Josephson coupled Moore-Read states

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GM, L. Hormozi, J. Slingerland, S.H. Simon, arxiv:1409:6339

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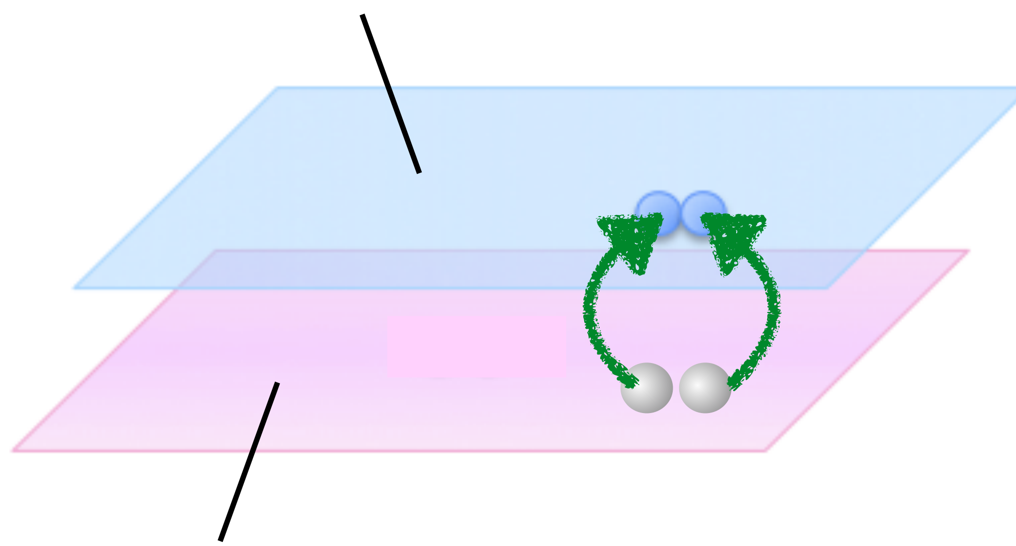
Idea

Moore-Read state ~ *p*-wave CF superconductor



Moore-Read state 2

What happens if Moore-Read states are coupled by tunnelling?

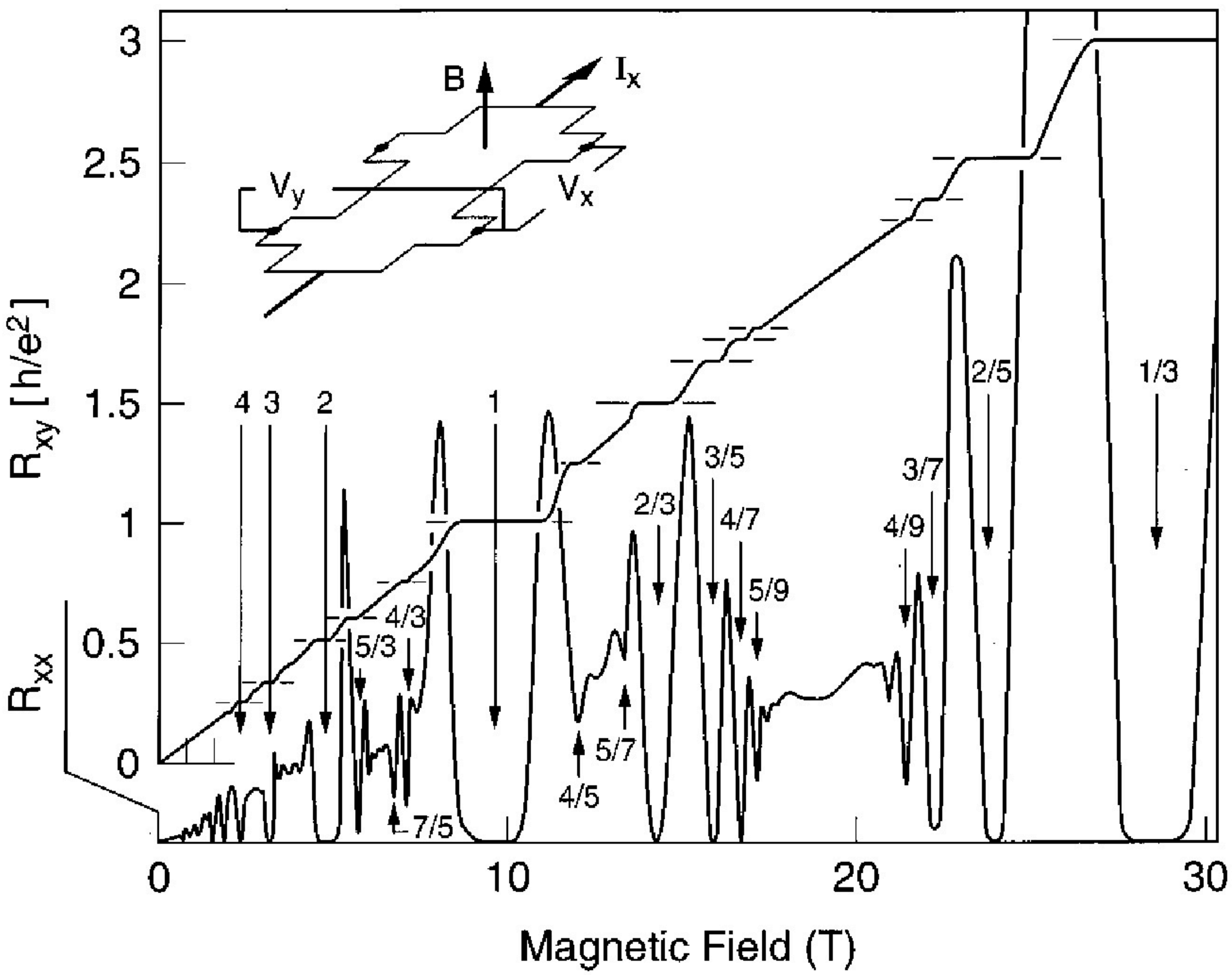
Outline

- Background:
  - Moore Read state as a superconductor of composite fermions
  - Josephson Effect

- How to couple Moore-Read states:
  - Single-particle tunnelling gap
  - “Orthogonality catastrophe" for tunnelling into a CF metal
  - Model Hamiltonians for Moore-Read bilayers

- Properties of the Josephson coupled Moore-Read state
  - Wave functions
  - Topological order of coupled Moore-Read states
  - Edge spectrum

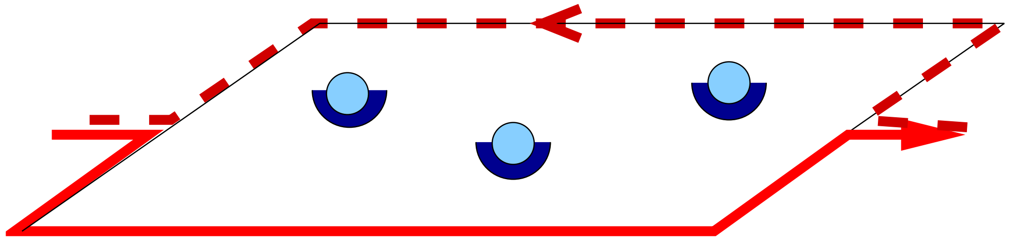
Reminder: Quantum Hall Effect



plateaux in Hall resistivity

incompressible quantum liquids

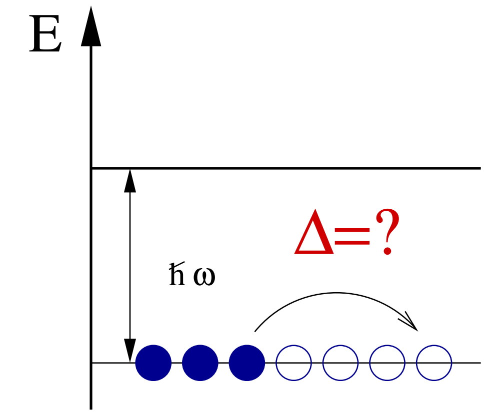
edge transport
& localisation in bulk



Fractional Quantum Hall Effect

- in transport, FQHE has same phenomenology as IQHE

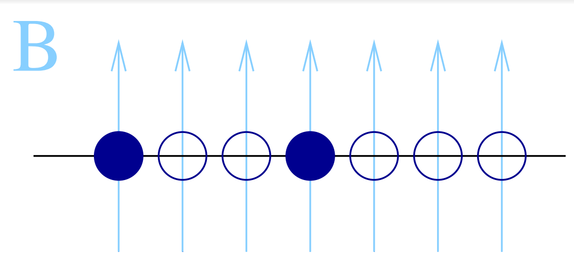
- IQHE: quantized plateaus $\leftrightarrow$ gapped excitations in bulk
- partially filled Landau-level (LL) $\Rightarrow$ naïvely expect degenerate groundstate & $\Delta \to 0$

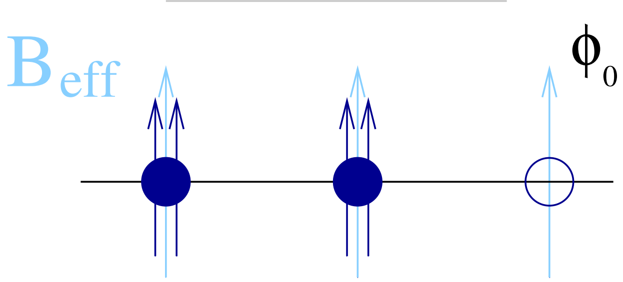


$\Rightarrow$ The nature of interactions determines the groundstate!

- Complicated many body problem in LLs

$$\mathcal{H} = \sum_{i<j} V(|\vec{r}_i - \vec{r}_j|)$$





$$B_{\text{eff}} = B - 2n\Phi_0$$

But: very successful trial wavefunctions exist: composite fermions with 'flux attached' [Jain 1989]
$\Rightarrow$ Effective problem in reduced magnetic field

Quantum Hall Wave Functions

Single particle states: (symmetric gauge: **A** ~ y **e**$_x$ + x **e**$_y$)

monomials $\phi_m \propto z^m$ $z = x + iy$

Many-particle states: (homogeneous) polynomials $\Psi(z_1, \ldots, z_N)$

Common wavefunctions:

IQHE: $$\Psi_{\nu=1} = \prod_{i<j}(z_i - z_j)$$ filled Landau level / Jastrow factor

FQHE: $$\Psi_{\nu=1/3} = \prod_{i<j}(z_i - z_j)^3$$ Laughlin state at filling 1/3

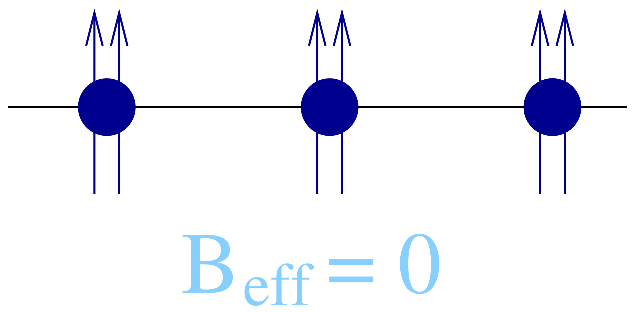
Flux Attachment

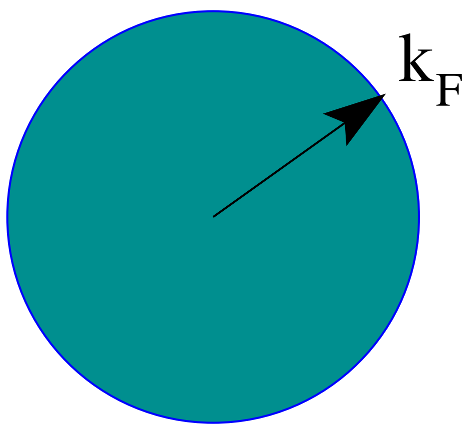
filled LL

The half-filled Landau level

- half filling: all flux attached to electrons in CF transformation

- CF non-interacting $\Rightarrow$ fill Fermi-sea
  $\Psi = \mathcal{P}_{\text{LLL}} \prod_{i<j}(z_i - z_j)^2 \Psi_{\text{FS}}^{\text{CF}}$
- But CF have interactions: screened Coulomb + Chern-Simons gauge field from flux-attachment





$\Rightarrow$ If CF have net attractive interaction, CF Fermi-sea is unstable to pairing & gap opens

- Pairing is a matter of interactions.
- Experimentally: $\nu = 1/2 \Rightarrow$ no QHE, but
  $\nu = 5/2 \Rightarrow$ QHE seen!

