CHERN INSULATORS

Emil J. Bergholtz
FU Berlin

Lectures on
"Numerical and analytical methods for strongly correlated systems"
Benasque, Spain, September 2 & 5, 2014

Two lectures: rough plan

- The quantum Hall effect
    - Crash course on integer and fractional effects
    - Why look for alternative realizations?
- Integer Chern insulators ~ lattice quantum Hall systems at zero field
    - Example lattice models
    - General properties, comparison with continuum Landau levels
    - Experiments!
- Fractional Chern insulators
    - Brief comments on challenge and methods
    - Relation to FQH states (adiabatic continuity, entanglement spectra, edge states, etc.)
    - Which FQH analogues to expect, when and why?
    - Competing instabilities
- Higher Chern numbers
    - Various constructions and why only some host FCIs
    - Topology + frustration: novel FCIs in surface bands of Weyl semi-metals
    - Experiments?

First: My collaborators

- On Chern insulators

  Jan Budich, Innsbruck
  Jens Eisert, Berlin

- On fractional Chern insulators

  Jörg Behrmann, Berlin
  Eliot Kapit, Oxford
  Zhao Liu, Princeton
  Roderich Moessner, Dresden
  Dmitry Kovrizhin, Cambridge
  Andreas Läuchli, Innsbruck
  Martin Rößner, Berlin
  Samuel Sanchez, Berlin/Copenhagen
  Maximilian Trescher, Berlin
  Masafumi Udagawa, Tokyo

- and recently on related projects

  Piet Brouwer (Berlin), Sebastian Diehl (Innsbruck), Heng Fan (Beijing), Masaaki Nakamura (Tokyo), Björn Sbierski (Berlin) Peter Zoller (Innsbruck)

Some preliminaries on the quantum Hall effect

Useful references:

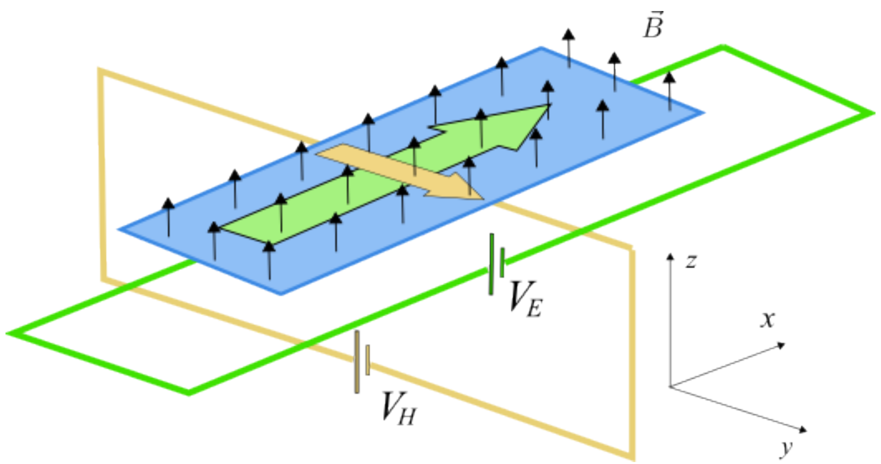
Steven M. Girvin,
The Quantum Hall Effect: Novel Excitations and Broken Symmetries
arXiv:cond-mat/9907002

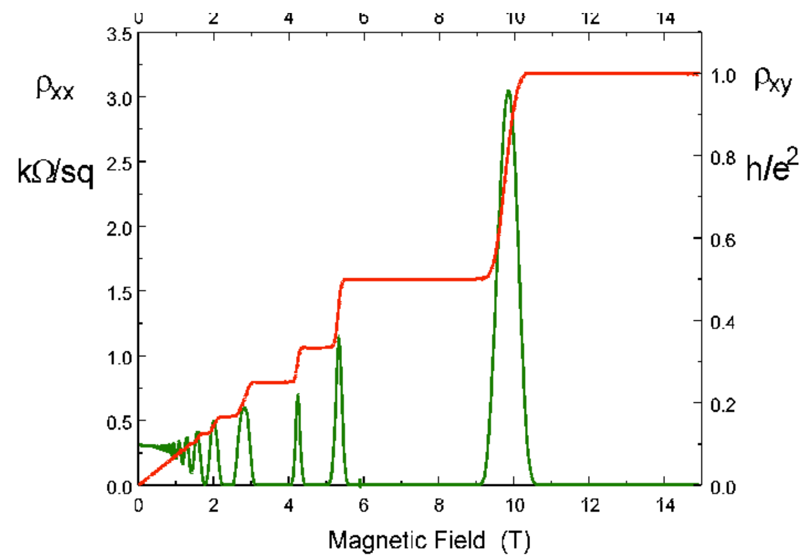
Chetan Nayak, Steven H. Simon, Ady Stern, Michael Freedman & Sankar Das Sarma,
Non-Abelian Anyons and Topological Quantum Computation
Rev. Mod. Phys. 80, 1083 (2008) [arXiv:0707.1889]

The quantum Hall effect

- Cold 2D electrons in a strong magnetic field.

(von Klitzling et al ’80)

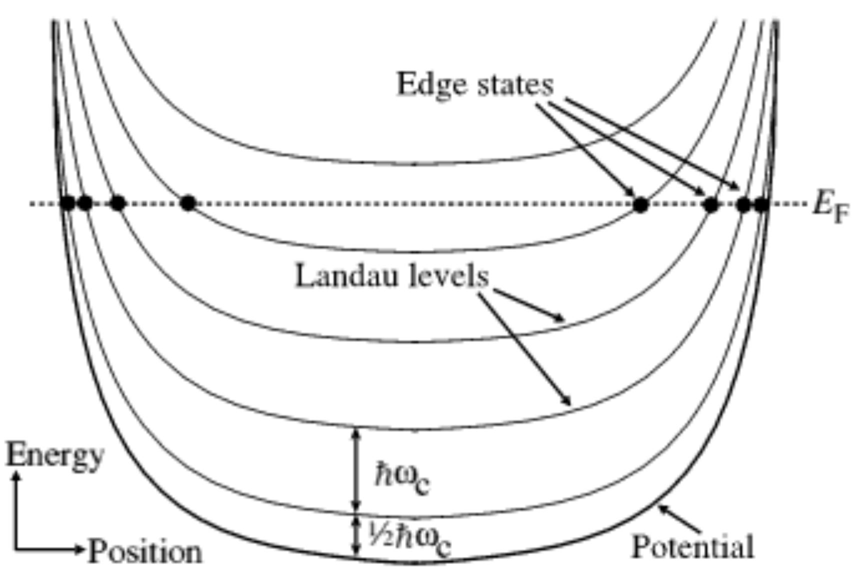




Quantization, IQHE:

