Anyons & Topological Phases

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University of Minnesota

Numerical and analytical methods
for strongly correlated systems

26/07/2014

Disclaimer

This talk is not about...

- experimental realizations
- symmetry-protected topological order
  ($\Rightarrow$ no topological insulators)
- edge-mode physics

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- about topological quantum order
- dealing with fine-tuned models
- considering bulk properties
- is a pedestrian approach to topological order

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Be aware of (over-) simplifications!

Topological order

Some characteristic features

- Gapped phases
- Ground-state degeneracy depends on topology of the system
- Ground states not distinguishable by local operators

  $\Rightarrow$ no local order parameter

  $\Rightarrow$ robustness of topological phases
- Long-range entanglement
- Elementary gapped excitations are so-called anyons

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$\rightsquigarrow$ Frederic Mila's talk

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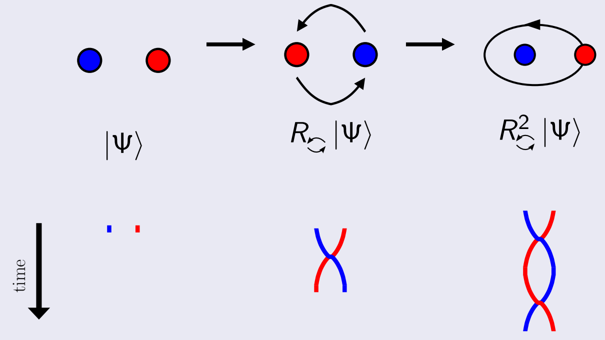
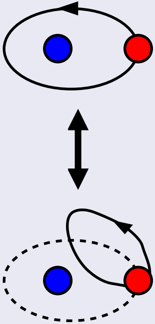
Outline

Anyons



3D

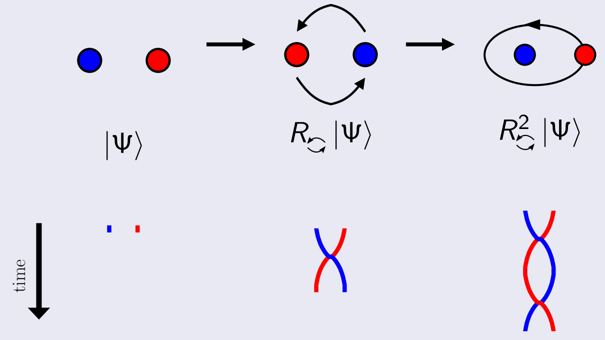


$$R_{\circlearrowleft}|\Psi\rangle = +|\Psi\rangle \Rightarrow \text{bosons}$$
$$R_{\circlearrowleft}|\Psi\rangle = -|\Psi\rangle \Rightarrow \text{fermions}$$

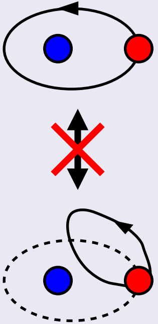
$$R^2_{\circlearrowleft} = \mathbb{1}$$

Leinaas, Myrheim, Il Nuovo Cim. B **37** 1 (1977)

Anyons



2D



$$R_{\circlearrowleft}|\Psi\rangle = e^{i\theta}|\Psi\rangle \Rightarrow \text{anyons (Abelian)}$$
$$R_{\circlearrowleft}|\Psi\rangle = U|\Psi\rangle \Rightarrow \text{anyons (non}-\text{Abelian)}$$

$$R^2_{\circlearrowleft} \neq \mathbb{1}$$

Leinaas, Myrheim, Il Nuovo Cim. B **37** 1 (1977)

the toric code

the ($\mathbb{Z}_2-$) toric code

A. Kitaev, Ann. Phys. **303**, 2 (2003)

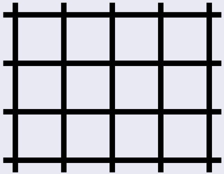
- the "Ising model" of topological order
- exactly solvable
- ground state topologically ordered
- realizes the same phase as e.g.
  - $\mathbb{Z}_2$ loop gas
  - quantum dimer model on Kagomé-lattice

the toric code

(aka $\mathbb{Z}_2$ gauge theory in $(2+1)D$)

Lattice

- Defined (here) on a square lattice
- Microscopic degrees of freedom:
    - located on the links
    - spin-$\frac{1}{2}$: $|\uparrow\rangle$, $|\downarrow\rangle$.