Moore & Read; Greiter, Wen & Wilczek

Review: BCS theory

Superconductor as a pair-condensate: BCS wavefunction

Grand Canonical:

$$|\mathrm{BCS}\rangle = \prod_{\mathbf{k}}{}' (u_k + v_k c^\dagger_{\mathbf{k}} c^\dagger_{-\mathbf{k}})|0\rangle$$

$$\propto \exp\left[\sum_{\mathbf{k}}{}' g_k c^\dagger_{\mathbf{k}} c^\dagger_{-\mathbf{k}}\right]|0\rangle \qquad g_k \equiv \frac{v_k}{u_k}$$

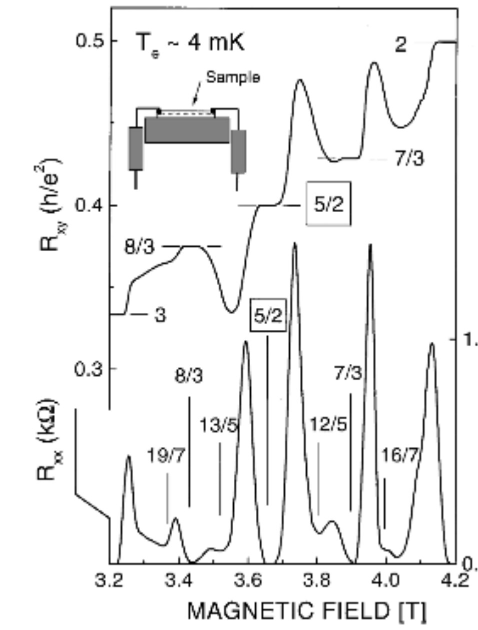
Fixed N / Real Space

$$\langle\{\mathbf{r}_i\}|\mathrm{BCS}\rangle = \mathrm{Pf}\left[\sum_{\mathbf{k}} g_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}_l - \mathbf{r}_m)}\right]_{l,m}$$

$$= \mathrm{Pf}\left[g(\mathbf{r}_l - \mathbf{r}_m)\right]_{l,m}$$

The Moore-Read State

Pairing instability to open gap in a half-filled Landau-level:
Trial state = Composite fermionized BCS state



$$\Psi^{MR}_{\nu=1/q} = \prod_{i<j}(z_i - z_j)^q \, \mathrm{Pf}\left[\frac{1}{z_i - z_j}\right]$$

Flux Attachment

BCS state

pair wave function $\qquad g(r) = \frac{1}{z} \propto r^l, \quad l = -1 \quad "p_x - ip_y"$

can write general pair WFs: $\qquad g(z_i - z_j) = \sum_{\mathbf{k}} g_{\mathbf{k}} \, \tilde{\phi}_{\mathbf{k}}(z_i) \, \tilde{\phi}_{-\mathbf{k}}(z_j)$

Note: bosons ↔ fermions $\qquad \Psi_F = \prod_{i<j}(z_i - z_j)\Psi_B$

Moore & Read 1991
Read & Green 2000
GM & Simon 2008

Parent Hamiltonians for FQHE

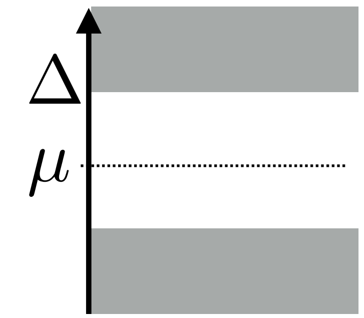
Easy to write parent Hamiltonians, especially for bosonic FQHE states:

State	Hamiltonian
Bosonic Laughlin state at ν=1/2 $$\Psi_{1/2} = \prod_{i<j} (z_i - z_j)^2$$	$$\mathcal{H} = \sum_{i<j} \delta^{(2)}(r_i - r_j) \equiv \hat{V}_0$$
Bosonic Moore-Read state at ν=1 $$\Psi^{\mathrm{MR}}_{\nu=1} = \prod_{i<j} (z_i - z_j) \mathrm{Pf}\left[\frac{1}{z_i - z_j}\right]$$	$$\mathcal{H} = \sum_{i<j<k} \delta^{(2)}(r_i - r_j)\delta^{(2)}(r_j - r_k)$$

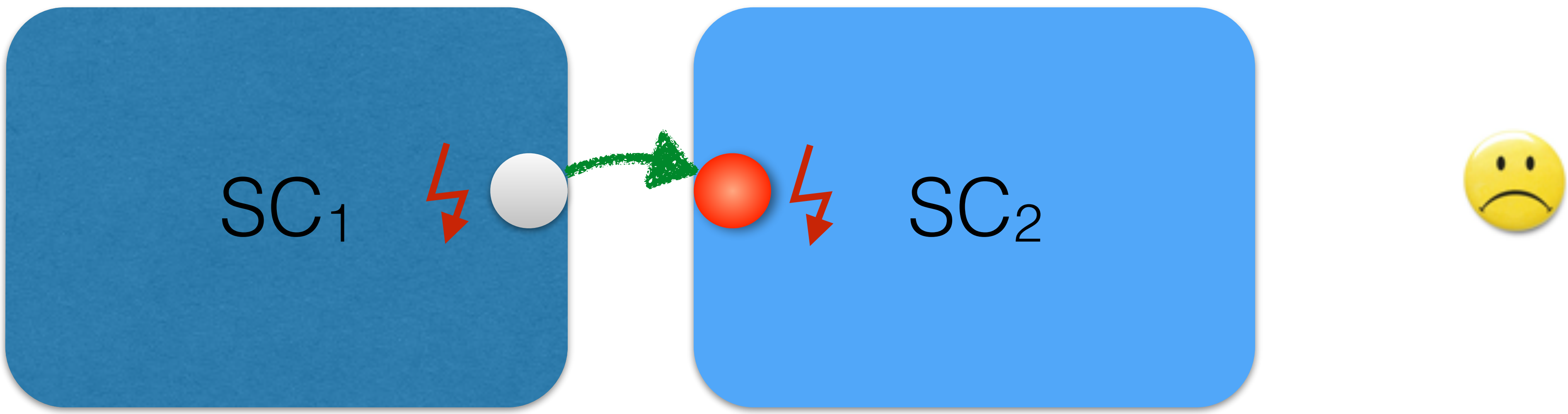
➜ vanishing properties select unique state at right filling factor

Josephson Effect for SC

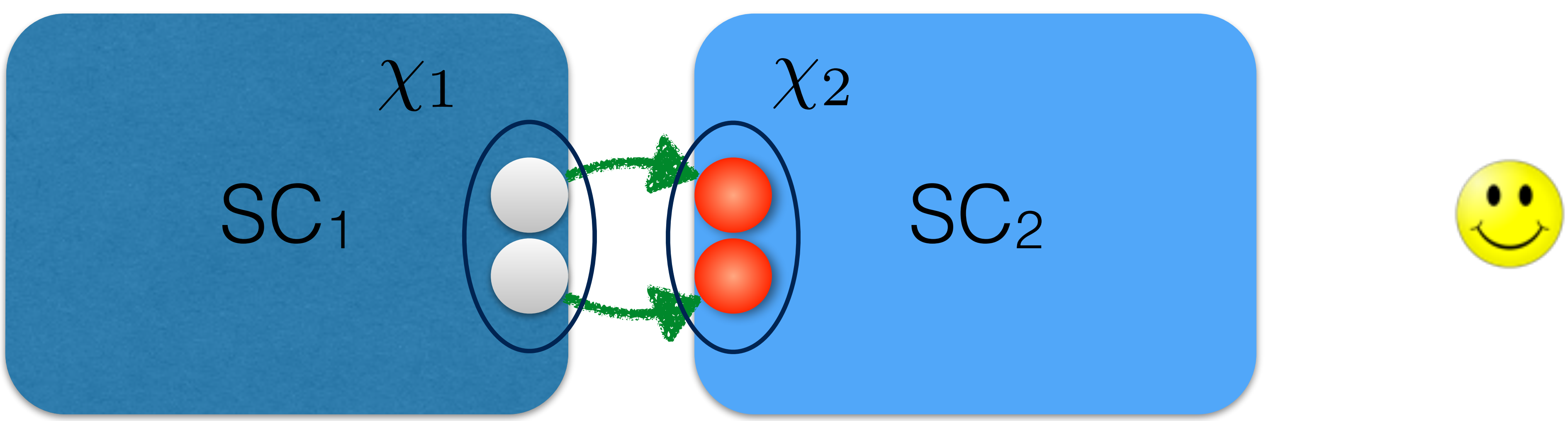
Two basic facts about tunnelling between superconductors:



1. Energy penalty 2Δ for tunnelling single particles



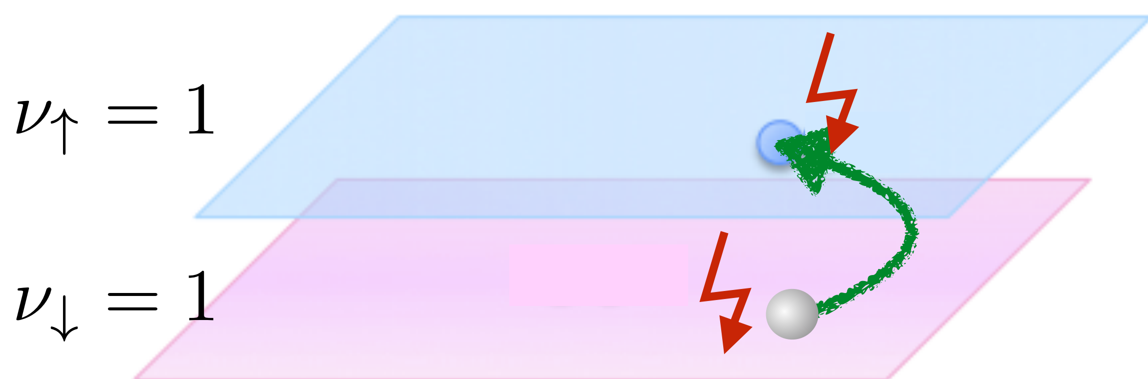
2. Pair-tunnelling is a low-energy process: driven by difference in SC phase



Josephson Coupled Moore-Read states

Can we couple composite fermion superconductors, too?