$R_K = h/e^2 = 25812.807557(18)\Omega$

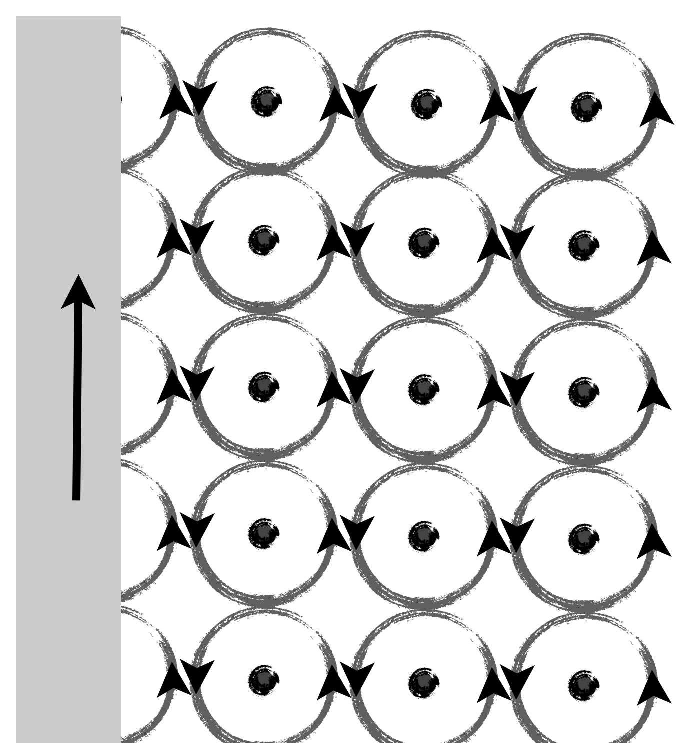
- Single-particle explanation (Laughlin ‘81, Halperin ’82,...)



Landau levels with bulk gap and protected edge states

One state per “flux quantum”

$\varphi_0 = hc/e$



Chern number

(Thouless et al ’82)

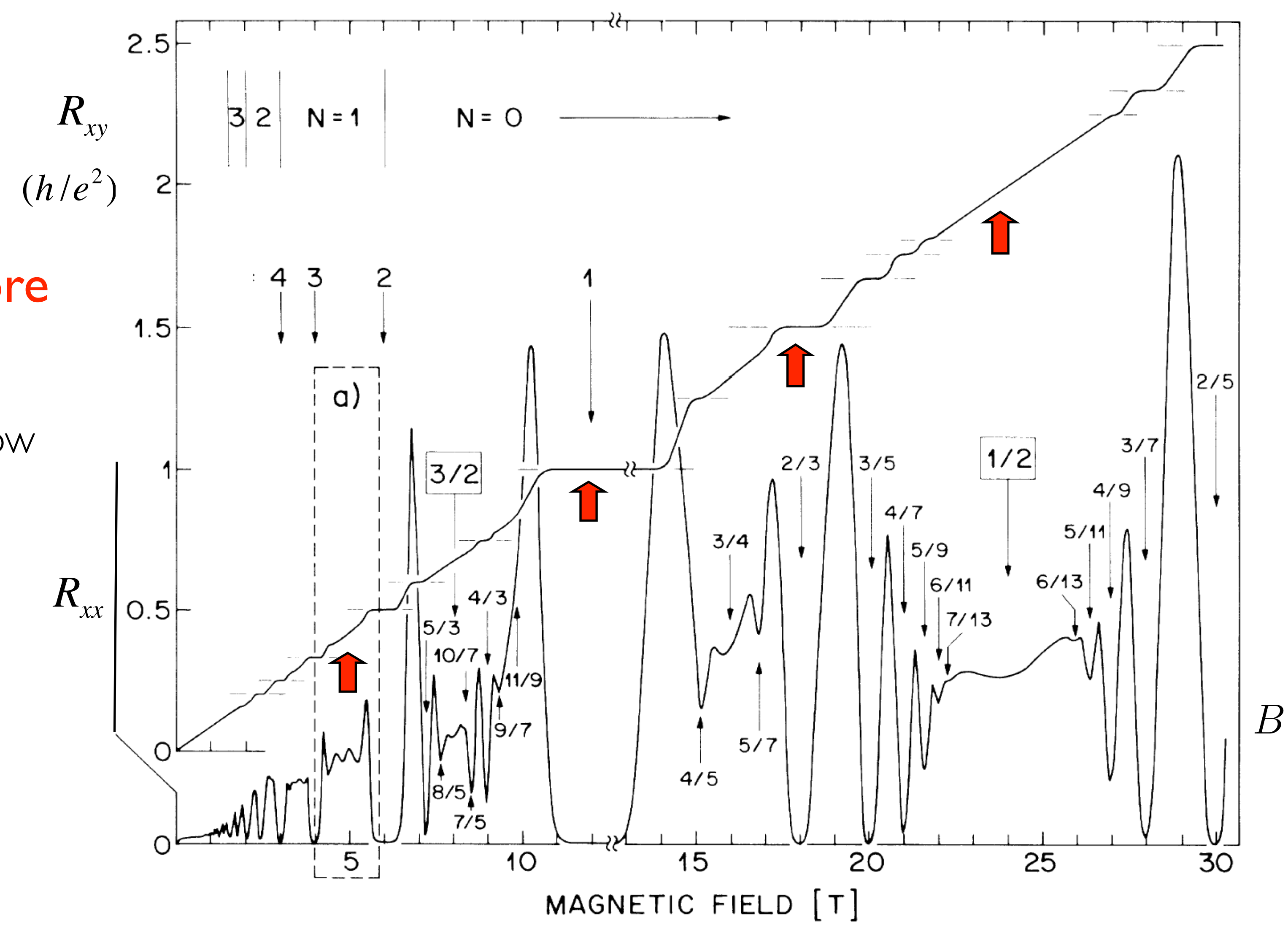
Integer & fractional quantum Hall effects

- Lower temperature, cleaner samples, stronger B-fields

....and much more

Stripes (high LL's)
Wigner Crystals (low filling).....

