so stable, accurate & fast as for OBC ...

● absence of a fully isometric gauge \Longrightarrow \text{need for generalized eival. problem} :(

● transfer matrices of long sub-chains: keep \( p \) dominant eivals/eivecs

\[ \Longrightarrow \text{costs } O(pm^3) \text{ vs. } O(m^5) \]

\[ \text{BUT } p \text{ often scales like } O(m) :(


Numeric

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Scanning through diverse regimes: TTN?

✓ binary TTN variational ansatz

๏ contraction costs are \( O(m^4) \) & overheads much smaller :)
๏ adaptive isometric gauge ==> always standard eigenvalue problem :)
๏ equal treatment for OBC & PBC's


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๏ observables are also fairly nicely reproduced (weakly site-dependent error)

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✓ binary TTN variational ansatz

✓ equal treatment for OBC & PBC’s
  ○ adaptive isometric gauge ==> always standard eigenvalue problem :)
  ○ contraction costs are $O(m^4)$ & overheads much smaller :)
  ○ observables are also fairly nicely reproduced (weakly site-dependent error)


Take-Home message

MF regime, i.e. low $\gamma = (gm)/(\hbar^2 n)$:
- interaction $\uparrow$ ==> barrier effect $\downarrow$
- (shorter density-density healing length ...)

LL regime, i.e. large $\gamma = (gm)/(\hbar^2 n)$:
- interaction $\uparrow$ ==> barrier effect $\uparrow$
- (faster decay of phase-phase correlations...)

- quantum fluctuations counteract the barrier screening by interactions
- existence of an optimal regime where the defects are less influential!
- need to choose extremals for an effective quantum state manipulation!

✓ results relevant also for: thin supercond. rings, photonic waveguides, ...
Open questions

- superpositions in time-of-flight momentum distributions

\[ n(k) = \int dx \int dx' \ e^{ik \cdot (x-x')} \rho_1(x, x') \]

- scaling of currents with ring size

- behaviour of currents with particle density (=> SF fraction?)

- finite temperature/entropy effects (relevant even in cold atoms)

- non-equilibrium dynamics due to barrier intensity/speed quench ...

Thanks to all of you for your attention!

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