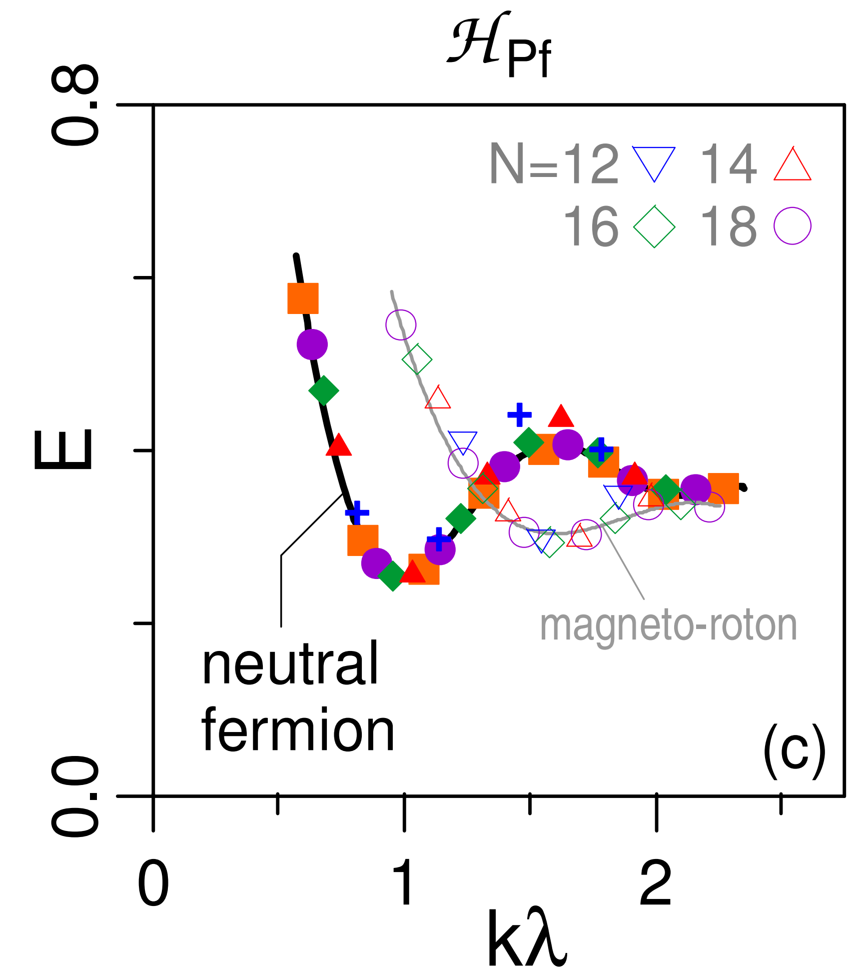
1. Gap for single-particle (neutral fermion or Bogoliubov) excitations



Single-particle tunnelling?

$$\mathcal{H}_{t1} = \int d^2r \left(\hat{c}^\dagger_\uparrow(r) \hat{c}_\downarrow(r) + h.c. \right)$$

➜ suppressed by finite (SC) gap!



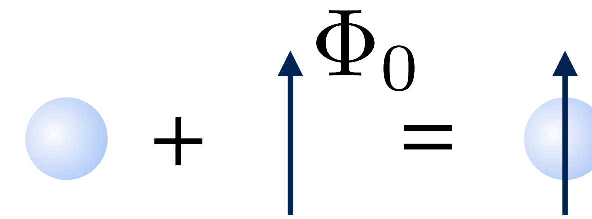
*GM, A. Wójs & N. Cooper, PRL **107** (2011)*

(magnetic length λ)

Josephson Coupled Moore-Read states

Can we couple composite fermion superconductors, too?

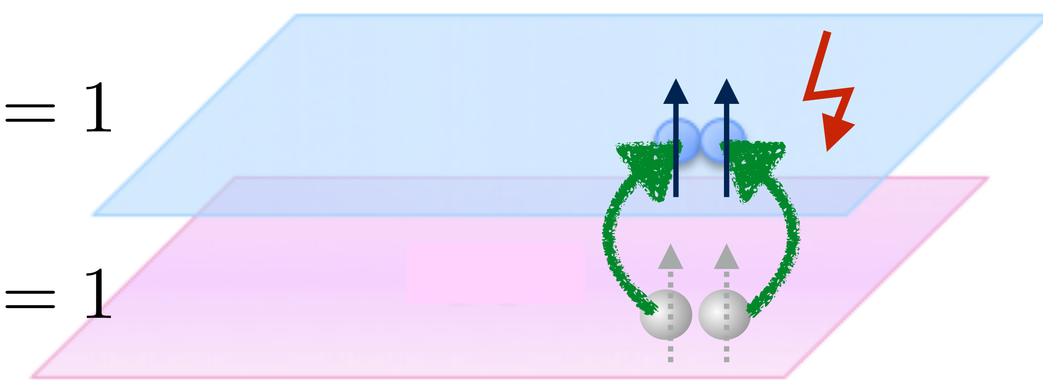
2. Pair-tunnelling between Moore-Read states



boson + flux = CF

$\nu_\uparrow = 1$

$\nu_\downarrow = 1$



$$\Psi^{\mathrm{MR}}_{\nu=1} = \prod_{i<j}(z_i - z_j)\mathrm{Pf}\left[\frac{1}{z_i - z_j}\right]$$

boson ‘sees’ all particles in same layer

Natural Hamiltonian for tunnelling pairs of bosons leaves flux per layer invariant

$$\mathcal{H}_{t2} = \int d^2r \left(\hat{c}^\dagger_\uparrow(r)c^\dagger_\uparrow(r)\hat{c}_\downarrow(r)c_\downarrow(r) + h.c.\right)$$

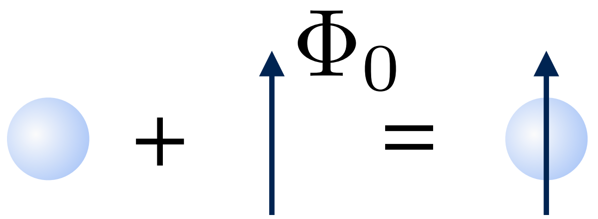
➜ pair-tunnelling still suppressed due to composite fermion correlation hole!

Josephson Coupled Moore-Read states

Helping pairs of composite fermions to tunnel

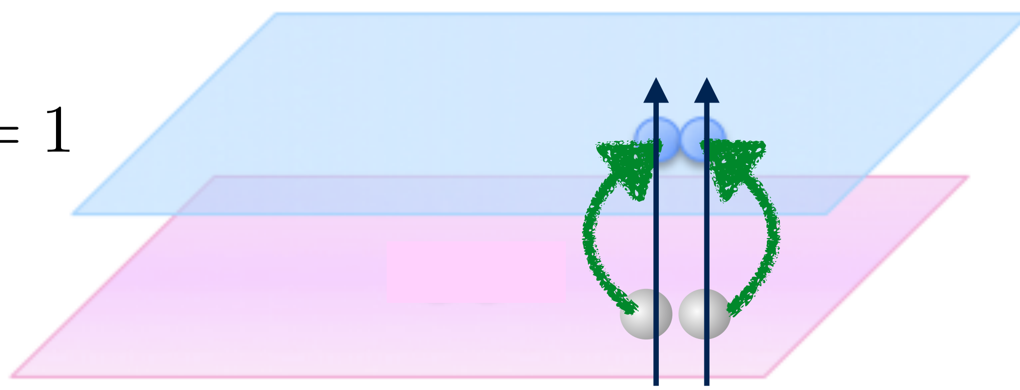
3. Opening up an inter-layer correlation hole

➜ boson should 'see' all particles in both layers



boson + flux = CF

$\nu_\uparrow + \nu_\downarrow = 1$



flux seen in both layers

➜ particles all correlated

➜ pairs can tunnel!

$$\Psi_0^{N_\uparrow,N_\downarrow} = \mathrm{Pf}\left(\frac{1}{z_i^\uparrow - z_j^\uparrow}\right)\prod_{i<j=1}^{N_\uparrow}(z_i^\uparrow - z_j^\uparrow) \times \mathrm{Pf}\left(\frac{1}{z_i^\downarrow - z_j^\downarrow}\right)\prod_{i<j=1}^{N_\downarrow}(z_i^\downarrow - z_j^\downarrow) \times \prod_{i=1}^{N_\uparrow}\prod_{j=1}^{N_\downarrow}(z_i^\uparrow - z_j^\downarrow)$$

layer 1   layer 2   intralayer-correlations

Parent Hamiltonian for Coupled Moore-Read layers

Add parent Hamiltonians to yield the coupled Moore-Read wavefunction:

$$\Psi_0^{N_\uparrow,N_\downarrow} = \mathrm{Pf}\left(\frac{1}{z_i^\uparrow - z_j^\uparrow}\right)\prod_{i<j=1}^{N_\uparrow}(z_i^\uparrow - z_j^\uparrow) \times \mathrm{Pf}\left(\frac{1}{z_i^\downarrow - z_j^\downarrow}\right)\prod_{i<j=1}^{N_\downarrow}(z_i^\downarrow - z_j^\downarrow) \times \prod_{i=1}^{N_\uparrow}\prod_{j=1}^{N_\downarrow}(z_i^\uparrow - z_j^\downarrow)$$

layer 1

layer 2

intralayer-
correlations

$$\hat{\mathcal{H}}_{3\text{-}2} = \sum_{i<j<k=1}^{N_\uparrow}\delta^{(2)}(z_i^\uparrow - z_j^\uparrow)\delta^{(2)}(z_j^\uparrow - z_k^\uparrow) + \sum_{i<j<k=1}^{N_\downarrow}\delta^{(2)}(z_i^\downarrow - z_j^\downarrow)\delta^{(2)}(z_j^\downarrow - z_k^\downarrow) + \sum_{i=1}^{N_\uparrow}\sum_{j=1}^{N_\downarrow}\delta^{(2)}(z_i^\uparrow - z_j^\downarrow)$$

Note: Can choose $N_\uparrow$, $N_\downarrow$ freely, zero energy states $N = N_\uparrow + N_\downarrow = N_\phi + 2$

➜ system has ground state degeneracy: $d = N/2 + 1$

Goldstone Mode

Degenerate ground state manifold $\Psi_{0,n} = \Psi_0^{N-2n,2n}$

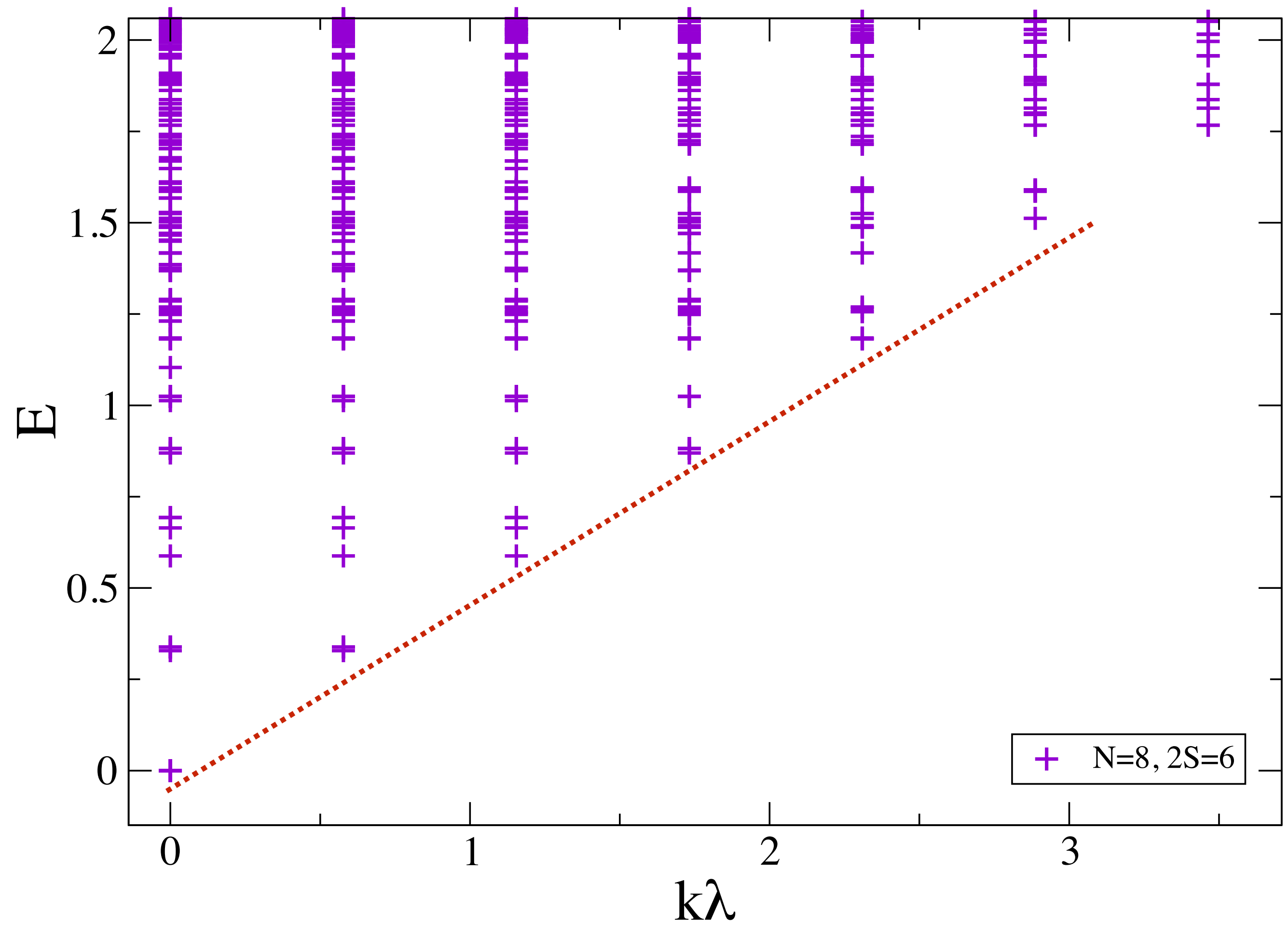
fluctuations in $N_\uparrow, N_\downarrow$

$$|\Psi\rangle = \sum_n c_n |\Psi_{0,n}\rangle$$

fluctuations in phase of superconducting order parameters

$$\Delta\chi = \chi_\uparrow - \chi_\downarrow$$

linearly dispersing collective excitations



exact diagonalization of $H_{3\text{-}2}$

➜ $H_{3\text{-}2}$ describes a gapless phase (so, not strictly a topological phase…)