(Willett et al1987)



Quantum Hall states

$$R_{xy} = \frac{h}{\nu e^2} \qquad R_{xx} = 0$$

h Planck's constant
e electron charge

$\nu = 1, 2, 3\ldots$ Integer QHE

von Klitzing, Dorda, Pepper 1980

$\nu = \frac{1}{3}, \frac{2}{5}\ldots$ Fractional QHE

Tsui, Störmer, Gossard 1982

$\nu = \frac{5}{2}, \frac{12}{5}\ldots$ Non-abelian Fractional QHE?

Willet et al 1987

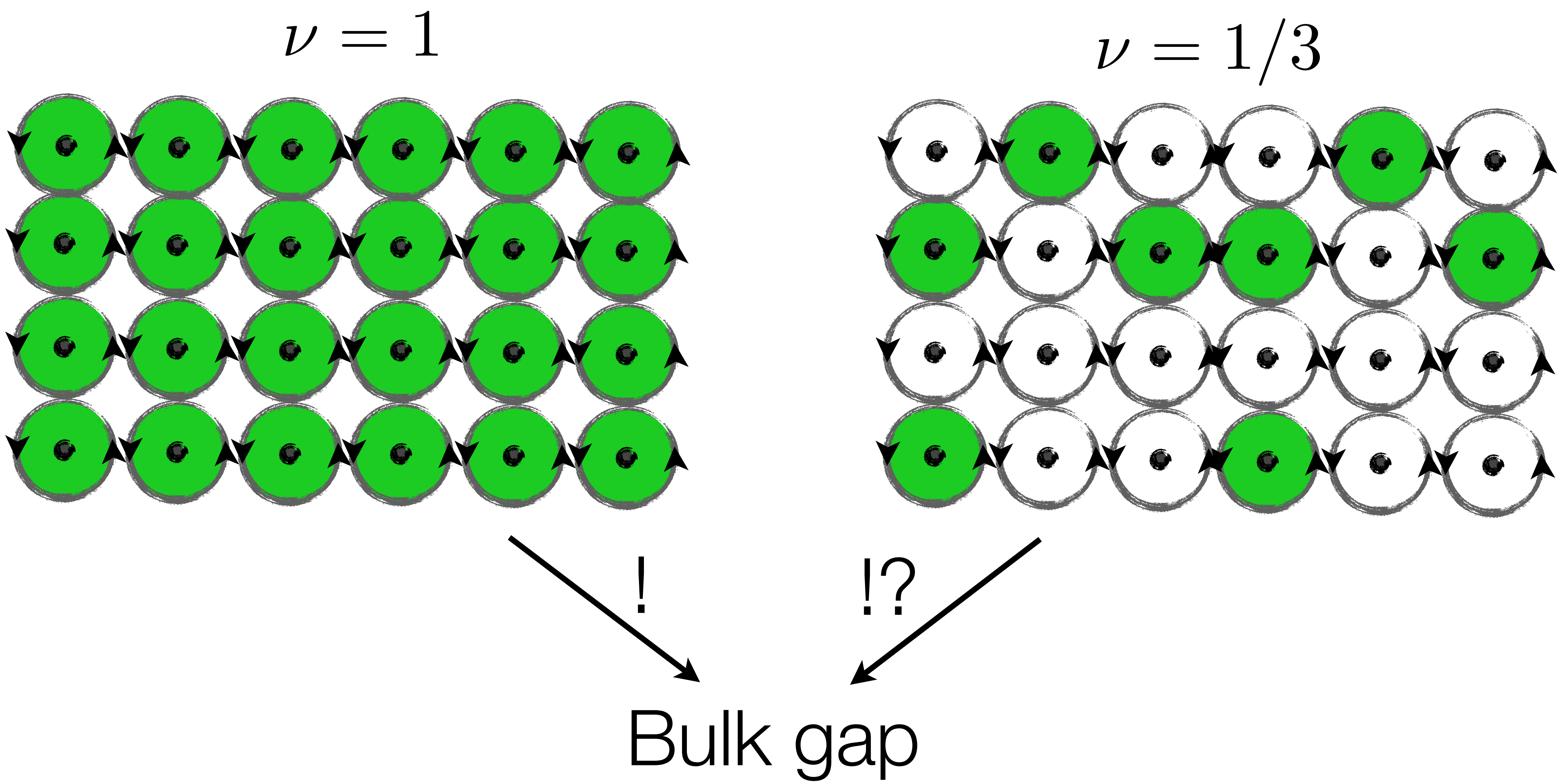
(incompressible quantum liquids)

Metallic states

$$\nu = \frac{1}{2}, \frac{1}{4}, \ldots$$

- Similar experimental signatures, but the integer explanation does not apply!