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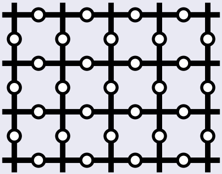

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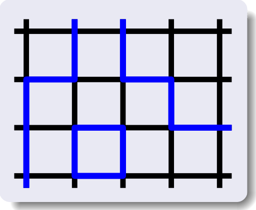


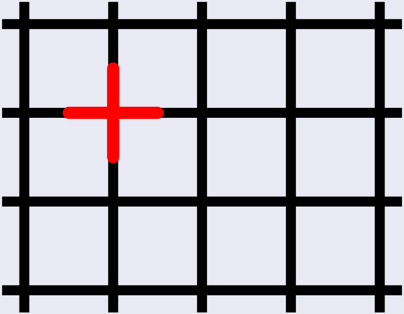
the toric code

The Hamiltonian

$$H_{TC} = -J \sum_{\text{vertices } s} A_s \; - \; J \sum_{\text{plaquettes } p} B_p$$

- $A_s = \prod_{i \in s} \sigma_i^z$
- eigenvalues are ± 1
  - $+1$: even number of $|\downarrow\rangle$
  - -1: odd number of $|\downarrow\rangle$

  $\Rightarrow$ charge at vertex s



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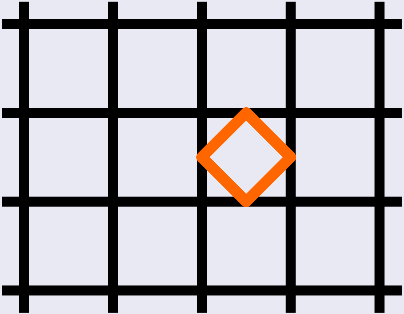
- eigenvalues are ± 1
  - $+1$: e.g. $|\square\rangle + |{\color{blue}\square}\rangle$
  - -1: e.g. $|\square\rangle - |{\color{blue}\square}\rangle$

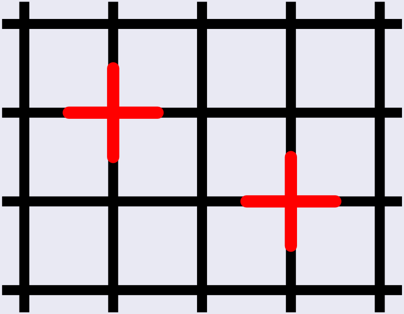
  $\Rightarrow$ flux at plaquette p



the toric code

Operator properties

- $[A_s, A_{s'}] \;= 0$



$\Rightarrow$ eigenvalues of A_s's and B_p's are conserved quantities

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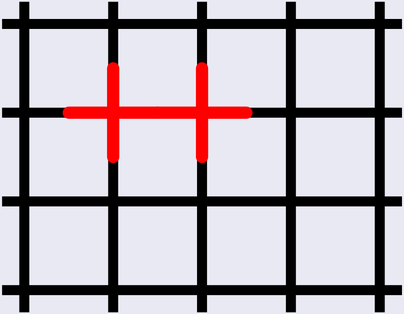

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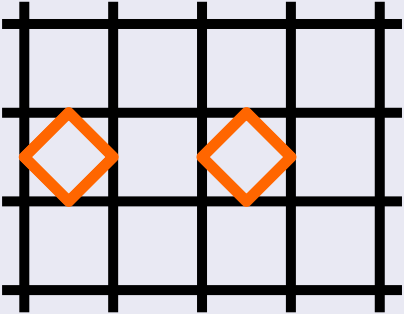

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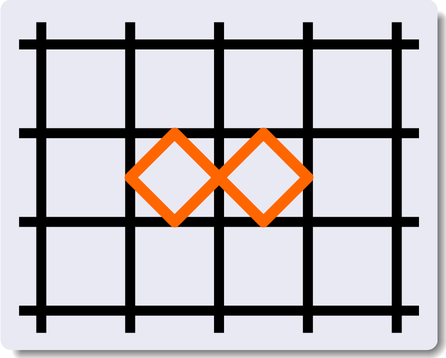

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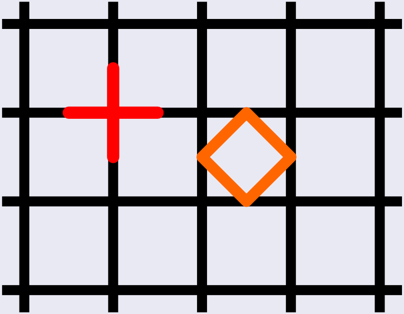

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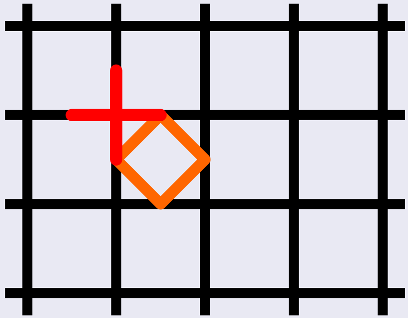

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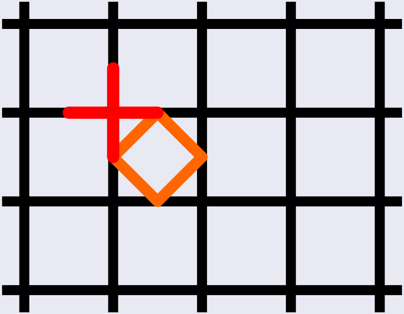

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A_s's and B_p's not only commuting but also frustration free
$\Rightarrow$ minimize each term separately