Turn up the Tunnelling

Response of the coupled Moore-Read system to Josephson tunnelling

$$\mathcal{H}^{\text{JC}}_{3\text{-}2}(t) = \mathcal{H}_{3\text{-}2} + t\mathcal{H}_{t2} \qquad t \ll 1$$

➜ minimize tunnelling energy: select a preferred superposition $\{c_n\}$

$$|\Psi\rangle = \sum_n c_n |\Psi_{0,n}\rangle \quad \rightarrow \quad \arg c_n - \arg c_{n-1} = \pi$$

➜ gap opened, Goldstone mode gapped linearly ~ t !

Which ground-state is selected? (linear perturbation theory, thermodynamic limit)

layer structure

spatial structure

$$\Psi_0(t>0) = 2^{-\frac{N}{2}} \mathrm{Pf}\left[\frac{|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle}{z_i - z_j}\right] \prod_{i,j}(z_i - z_j)$$

Identifying the new ground state

Unique, gapped ground state of $\mathcal{H}^{\text{JC}}_{3\text{-}2}(t) = \mathcal{H}_{3\text{-}2} + t\mathcal{H}_{t2}$ $\quad t \ll 1$

layer structure

spatial structure

$$\Psi_0(t>0) = 2^{-\frac{N}{2}}\,\text{Pf}\left[\frac{|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle}{z_i - z_j}\right]\prod_{i,j}(z_i - z_j)$$

Change basis $\quad \sigma_x \quad |\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle) \quad \sigma_z$

$$\Psi_0(t>0) = 2^{-\frac{N}{2}}\,\text{Pf}\left[\frac{|+-\rangle + |-+\rangle}{z_i - z_j}\right]\prod_{i,j}(z_i - z_j)$$

(Cauchy) $$= \prod_{i<j}^{N/2}(z_i^+ - z_j^+)^2 \prod_{i<j}^{N/2}(z_i^- - z_j^-)^2 \equiv \Psi_{220}^{\pm}$$

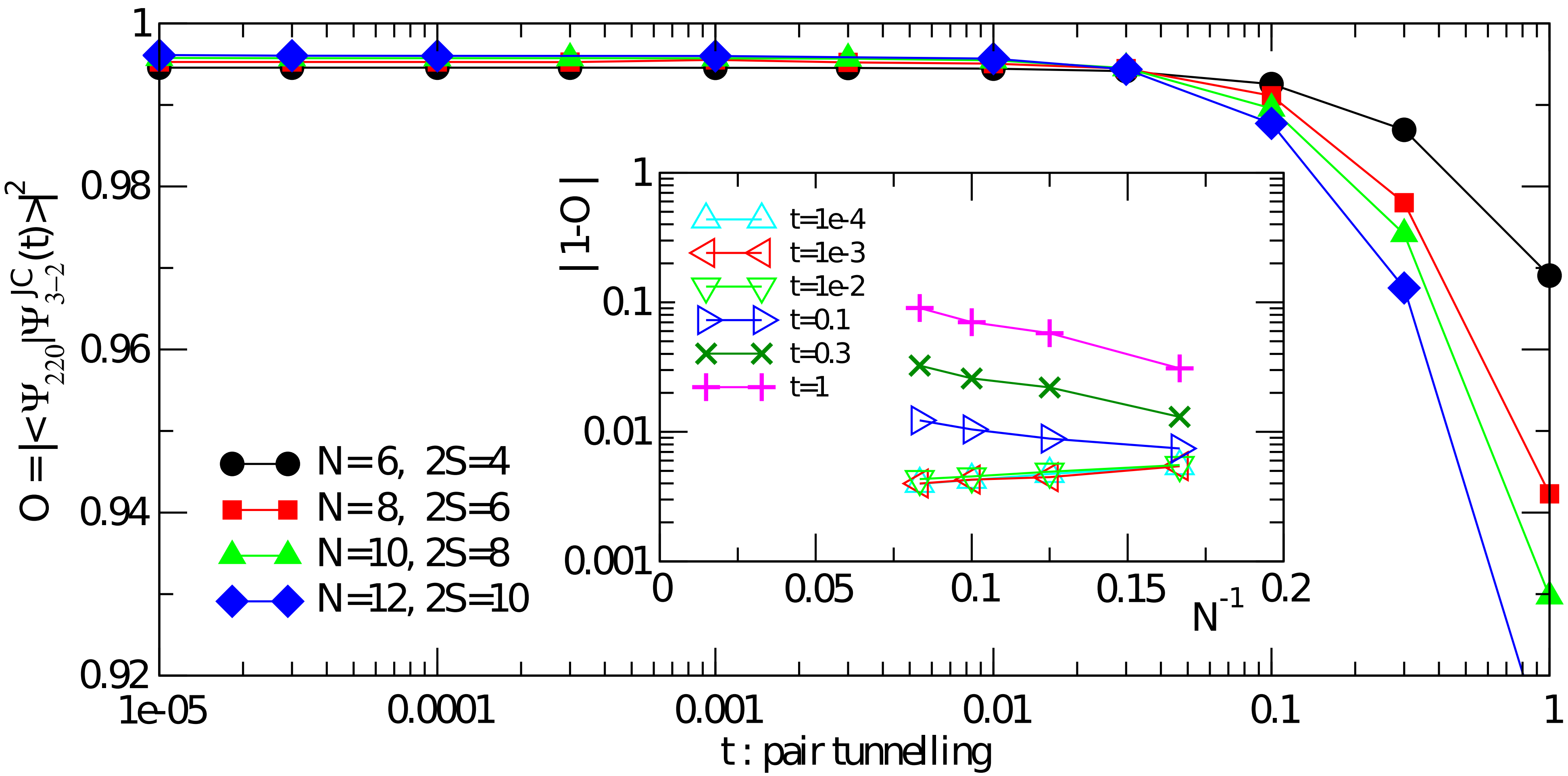
Parent Hamiltonian in the σ_x basis

σ_z basis	σ_x basis
3-body a) $\mathcal{H}^{\mathrm{JC}}_{3\text{-}2}(t) = \mathcal{H}_{3\text{-}2} + t\mathcal{H}_{t2}$	
2-body b) $\mathcal{H}^{\uparrow\downarrow}_{220} = \hat{V}_0^{\uparrow\uparrow} + \hat{V}_0^{\downarrow\downarrow} + \hat{V}_0^{\uparrow\downarrow} + \hat{V}_0^{\mathrm{tun}}$ ⟵	2-body $\mathcal{H}^{+-}_{220} = \hat{V}_0^{++} + \hat{V}_0^{--}$ ↑
Superposition of coupled Moore-Read states $\sum_n (-1)^n \Psi_{0,n}$	Halperin 220 state $\Psi^{\pm}_{220}$

Compare ground states of H3-2 vs H220

a) $\mathcal{H}^{\mathrm{JC}}_{3\text{-}2}(t) = \mathcal{H}_{3\text{-}2} + t\mathcal{H}_{t2}$

b) $\mathcal{H}^{\uparrow\downarrow}_{220} = \hat{V}_0^{\uparrow\uparrow} + \hat{V}_0^{\downarrow\downarrow} + \hat{V}_0^{\uparrow\downarrow} + \hat{V}_0^{\mathrm{tun}}$



➜ confirm ground states identical (up to finite size effects)

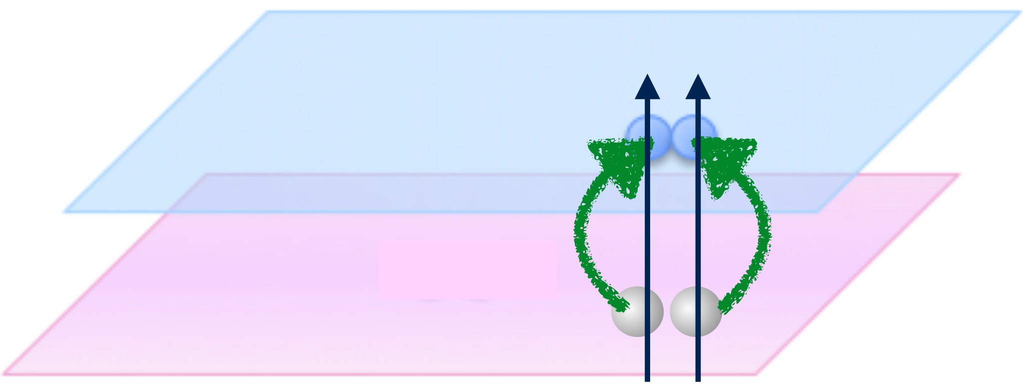
Summary so far…

Recipe for Josephson coupling of composite fermion superconductors:

1) No single particle tunnelling

2) No pair tunnelling without correlation hole

3) With extra inter-layer correlations, get a degenerate ground state



4) Even infinitesimal tunnelling opens up a gap

$\nu_{\uparrow} + \nu_{\downarrow} = 1$

5) The new ground state is in the Halperin-220 phase

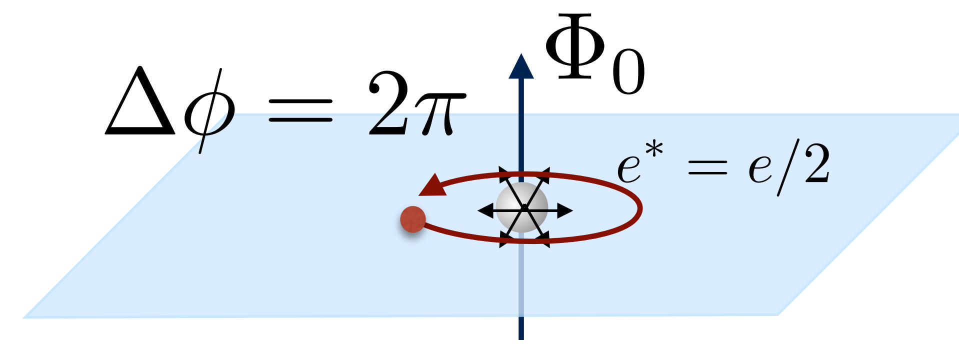
➜ Get simple two-body Hamiltonian that generates superposition of coupled Moore-Read states (Moore-Read usually requires 3-body Hamiltonian!)