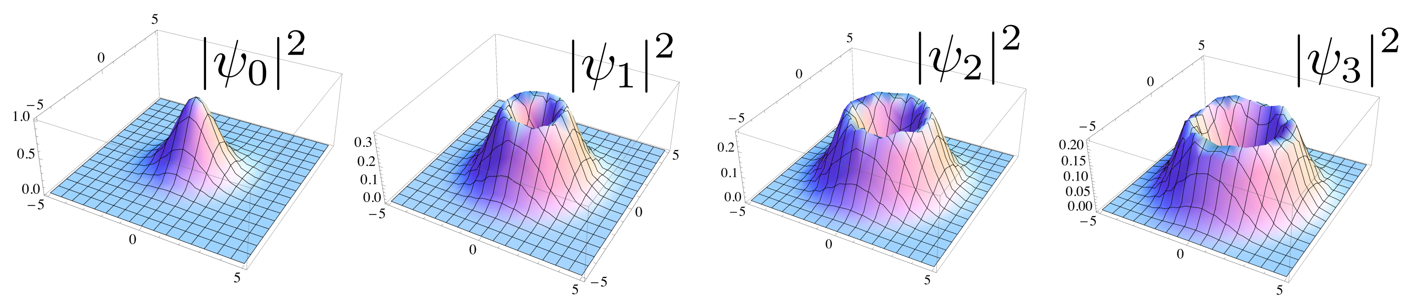
Fractional quantum Hall effect



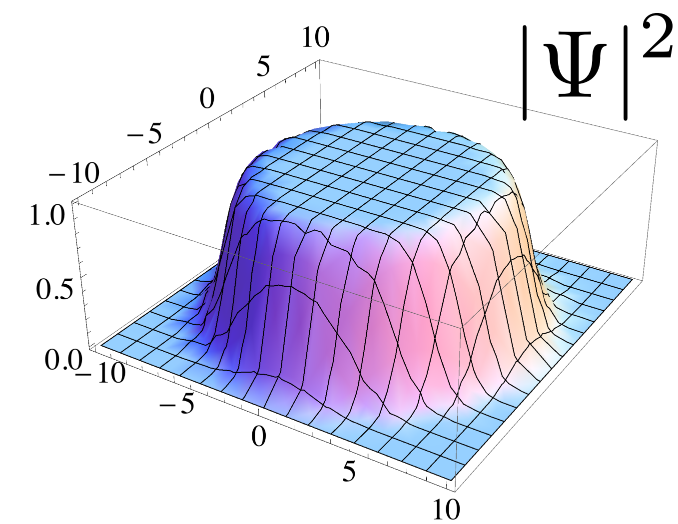
- ***Flat band***, partial filling ⟶ Interactions within the band determine everything!
- Wigner crystals only at very low filling, below 1/7 ("Topological obstruction")

Fractional quantum Hall effect

- Single-particle states: $\psi_m \propto z^m e^{-|z|^2/4\ell_B^2}$ $\mathbf{A} = \frac{B}{2}(-y\hat{x} + x\hat{y})$



- Filled Landau level $\Psi_{\nu=1} = \prod_{i<j}(z_i - z_j)e^{-\sum_i |z_i|^2/4\ell_B^2}$

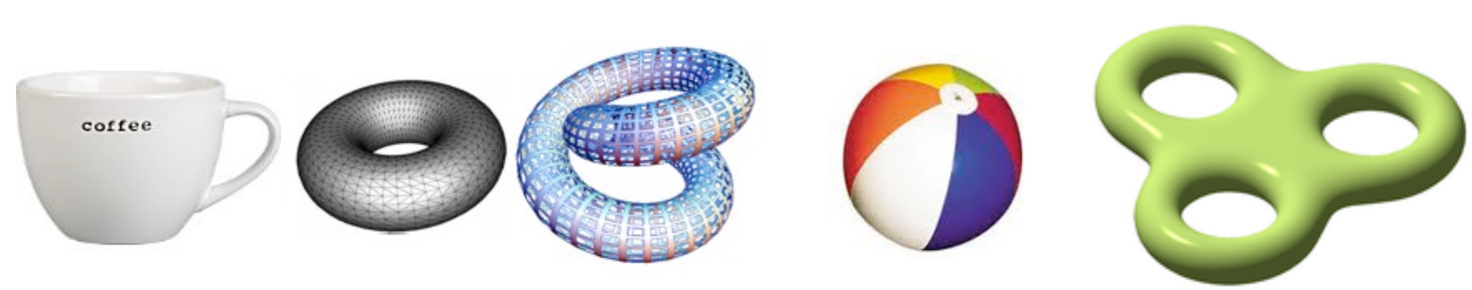


- Laughlin! $\Psi_{\nu=1/3} = \prod_{i<j}(z_i - z_j)^3 e^{-\sum_i |z_i|^2/4\ell_B^2}$

Filling fraction: $\nu = \frac{N}{3(N-1)+1} \to \frac{1}{3}$

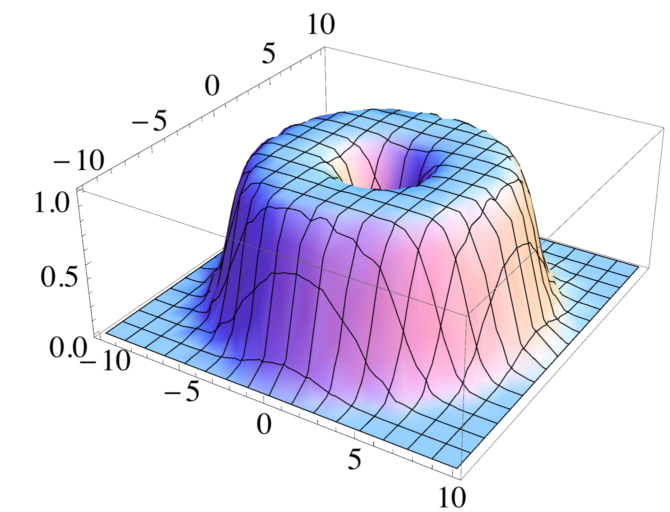
Unique vanishing properties $\longrightarrow$ Gap! (well,...)

Topological order!



Fractionalization

- (Fractional) excitations



$$\Psi_{\nu=1/3,\ 3\ qh's} = \prod_i (z_i - w)^3 \prod_{i<j} (z_i - z_j)^3 e^{-\sum_i |z_i|^2/4\ell_B^2}$$

  - Hole with charge e*=e at w (fermionic statistics).