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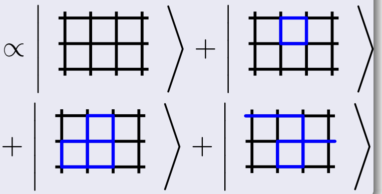
$$|\mathbf{gs}\rangle \propto \prod_p \frac{\mathbb{1} + B_p}{2} \prod_s \frac{\mathbb{1} + A_s}{2} |\mathrm{ref}\rangle$$



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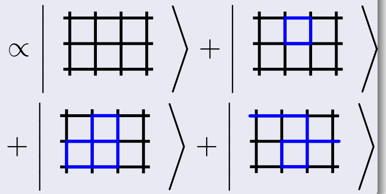


loop gas without string tension

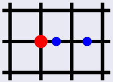
excitations

charge-excitations

- $\sigma_i^x \frac{\mathbb{1}+A_s}{2} = \frac{\mathbb{1}-A_s}{2}\sigma_i^x$

$\Rightarrow$ string of σ_i^x creates charges at its endpoints

$X\,|\mathbf{gs}\rangle =$



flux-excitations

- $\sigma_i^z \frac{\mathbb{1}+B_p}{2} = \frac{\mathbb{1}-B_p}{2}\sigma_i^z$

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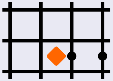
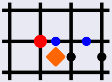


composite-excitations

- endpoints of open X, Z-string yield charge-flux composite

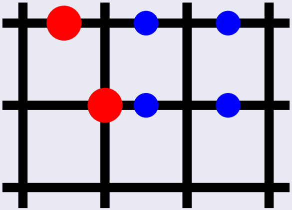
$\Rightarrow$ open strings entering region defining particle type

$XZ\,|\mathbf{gs}\rangle =$



exchange-statistics

- charges are (hardcore-) bosons
- fluxes are (hardcore-) bosons
- charges and fluxes are mutual semions
- composites are fermions



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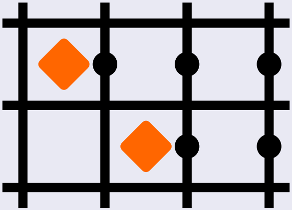

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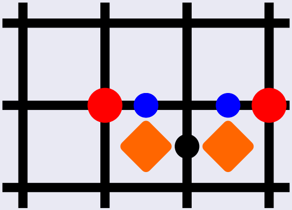

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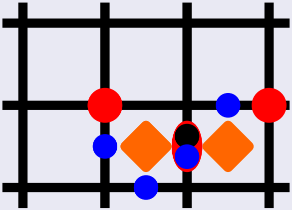

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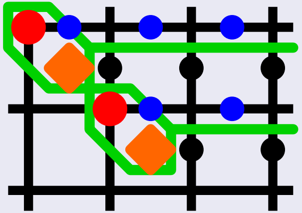
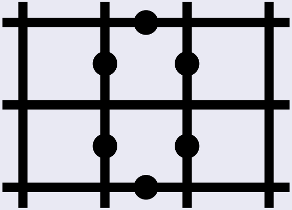


measuring non-local fluxes

- idea: measure total flux through several plaquettes
- corresponding operator $W_{p_1,\ldots,p_n}$ defined analogously to B_p
- e.g. $W_{p_1,p_2} = B_{p_1} B_{p_2}$

$\Rightarrow$ for contractible loops:
$$W_{p_1,\ldots,p_n} = \prod_{i\in\{1,\ldots,n\}} B_{p_i}$$



- analogous for charges
- analogous for composites

no non-contractible loop
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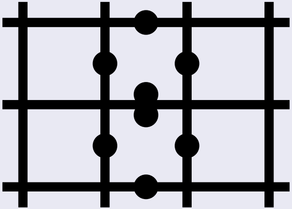

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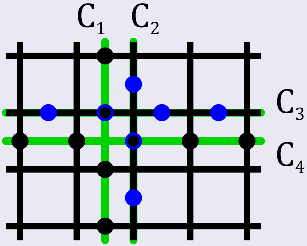
measuring global fluxes/charges on the torus

- non-contractible loops $\mathcal{C}_i$
- operators W_i are not product of A_s, B_p
- $[W_i, H_{TC}] = 0$
  $\Rightarrow$ new good quantum numbers
- but only two of them as
  $[W_1, W_3] \neq 0 \neq [W_2, W_4]$



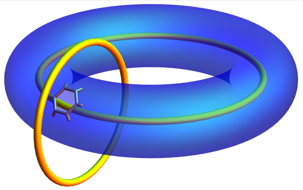
choose e.g.

- W_1, W_2 as quantum numbers
- W_3, W_4 as "raising/lowering" operators

interpretation of global operators

interpretation

- W_1/W_2 measure charge/flux through one hole of the torus
- W_3/W_4 transport charge/flux through this hole



Topology $\leftrightarrow$ non-contractible loops
$\leftrightarrow$ non-trivial string operators per excitation
$\leftrightarrow$ ground-state degeneracy

Summary Example 1

so far:

- point-like anyons in $2D$
- toric code as example for a gauge theory
- excitations can be pictured as endpoints of (tensionless) string
- string formalism allows to capture non-local features
- topology gives rise to e.g. non-trivial ground-state degeneracy

Notice:

Blackboard pictures missing!