Quasihole Excitations

Adding flux generates quasihole excitations / vortices

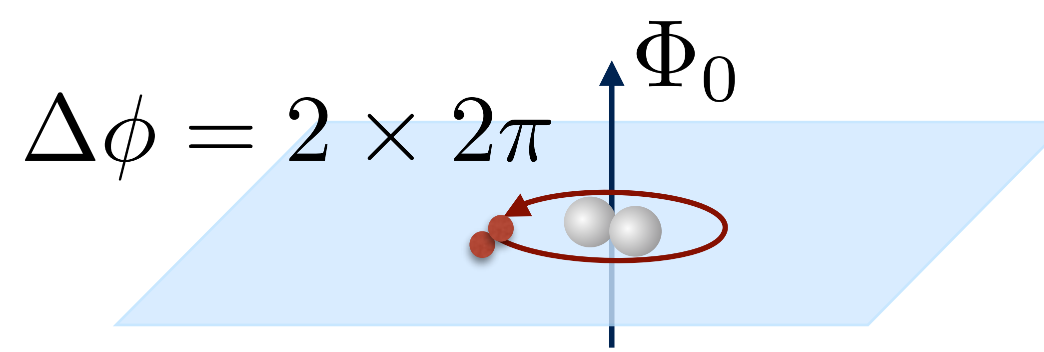
Reminder: Laughlin

$$\Psi^{\mathrm{qh}}_{\nu=1/2} = \prod_i (z_i - w) \prod_{i<j} (z_i - z_j)^2$$



Moore-Read state

Extra flux seen by Cooper pairs



➜ pick up twice the phase, fundamental flux quantum is halved! $\Phi_0^{SC} = \Phi_0/2$

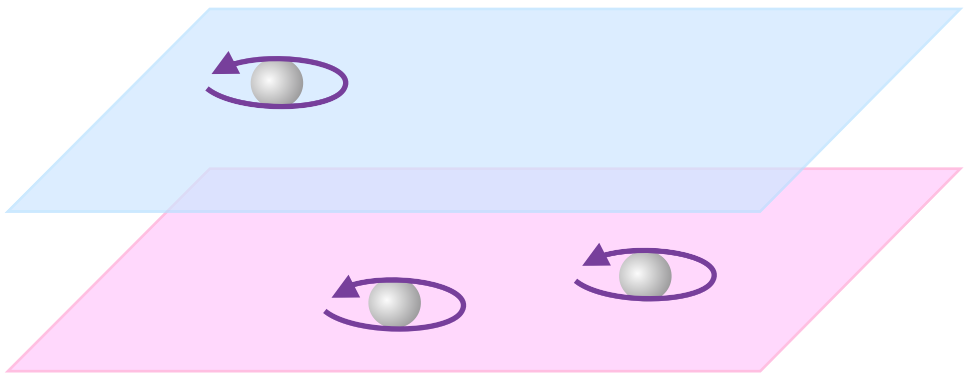
$$\Psi^{2\mathrm{qh}}(w_1, w_2) = \prod_{i<j=1}^{N} (z_i - z_j) \times \mathrm{Pf}\left(\frac{(z_i - w_1)(z_j - w_2) + i \leftrightarrow j}{z_i - z_j} \right)$$

➜ only one member of Cooper-pair needs to see the flux - hole fractionalizes!

Quasihole Excitations: Coupled MR

Simple Parent Hamiltonian: Can still solve for all quasiholes states exactly!

A) $\mathcal{H}_{3\text{-}2}$ no tunnelling



➜ independent quasiholes positions $\{w_k^\uparrow\}, \{w_l^\downarrow\}$ in layer 1 and layer 2

$$\Psi^{\mathrm{qh}}_{3\text{-}2}(\{z_i^\uparrow\},\{z_j^\downarrow\};\{w_k^\uparrow\},\{w_l^\downarrow\}) = \Psi^{\mathrm{qh},\,\nu=1}_{m_1^\uparrow,\dots m_{N_\uparrow}^\uparrow}(z_1^\uparrow,\dots z_{N_\uparrow}^\uparrow; w_1^\uparrow,\dots w_{2n_\uparrow}^\uparrow)$$

$$\times\, \Psi^{\mathrm{qh},\,\nu=1}_{m_1^\downarrow,\dots m_{N_\downarrow}^\downarrow}(z_1^\downarrow,\dots z_{N_\downarrow}^\downarrow; w_1^\downarrow,\dots w_{2n_\downarrow}^\downarrow)$$

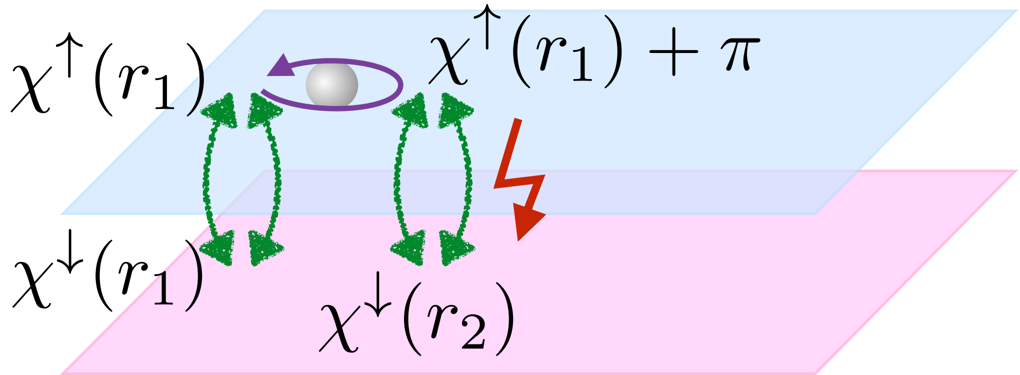
$$\times \prod_{i,j}(z_i^\uparrow - z_j^\downarrow)$$

Quasihole Excitations: Coupled MR

Simple Parent Hamiltonian: Can still solve for all quasiholes states exactly!

B) $\mathcal{H}^{\mathrm{JC}}_{3\text{-}2}(t)$, $\quad t \neq 0$ ➜ enforce constant phase relation of SC OP

or $\mathcal{H}^{\uparrow\downarrow}_{220}$



Quasihole Excitations: Coupled MR

Simple Parent Hamiltonian: Can still solve for all quasiholes states exactly!

B) $\mathcal{H}^{\mathrm{JC}}_{3\text{-}2}(t), \quad t \neq 0$ ➜ enforce constant phase relation of SC OP

or $\mathcal{H}^{\uparrow\downarrow}_{220}$

➜ quasiholes energetically confined, identical # and positions in layer 1 and 2

$$w_k^{\uparrow} = w_k^{\downarrow} \equiv w_k, \quad k = 1, \ldots, 2n$$

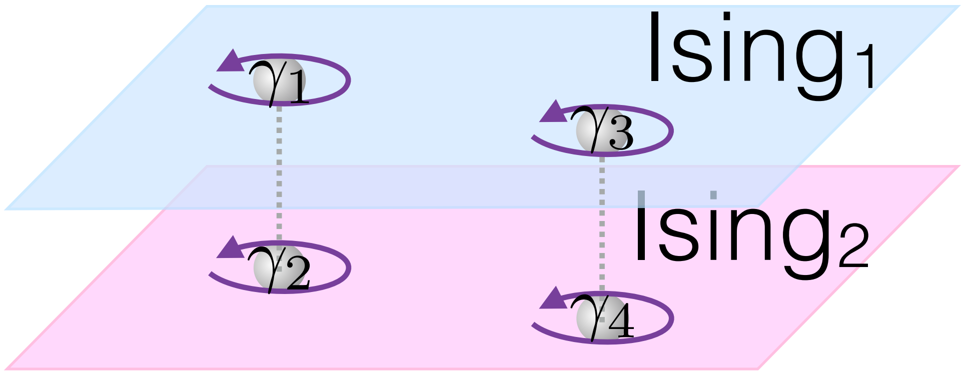
$$\Psi^{\mathrm{qh}}_{\mathrm{JC}}(\{z_i^{\uparrow}\}, \{z_j^{\downarrow}\}; \{w_k\}) = \Psi^{\mathrm{qh},\,\nu=1}_{m_1^{\uparrow},\ldots,m_{N_\uparrow}^{\uparrow}}(z_1^{\uparrow}, \ldots, z_{N_\uparrow}^{\uparrow}; w_1, \ldots, w_{2n}) \times \Psi^{\mathrm{qh},\,\nu=1}_{m_1^{\downarrow},\ldots,m_{N_\downarrow}^{\downarrow}}(z_1^{\downarrow}, \ldots, z_{N_\downarrow}^{\downarrow}; w_1, \ldots, w_{2n}) \times \prod_{i,j}(z_i^{\uparrow} - z_j^{\downarrow})$$

Topological order from state counting

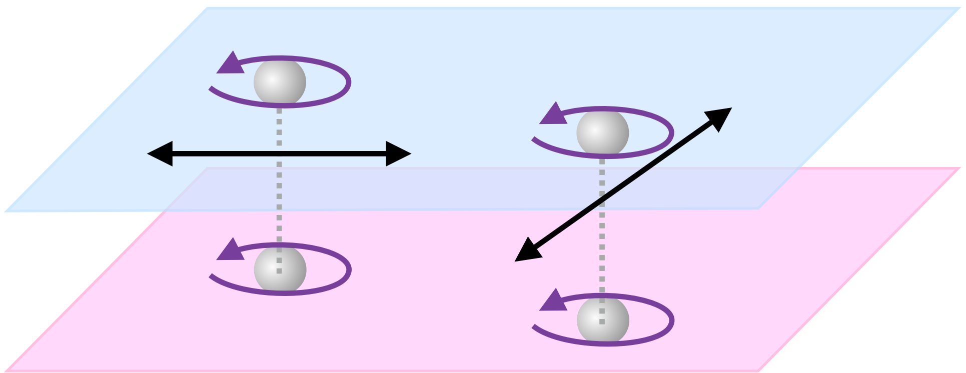
Degeneracy of quasihole states informs about topological order

Two contributions to degeneracy:

1) Topological degeneracy for non-Abelian character of quasiparticles (in each layer)



2) Positional degeneracy from different placement of the quasiparticles in the plane



Counting: Moore-Read

1) write trial state with *n* flux added $\quad N_\phi = N - 2 + n$

(Symmetric) polynomial in QH coord's $\{w\}$

antisymmetrizer

$$\Psi^{\mathrm{qh},\nu=1}_{\mathrm{MR},m_1,\ldots m_p}(z_1,\ldots z_N;w_1,\ldots w_{2n}) \propto \prod_{i<j}(z_i - z_j)\mathcal{A}_z \prod_{k=1}^{p} z_k^{m_k} \prod_{l=1}^{\frac{N-p}{2}} \frac{\Phi\big(z_{p+2l-1}, z_{p+2l}; w_1, \ldots w_{2n}\big)}{z_{p+2l-1} - z_{p+2l}}$$

Read & Rezayi 1996

broken pairs in single orb's

Cooper pairs in condensate

2) Count polynomials of this type (expanding in elementary symmetric pol's)

- broken pairs: $0 \le m_1 < m_2 < \ldots < m_p \le n-1$ ➜ $d_{\mathrm{topo}}(n,p) = \binom{n}{p}$
- quasihole positions: Φ has maximum degree $r = (N-p)/2$ in w

➜ $d_{\mathrm{orb}}(N,n,p) = \binom{r+2n}{2n}$

Counting: Coupled Moore-Read state

1) write trial state with *n* flux added $N_\phi = N - 2 + n$

$$\Psi^{\text{qh}}_{\text{JC}}(\{z_i^\uparrow\},\{z_j^\downarrow\};\{w_k\}) = \Psi^{\text{qh},\,\nu=1}_{m_1^\uparrow,\dots,m_{N_\uparrow}^\uparrow}(z_1^\uparrow,\dots,z_{N_\uparrow}^\uparrow;w_1,\dots,w_{2n})$$

$$\times\,\Psi^{\text{qh},\,\nu=1}_{m_1^\downarrow,\dots,m_{N_\downarrow}^\downarrow}(z_1^\downarrow,\dots,z_{N_\downarrow}^\downarrow;w_1,\dots,w_{2n})$$

$$\times\prod_{i,j}(z_i^\uparrow - z_j^\downarrow)$$

2) Count polynomials of this type (expanding in elementary symmetric pol's)