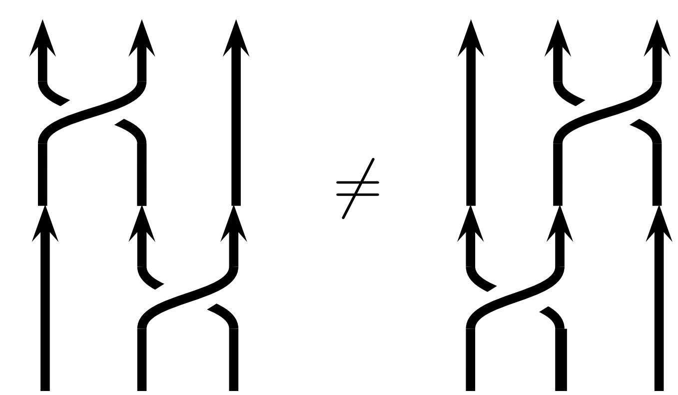
  - Easily splits in 3 equal pieces:

$$\to \prod_i (z_i - w_1) \prod_i (z_i - w_3) \prod_i (z_i - w_3) \prod_{i<j} (z_i - z_j)^3 e^{-\sum_i |z_i|^2/4\ell_B^2}$$

  - Abelian anyons with charge e*=e/3 and fractional statistics!

  - e*=e/3 observed in shot noise! (two groups ’97)

- States with non-Abelian excitations conjectured
  - Majorana fermions at filling 5/2
  - Fibonacci anyons at 12/5 filling!?



- The dream: topological quantum computation

$1,\ 0 \quad \to \quad \alpha|0\rangle + \beta|1\rangle$ Immune to noise, decoherence etc (Kitaev ’97)

The fractional quantum Hall effect has essentially all phenomena we can dream of -- why look further?

- So far: No “topological quantum computer” in service, no Nobel prize for non-Abelian anyons.

  - Despite the first observation of the 5/2 state in 1987, and more recently many other suggested realizations...

- Extreme conditions needed

  - Very strong magnetic fields, typically 10-30 Tesla
  - Extremely clean samples are needed, especially for possible non-Abelian states
  - Very cold, less than one Kelvin $\Delta E \sim e^2/\ell_B$

- Are there alternative realizations?

Wish list:

- Zero (or at least weak) magnetic field
- Larger gaps (shorter characteristic length scales)
- Even richer phenomena

Chern insulators

~ lattice quantum Hall systems at zero field

Useful references:

S. A. Parameswaran, R. Roy & S. L. Sondhi
Fractional Quantum Hall Physics in Topological Flat Bands
C. R. Physique 14, 816 (2013) [arXiv:1302.6606]

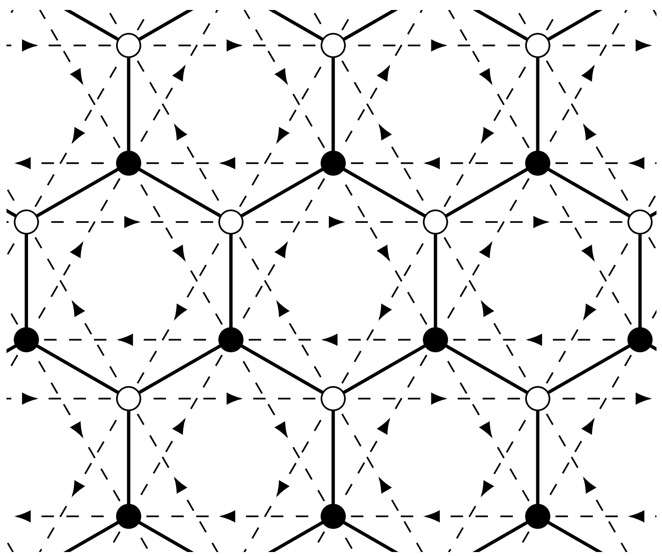
E. J. Bergholtz & Z. Liu
Topological Flat Band Models and Fractional Chern Insulators
Int. J. Mod. Phys. B 27, 1330017 (2013) [arXiv:1308.0343]

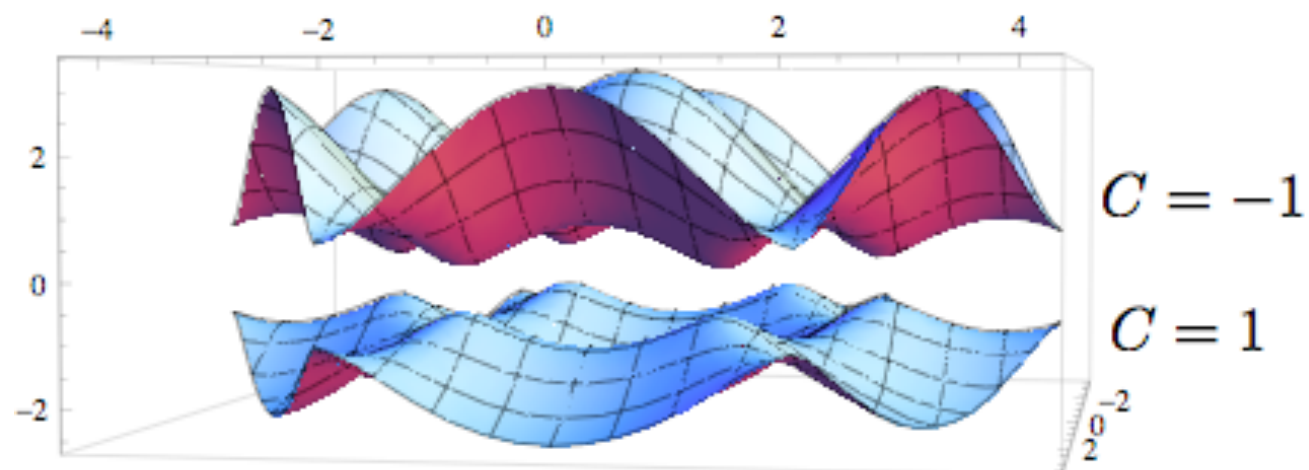
L. Chen, T. Mazaheri, A. Seidel, & X. Tang,
The impossibility of exactly flat non-trivial Chern bands in strictly
local periodic tight binding models
J. Phys. A: Math. Theor. 47, 152001 (2014) [arXiv:1311.4956]

The Haldane model

F.D.M. Haldane,
Model for a Quantum Hall Effect without Landau Levels: Condensed-Matter Realization of the "Parity Anomaly"
Phys. Rev. Lett. 61, 2015 (1988).