- broken pairs $p_\uparrow, p_\downarrow$ for $\uparrow$and$\downarrow$ layers: ➜ $d_{\text{topo}}(n,p_\uparrow,p_\downarrow) = \binom{n}{p_\uparrow}\times\binom{n}{p_\downarrow}$
- quasihole positions: $\Phi_\uparrow\times\Phi_\downarrow$ has maximum degree $r = (N - p_\uparrow - p_\downarrow)/2$

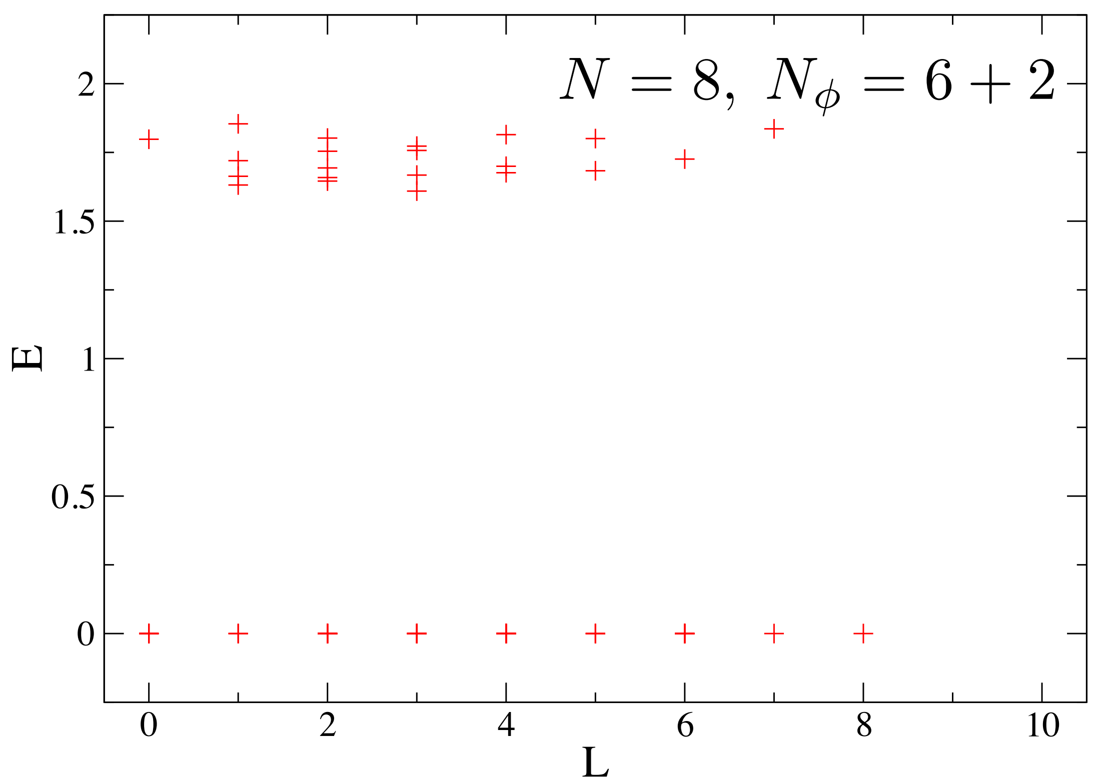
$$➜\ d_{\text{orb}}(N,n,p_\uparrow,p_\downarrow) = \binom{r+2n}{2n}$$

Finite size results (ED)

Check state counts match ED of model Hamiltonian $\mathcal{H}_{220}^{\uparrow\downarrow}$

Typical spectrum:

- large gap compared to characteristic separation of excited states
- zero-energy manifold of quasihole states, with N_s *even* or *odd*, yielding two separate sectors



- angular momentum structure for N=8 bosons, n=2 add'l flux quanta:

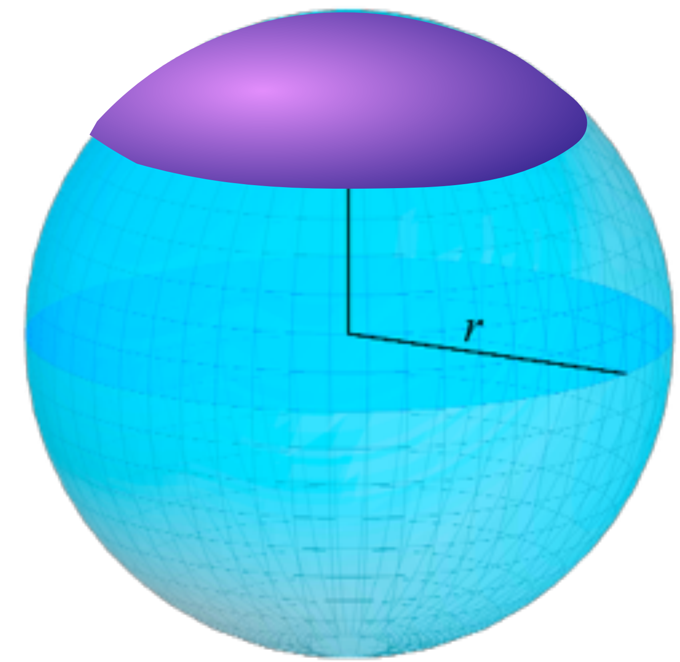
$$\mathcal{L}^{\mathrm{JC}}_{\mathrm{even}}(N = 8, n = 2) = 0^4 \oplus 2^5 \oplus 3^2 \oplus 4^5 \oplus 5^1 \oplus 6^3 \oplus 8^1$$

$$\mathcal{L}^{\mathrm{JC}}_{\mathrm{odd}}(N = 8, n = 2) = 0^1 \oplus 1^2 \oplus 2^3 \oplus 3^4 \oplus 4^3 \oplus 5^2 \oplus 6^2 \oplus 7^1$$

➜ all 39 states found in the correct angular momentum sectors

Edge spectrum of the coupled Pfaffian

Limit of huge correlation hole yields edge spectrum



- droplet/sphere $N_\phi = N - 2 + n$
- take $N \to \infty, \quad n \to \infty, \quad n \ll N$
- measure momentum 'relative to edge'

$$\Delta m = L_z^{\max} - L_z$$

Character of edge for coupled Moore-Read state (checked against data):

$$\chi^{\mathrm{JC}} = \left[(\chi_+^{\mathrm{MW}})^2 + (\chi_-^{\mathrm{MW}})^2\right] \chi^{\mathrm{Bose}} = \chi_0^{U(1)_4} \times \chi^{\mathrm{Bose}} = \chi_0 \times \chi^{\mathrm{Bose}} \times \chi^{\mathrm{Bose}}$$

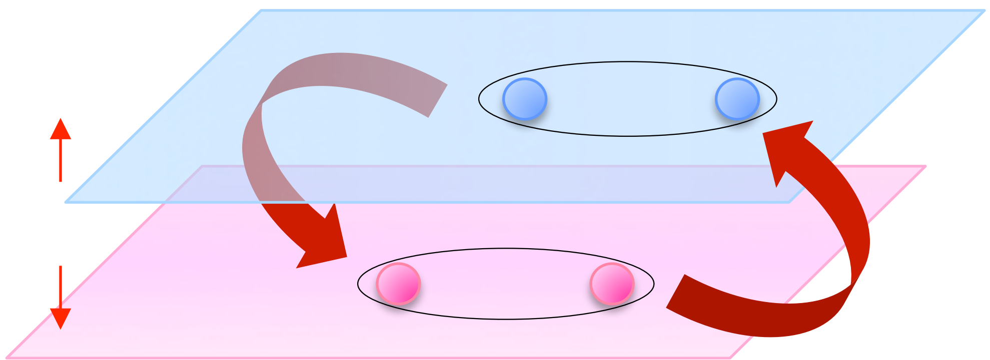
➜ edge differs from Halperin 220-state!

$$\chi_{220} = \chi^{\mathrm{Bose}} \times \chi^{\mathrm{Bose}}$$

Edge States of the Coupled Pfaffian

Charged sector $U(1)_4$ x $U(1)_4$ Neutral sector

$$H(r,t)=\sum_{i,j=1}^{N}\delta(z_i-z_j)\sum_{s\neq s'=\uparrow,\downarrow}\Big(|ss\rangle\langle ss|+r(|ss'\rangle\langle ss'|+|ss'\rangle\langle s's|)+\boxed{t|ss\rangle\langle s's'|\Big)_{ij}}$$



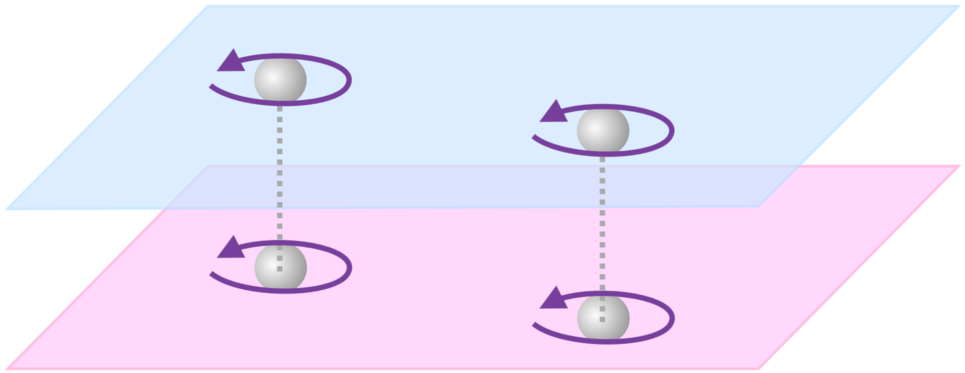
Total Charge = fixed → No V_4 operators can be added → U(1)

No restriction on the neutral sector → V_4 operators can be added → $U(1)_4$

→ Edge theory: U(1) x $U(1)_4$

Summary

- Moore-Read states can be coupled by Josephson tunnelling, when suitable interlayer correlations are added to the system
- There is an exact parent Hamiltonian for the Josephson coupled Moore-Read states, involving only two-body interactions
- The Josephson term gaps out the Goldstone mode and pins quasiholes in both layers together
- The resulting phase is in the universality class of the (Abelian) Halperin 220 state, formally a $U(1)_2 \times U(1)_2$ CFT in the bulk



- One cannot conclude about the CFT seen at the edge directly from knowing the bulk CFT. Particle number and charge conservation restrict which part of the bulk CFT is seen at the edge.

GM, L. Hormozi, J. Slingerland, S.H. Simon, arxiv:1409:6339; see also PRL ***108*** *(2012).*

CFT Description – Review of MR state

Reminder: Single layer Moore-Read pfaffian state …

Moore and Read, Nucl. Phys. B360 362 (1991)

$$\Psi_{gs} = Pf\left(\frac{1}{z_i - z_j}\right)\prod_{i<j=1}^{N}(z_i - z_j)$$

Ising CFT: $\{1,\ \sigma\ ,\psi\ \}$

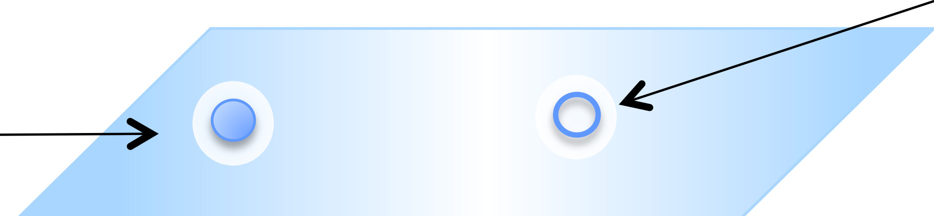
Neutral part: topological properties

$\mathrm{U}(1)_4$: $\{1,\ e^{i\phi_c},\ e^{i\phi_c/2}, e^{-i\phi_c/2}\ \}$

Charged part

Electron operators

$\psi_e(z\ ) = \psi\ \ e^{i\phi_c}$



$\psi_{qh}(\eta) = \sigma\ \ e^{\pm i\phi_c/2}$

Smallest-charge quasihole operator $q_{qh} = e/2$

$$\Psi_{gs}(\{z_i\}) = \left\langle \prod_{i=1}^{N}\psi_e(z_i)\right\rangle$$

$$\Psi_{qh}(\{z_i\},\{\eta_j\}) = \left\langle \prod_{i=1}^{N}\psi_e(z_i)\prod_{j=1}^{n}\psi_{qh}(\eta_j)\right\rangle$$

Not all 12 sectors of Ising x $\mathrm{U}(1)_4$ lead to valid wavefunctions

Final topological theory: $\mathrm{SU}(2)_2$

CFT Description – Coupled MR state

$$\Psi_{gs} = Pf\left(\frac{1}{z_i^{\uparrow} - z_j^{\uparrow}}\right)\prod_{i<j=1}^{N^{\uparrow}}(z_i^{\uparrow} - z_j^{\uparrow})\, Pf\left(\frac{1}{z_i^{\downarrow} - z_j^{\downarrow}}\right)\prod_{i<j=1}^{N^{\downarrow}}(z_i^{\downarrow} - z_j^{\downarrow}) \prod_{i=1}^{N^{\uparrow}}\prod_{j=1}^{N^{\downarrow}}(z_i^{\uparrow} - z_j^{\downarrow})$$

Each layer is like a Moore-Read pfaffian state …

2 copies of *Ising* CFT: $\{1, \sigma^{\uparrow}, \psi^{\uparrow}\} \times \{1, \sigma^{\downarrow}, \psi^{\downarrow}\}$

Neutral part: topological properties

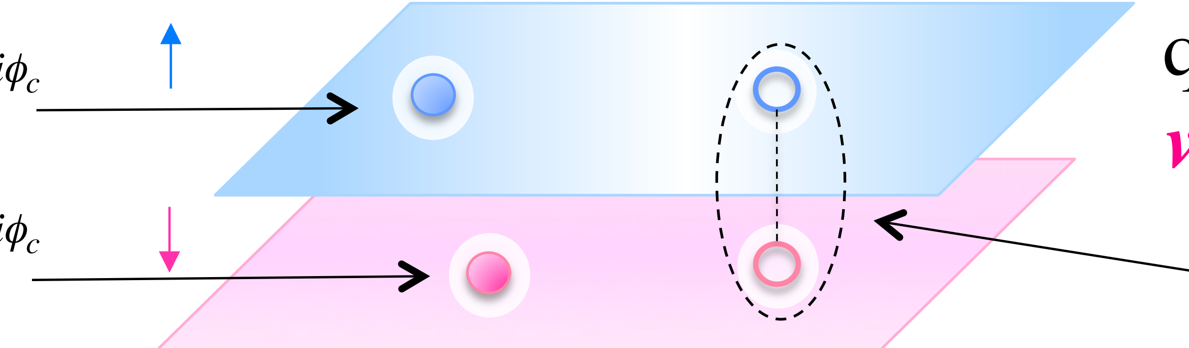
$\mathrm{U}(1)_4$: $\{1, e^{i\phi_c}, e^{i\phi_c/2}, e^{-i\phi_c/2}\}$

Charge

Electron operators

$\psi_e^{\uparrow}(z^{\uparrow}) = \psi_{\uparrow}\, e^{i\phi_c}$

$\psi_e^{\downarrow}(z^{\downarrow}) = \psi_{\downarrow}\, e^{i\phi_c}$



Smallest-charge $q_{qh} = e/2$ quasihole operators with ***valid*** wavefunctions:

$\psi_{qh}(\eta) = \underline{\sigma_{\uparrow}\sigma_{\downarrow}}\, e^{\pm i\phi_c/2}$

same position

Excitation with σ or σ separated are not energetically favorable.

→ Sectors with odd numbers of σ or σ operators are *confined*.

Topological Properties – neutral part

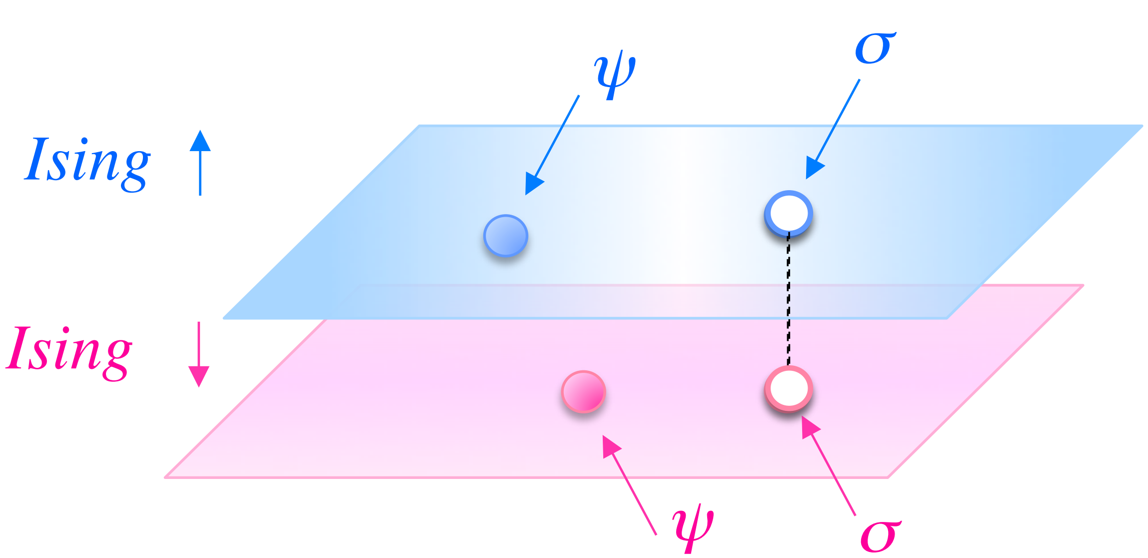
Let us ignore the charge part and just focus on the neutral part.

Topological sectors of *Ising* x *Ising*:

↓ \ ↑	1	σ	ψ
1	$1\;1$	$\cancel{1\;\sigma}$	$1\;\psi$
σ	$\cancel{\sigma\;1}$	$\sigma\;\sigma$	$\cancel{\sigma\;\psi}$
ψ	$\psi\;1$	$\cancel{\psi\;\sigma}$	$\psi\;\psi$

Visible in edge spectrum!

If we cross out the sectors that involve an odd number of σ or σ :



In this reduced subspace certain sectors are topologically identical.

Condensing $\psi\;\psi$

		$\mathrm{U}(1)_4$
$1\;1 \quad h=0$ $\psi\;\psi \quad h=1$	$\to 1$	1
$1\;\psi \quad h=1/2$ $\psi\;1 \quad h=1/2$	$\to \psi$	$e^{i\phi}$
$\sigma\;\sigma \quad h=1/8$	$\to a$	$e^{i\phi/2}$
	$\to a^*$	$e^{-i\phi/2}$

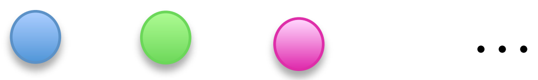
$\sigma\;\sigma$ sector has to split

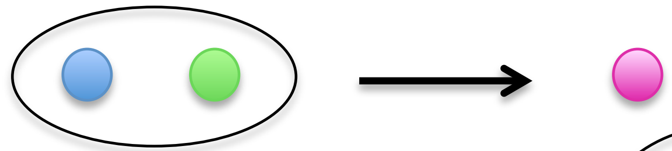
Bais and Slingerland, Phys. Rev. B 79, 045316 (2009)

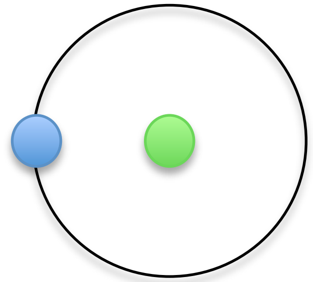
Topological Bose Condensation

- Start with an anyon theory

Primary fields of a CFT

1. Anyon labels: $\{1, a, b, c \ldots\}$ 

2. Fusion rules: $a \times b = c + ?$ 

3. Braiding rules: $e^{2\pi i(h_c - h_a - h_b)}$ 

Conformal dimensions of the primary fields of the CFT

- Identify a boson $\longrightarrow$ $h_b =$ integer

- Condense the boson $\longrightarrow$ $b = 1$

- Identify the remaining sectors such that they form a new consistent anyon theory

Condensation of $(\psi\ \psi)$

0	$(1\ 1)$	$\longrightarrow$	$\boxed{(1\ 1) = (\psi\ \psi)}$
$\boxed{1/16}$	$(1\ \sigma) \times (\psi\ \psi) = (\psi\ \sigma)$	$\longrightarrow$	$(1\ \sigma)$, $(\psi\ \sigma)$ are *confined*
1/2	$(1\ \psi) \times (\psi\ \psi) = (\psi\ 1)$	$\longrightarrow$	$\boxed{(1\ \psi) = (\psi\ 1)}$
$\boxed{1/16}$	$(\sigma\ 1) \times (\psi\ \psi) = (\sigma\ \psi)$	$\longrightarrow$	$(\sigma\ 1) = (\sigma\ \psi)$ are *confined*
1/8	$(\sigma\ \sigma) \times (\psi\ \psi) = (\sigma\ \sigma)$	$\longrightarrow$	$\boxed{(\sigma\ \sigma)_1}$, $\boxed{(\sigma\ \sigma)_2}$
$\boxed{9/16}$	$(\sigma\ \psi)$		
1/2	$(\psi\ 1)$		
$\boxed{9/16}$	$(\psi\ \sigma)$		
1	$(\psi\ \psi)$		

$\Downarrow$

$U(1)_4$

Ising fusion rules:

$\sigma \times \psi = \sigma$

$\psi \times \psi = 1$

$\sigma \times \sigma = 1 + \psi$

Splitting of $(\sigma\ \sigma)$

Before splitting

$$(\sigma\ \sigma) \times (\sigma\ \sigma) = (1\ 1) + (1\ \psi) + (\psi\ 1) + (\psi\ \psi)$$

$$(\sigma\ \sigma) \begin{cases} (\sigma\ \sigma)_1 \\ (\sigma\ \sigma)_2 \end{cases}$$

$$(\sigma\ \sigma)_1 \times (\sigma\ \sigma)_1 = (\sigma\ \sigma)_2 \times (\sigma\ \sigma)_2 = (1\ \psi) = (\psi\ 1)$$

$$(\sigma\ \sigma)_1 \times (\sigma\ \sigma)_2 = (\sigma\ \sigma)_2 \times (\sigma\ \sigma)_1 = (\psi\ \psi) = (1\ 1)$$

→ After splitting, the $(\sigma\ \sigma)$ sectors become Abelian!

Ising fusion rules:

$\sigma \times \psi = \sigma$

$\psi \times \psi = 1$

$\sigma \times \sigma = 1 + \psi$

Condensation of $(\psi\ \psi)$

0 $(1\ 1)$ $\longrightarrow$ $(1\ 1) = (\psi\ \psi)$

$1/16$ $(1\ \sigma) \times (\psi\ \psi) = (\psi\ \sigma)$ $\longrightarrow$ $(1\ \sigma)$, $(\psi\ \sigma)$ are *confined*

$1/2$ $(1\ \psi) \times (\psi\ \psi) = (\psi\ 1)$ $\longrightarrow$ $(1\ \psi) = (\psi\ 1)$

$1/16$ $(\sigma\ 1) \times (\psi\ \psi) = (\sigma\ \psi)$ $\longrightarrow$ $(\sigma\ 1) = (\sigma\ \psi)$ are *confined*

$1/8$ $(\sigma\ \sigma) \times (\psi\ \psi) = (\sigma\ \sigma)$ $\longrightarrow$ $(\sigma\ \sigma)_1$, $(\sigma\ \sigma)_2$

$9/16$ $(\sigma\ \psi)$

$1/2$ $(\psi\ 1)$

$9/16$ $(\psi\ \sigma)$

1 $(\psi\ \psi)$

$U(1)_4$

Ising fusion rules:

$$\sigma \times \psi = \sigma$$
$$\psi \times \psi = 1$$
$$\sigma \times \sigma = 1 + \psi$$

Topological Order of the Coupled Pfaffian

Full theory: $U(1)_4$ x $U(1)_4$

Electron operators:

$$\psi_e^{\uparrow}(z^{\uparrow}) = \sqrt{2}\cos\phi_n(z^{\uparrow})e^{i\phi_c(z^{\uparrow})}$$
$$\psi_e^{\downarrow}(z^{\downarrow}) = i\sqrt{2}\sin\phi_n(z^{\downarrow})e^{i\phi_c(z^{\downarrow})}$$

=

$$\psi_e^{\uparrow} = \psi_{\uparrow}\, e^{i\phi_c}$$
$$\psi_e^{\downarrow} = \psi_{\downarrow}\, e^{i\phi_c}$$

Only those sectors of $U(1)_4$ x $U(1)_4$ that are local with respect to these electron operators will lead to valid wavefunctions.