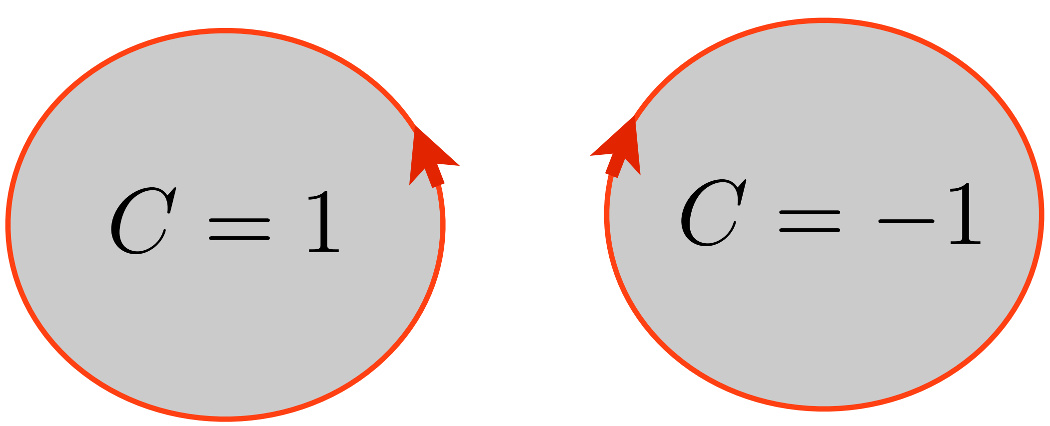
- Spinless ‘graphene’ + complex next nearest neighour hopping



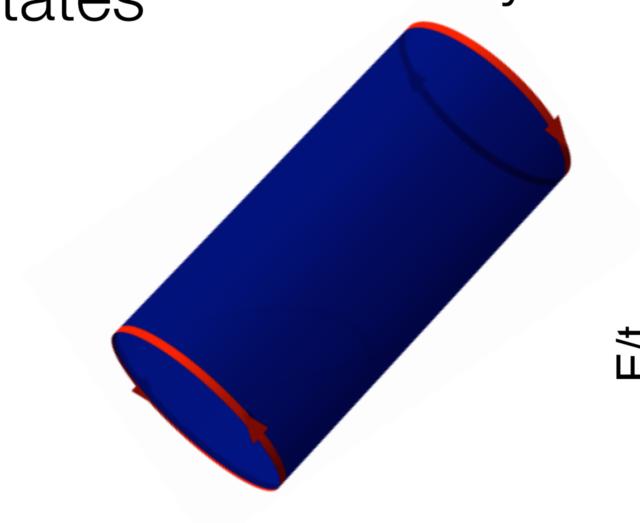


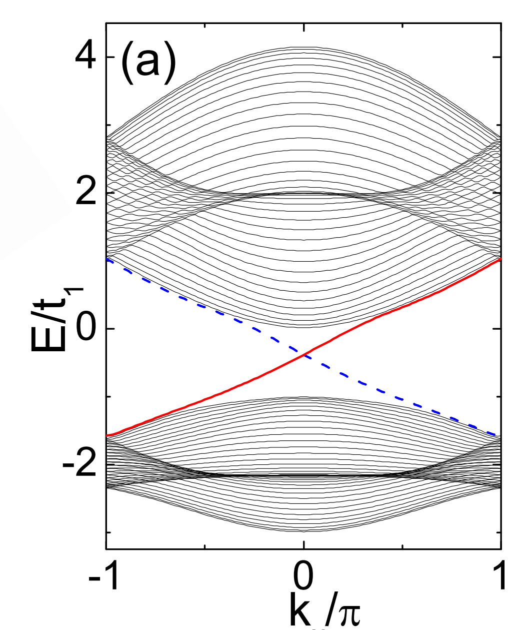
  - Zero average magnetic field

- Topologically protected gapless chiral edge states
  - Bulk-boundary correspondence



cylinder spectra





  - Quantized Hall response: $\sigma_{xy} = C\frac{e^2}{h}$

The Dirac model

See e.g., Qi, Hughes & Zhang, PRB 78, 195424 (2008)

- Generic two-band model, formulated directly in reciprocal space

$$\mathcal{H}(\mathbf{k}) = \mathbf{d}(\mathbf{k}) \cdot \boldsymbol{\sigma} \qquad \mathcal{H} = \sum_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\alpha} \mathcal{H}(\mathbf{k})_{\alpha\beta} c_{\mathbf{k}\beta}$$

- Diagonalized by squaring: $E(\mathbf{k}) = \pm|\mathbf{d}(\mathbf{k})|$

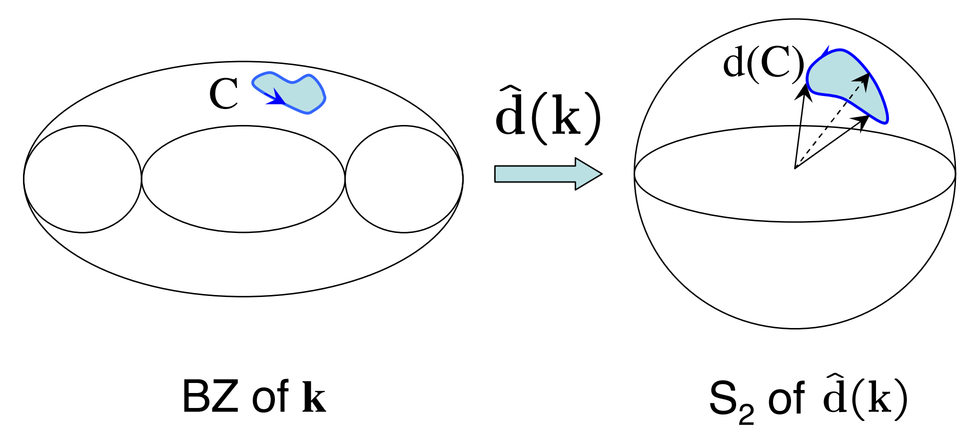
- Flat band model obtained by : $\mathbf{d}(\mathbf{k}) \to \hat{\mathbf{d}}(\mathbf{k}) = \dfrac{\mathbf{d}(\mathbf{k})}{|\mathbf{d}(\mathbf{k})|}$