→ The rest are confined.

Adding Charge – further reduction

Full theory: $U(1)_4 \times U(1)_4$

$U(1)_4$ \ $U(1)_4$	1	a	a^*	ψ
1	$1\ 1$	$\cancel{1\ a}$	$\cancel{1\ a^*}$	$1\ \psi$
a	$\cancel{a\ 1}$	$a\ a$	$a\ a^*$	$\cancel{a\ \psi}$
a^*	$\cancel{a^*\ 1}$	a^*a	a^*a^*	$\cancel{a^*\ \psi}$
ψ	$\psi\ 1$	$\cancel{\psi\ a}$	$\cancel{\psi\ a^*}$	$\psi\ \psi$

Excluding the confined sectors, we end up with these 8: → $U(1)_2 \times U(1)_2$

In this reduced subspace, only 4 sectors are topologically distinct.

$1\ 1$ $\psi\ \psi$	$h = 0$ $h = 1$	→ 1	$1 \quad 1$
$1\ \psi$ $\psi\ 1$	$h = 1/2$	→ ψ	$e^{\frac{i\phi_1}{\sqrt{2}}} \quad e^{\frac{i\phi_2}{\sqrt{2}}}$
$a\ \ a$ $a^*\ a^*$	$h = 1/4$	→ e	$e^{\frac{i\phi_1}{\sqrt{2}}} \quad 1$
$a^*\ a$ $a\ \ a^*$	$h = 1/4$	→ m	$1 \quad e^{\frac{i\phi_2}{\sqrt{2}}}$

Adding Charge – further reduction

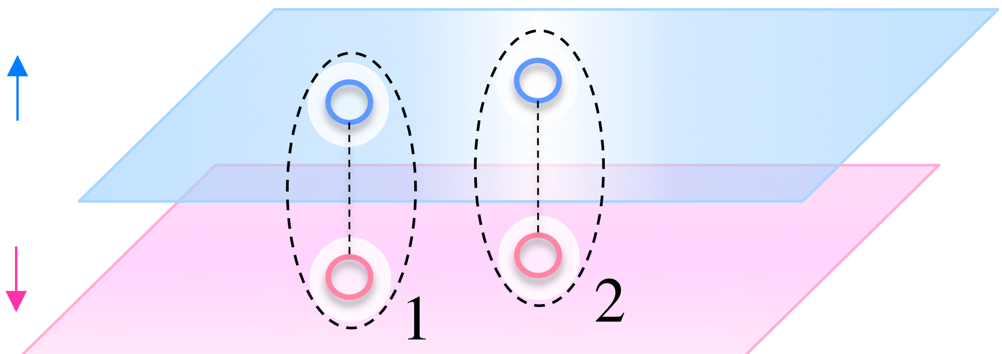
Full theory: $U(1)_4 \times U(1)_4$

$U(1)_4 \backslash U(1)_4$	1	a	a^*	ψ
1	$1\;1$	$\cancel{1\;a}$	$\cancel{1\;a^*}$	$1\;\psi$
a	$\cancel{a\;1}$	$a\;a$	$a\;a^*$	$\cancel{a\;\psi}$
a^*	$\cancel{a^*1}$	a^*a	a^*a^*	$\cancel{a^*\psi}$
ψ	$\psi\;1$	$\cancel{\psi\;a}$	$\cancel{\psi\;a^*}$	$\psi\;\psi$

Excluding the confined sectors, we end up with these 8: $\rightarrow U(1)_2 \times U(1)_2$

$\psi = e \times m$

$$\psi^1_{qh}(\eta) = \left(\sigma_\uparrow\sigma_\downarrow\right)_1 e^{i\phi_c/2}$$



$$\psi^2_{qh}(\eta') = \left(\sigma_\uparrow\sigma_\downarrow\right)_2 e^{i\phi_c/2}$$

			$U(1)_2 \times U(1)_2$
$1\;1$, $\psi\;\psi$	$h=0$, $h=1$	$\rightarrow 1$	$1 \quad 1$
$1\;\psi$, $\psi\;1$	$h=1/2$	$\rightarrow \psi$	$e^{\frac{i\phi_1}{\sqrt{2}}} \quad e^{\frac{i\phi_2}{\sqrt{2}}}$
$a\;a$, $a^*\;a^*$	$h=1/4$	$\rightarrow e$	$e^{\frac{i\phi_1}{\sqrt{2}}} \quad 1$
$a^*\;a$, $a\;a^*$	$h=1/4$	$\rightarrow m$	$1 \quad e^{\frac{i\phi_2}{\sqrt{2}}}$

Adding Charge – further reduction

Full theory: $U(1)_4 \times U(1)_4$

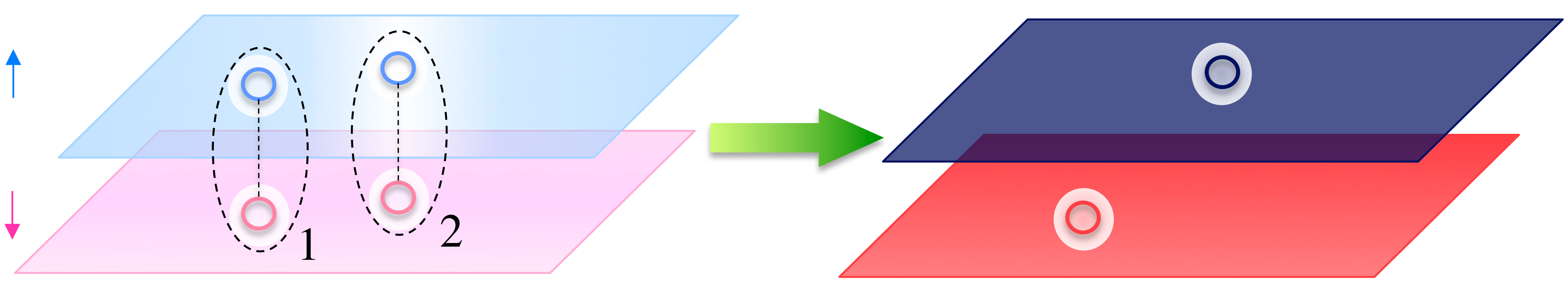
$U(1)_4$ \ $U(1)_4$	1	a	a^*	ψ
1	$1\ 1$	$\cancel{1\ a}$	$\cancel{1\ a^*}$	$1\ \psi$
a	$\cancel{a\ 1}$	$a\ a$	$a\ a^*$	$\cancel{a\ \psi}$
a^*	$\cancel{a^*1}$	a^*a	a^*a^*	$\cancel{a^*\ \psi}$
ψ	$\psi\ 1$	$\cancel{\psi\ a}$	$\cancel{\psi\ a^*}$	$\psi\ \psi$

$U(1)_2 \times U(1)_2$

1	1
$e^{\frac{i\phi_1}{\sqrt{2}}}$	$e^{\frac{i\phi_2}{\sqrt{2}}}$
$e^{\frac{i\phi_1}{\sqrt{2}}}$	1
1	$e^{\frac{i\phi_2}{\sqrt{2}}}$

$\psi = e \times m$

$$\psi^1_{qh}(\eta) = \left(\sigma_\uparrow \sigma_\downarrow\right)_1 e^{i\phi_c/2}$$



$$\psi^2_{qh}(\eta') = \left(\sigma_\uparrow \sigma_\downarrow\right)_2 e^{i\phi_c/2}$$

(220) state in a different basis

Edge States of the Coupled Pfaffian

U(1)

k	count	
.	.	.
.	.	.
.	.	.
5	7	$a_{-5}/\ a_{-4}a_{-1}/\ a_{-3}a_{-2}\ /\ a_{-3}a_{-1}^{2}/\ a_{-2}^{2}a_{-1}/\ a_{-2}a_{-1}^{3}/a_{-1}^{5}\ \lvert 0\rangle$
4	5	$a_{-4}/\ a_{-3}a_{-1}\ /\ a_{-2}^{2}/\ a_{-2}a_{-1}^{2}\ /\ a_{-1}^{4}\ \lvert 0\rangle$
3	3	$a_{-3}/\ a_{-2}\ a_{-1}/\ a_{-1}^{3}\ \lvert 0\rangle$
2	2	$a_{-2}/\ a_{-1}^{2}\ \lvert 0\rangle$
1	1	$a_{-1}\ \lvert 0\rangle$
0	1	$\lvert 0\rangle$

a_{-n}^{m} places m bosons in an orbital with momentum n

Edge States of the Coupled Pfaffian

$\mathrm{U}(1)_4$

Additional bosonic descendent fields should be included in the counting

$V_{4n} = e^{i2n\phi}$ $\quad k = 2n^2 \quad n = 1, 2, \ldots$

k	count	
.	.	.
.	.	.
.	.	.
5	7	$a_{-5}/\ a_{-4}a_{-1}/\ a_{-3}a_{-2}\ /\ a_{-3}a_{-1}^{\ 2}/\ a_{-2}^{\ 2}a_{-1}/\ a_{-2}a_{-1}^{\ 3}/a_{-1}^{\ 5}\ \lvert 0\rangle$
4	5	$a_{-4}/\ a_{-3}a_{-1}\ /\ a_{-2}^{\ 2}/\ a_{-2}a_{-1}^{\ 2}\ /\ a_{-1}^{\ 4}\ \lvert 0\rangle$
3	3	$a_{-3}/\ a_{-2}\ a_{-1}/\ a_{-1}^{\ 3}\ \ \lvert 0\rangle$
2	2	$a_{-2}/\ a_{-1}^{\ 2}\ \lvert 0\rangle$
1	1	$a_{-1}\ \lvert 0\rangle$
0	1	$\lvert 0\rangle$

$a_{-n}^{\ m}$ places m bosons in an orbital with momentum n

Edge States of the Coupled Pfaffian

$U(1)_4$

Additional bosonic descendent fields should be included in the counting

$V_{4n} = e^{i2n\phi}$ $\quad k = 2n^2 \quad n = 1, 2, \ldots$

k	count	
.	.	.
.	.	.
.	.	.
5	7 + 3	$a_{-5}/\, a_{-4}a_{-1}/\, a_{-3}a_{-2}\,/\, a_{-3}a_{-1}^{\;2}/\, a_{-2}^{\;2}a_{-1}/\, a_{-2}a_{-1}^{\;3}/a_{-1}^{\;5}/\, V_4\, a_{-3}/\, V_4\, a_{-2}\, a_{-1}^{\;2}/\, V_4\, a_{-1}^{\;2}\, \|0\rangle$
4	5 + 2	$a_{-4}/\, a_{-3}a_{-1}\,/\, a_{-2}^{\;2}/\, a_{-2}a_{-1}^{\;2}\,/\, a_{-1}^{\;4}\,/\, V_4\, a_{-2}/\, V_4\, a_{-1}^{\;2}\;\; \|0\rangle$
3	3 + 1	$a_{-3}/\, a_{-2}\, a_{-1}/\, a_{-1}^{\;3}\,/\, V_4\, a_{-1}\;\|0\rangle$
2	2 + 1	$a_{-2}/\, a_{-1}^{\;2}/\, V_4\;\|0\rangle$
1	1	$a_{-1}\;\|0\rangle$
0	1	$\|0\rangle$

The addition of these operators corresponds to adding new particles to the system.

In a system with fixed total particle number *the level* does not appear!