- Geometric interpretation of the Chern number as the wrapping of a sphere

$$C = \frac{1}{4\pi} \int dk_x \int dk_y \, \hat{\mathbf{d}} \cdot \left(\frac{\partial \hat{\mathbf{d}}}{\partial k_x} \times \frac{\partial \hat{\mathbf{d}}}{\partial k_y} \right)$$

Chern number: robust, integer

Berry curvature, sensitive to details



Example, nearest neighbor model:

$$\begin{cases} d_x(\mathbf{k}) = \sin k_x \\ d_y(\mathbf{k}) = \sin k_y \\ d_z(\mathbf{k}) = m + \cos k_x + \cos k_y \end{cases} \Rightarrow \quad C = \begin{cases} 1 & \text{for } 0 < m < 2 \\ -1 & \text{for } -2 < m < 0 \\ 0 & \text{otherwise} \end{cases}$$

The Kapit-Mueller model

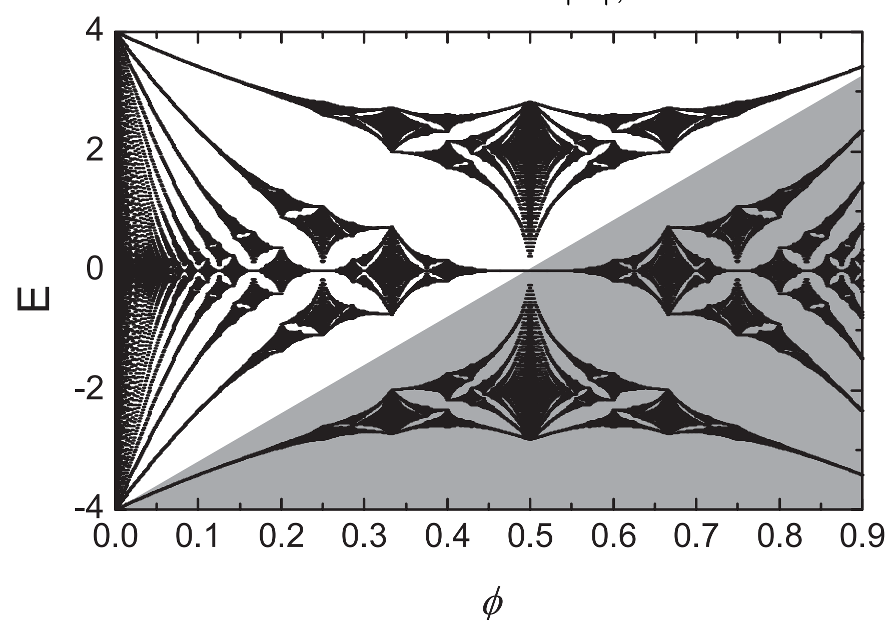
E. Kapit and E. Mueller
Exact Parent Hamiltonian for the Quantum Hall States in a Lattice
Phys. Rev. Lett. 105, 215303 (2010).

- Modified Hofstadter model: particles hopping on a lattice with ϕ flux quanta piercing each unit cell.

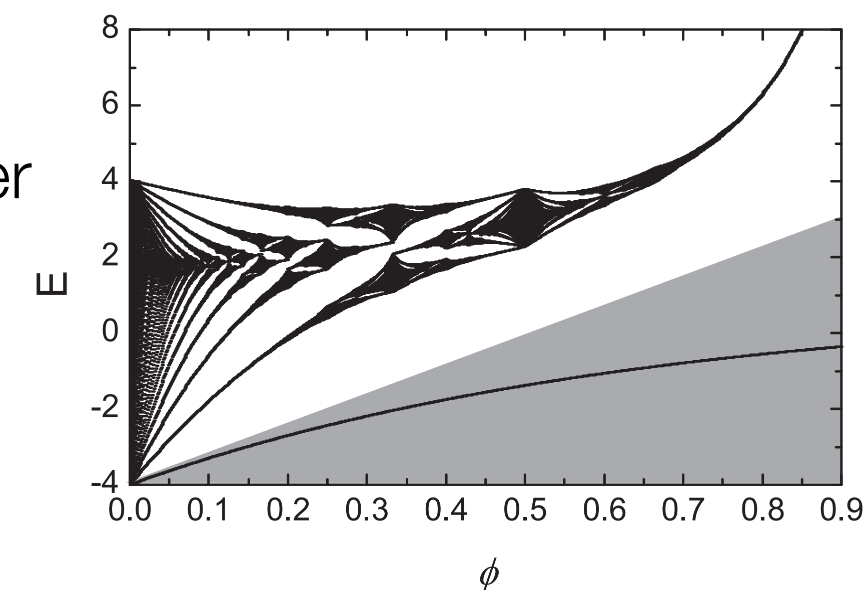
$$H_0 = -\sum_{j \neq k} \left(J\left(z_j, z_k\right) a_j^{\dagger} a_k + \text{H.c.} \right) \qquad J\left(z_j, z_k\right) = J_0 W(z) e^{(\pi/2)(z_j z^* - z_j^* z)\phi},\ z \equiv z_j - z_k$$

$$w(z) = \delta_{|z|,1} \qquad W(z) = (-1)^{1+x+y+xy} e^{-(\pi/2)(1-\phi)|z|^2}$$

Hofstadter (1976)



Kapit Mueller



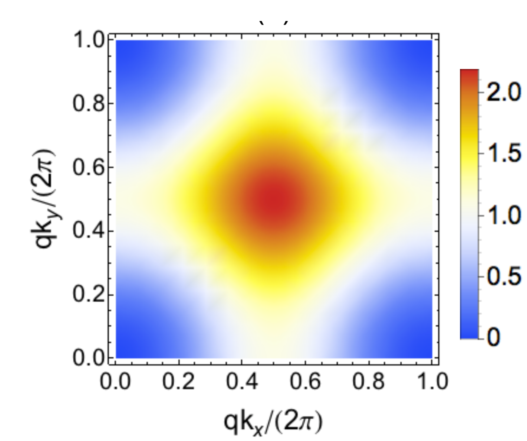
- Discretized Landau level wave functions span the lowest band!

$$\psi_n\left(z_j\right) = z_j^n \exp\left(-\frac{\pi\phi\left|z_j\right|^2}{2} \right)$$

- Onsite interactions give exact model FQH states of bosons!

- Simple single particle states not orthogonal

- Non-uniform Berry curvature and modified excitation spectra

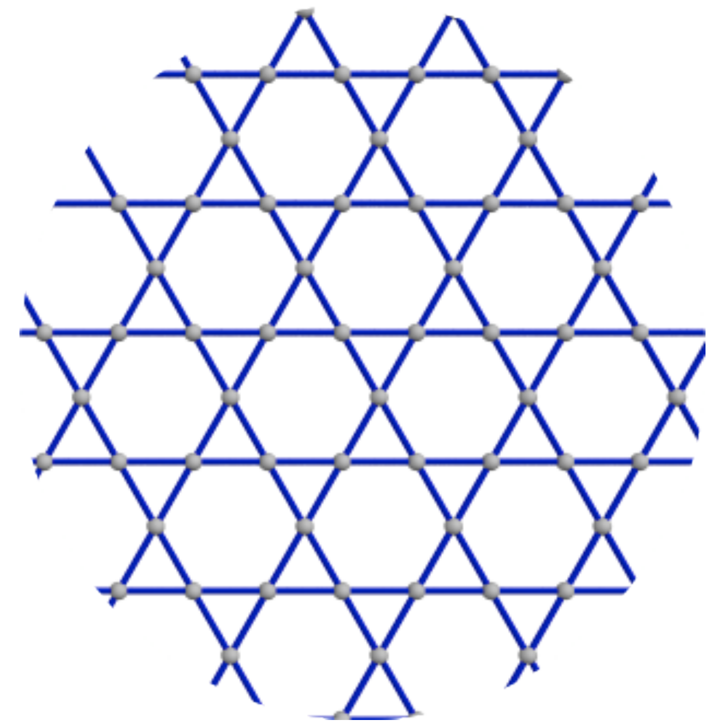


$\phi = 1/2$

Brief detour: Flat bands due to frustration

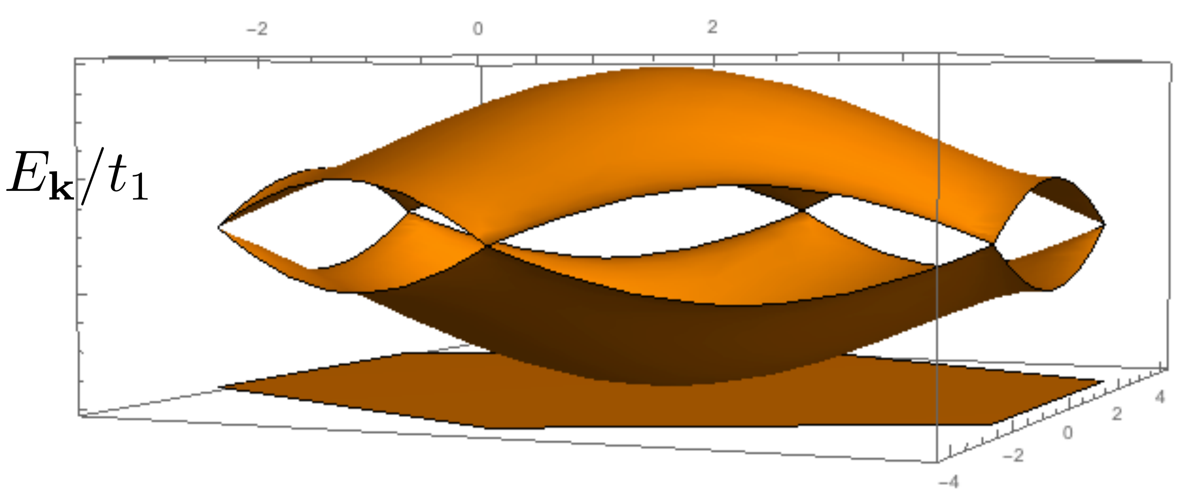
- Exactly flat bands are easy to find in geometrically frustrated lattice models

  - Example: nearest neighbor hopping on a kagome lattice



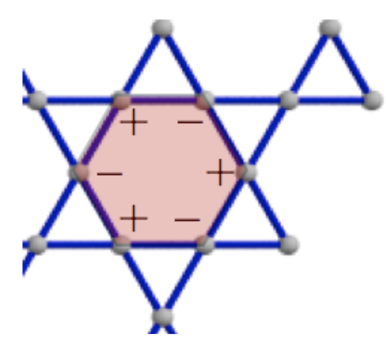
$$H = t_1 \sum_{\langle i,j \rangle} c_i^\dagger c_j$$

Fourier transformed: $$\mathcal{H}_{\mathbf{k}} = t_1 \begin{pmatrix} 0 & 1+e^{ik_1} & 1+e^{ik_2} \\ 1+e^{-ik_1} & 0 & 1+e^{-ik_3} \\ 1+e^{-ik_2} & 1+e^{ik_3} & 0 \end{pmatrix}$$



"Graphene + a flat band"

- Flat band understood in terms of localized modes



not Wannier functions!

- These bands are *not topological*!

  - Touching points and thereby no well-defined Chern numbers

- But good general knowledge and the same ideas will also be useful for creating new topological bands (second lecture)

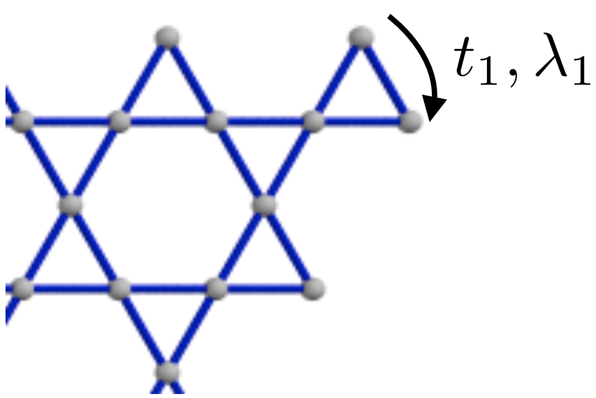
Topological Flat Band Models

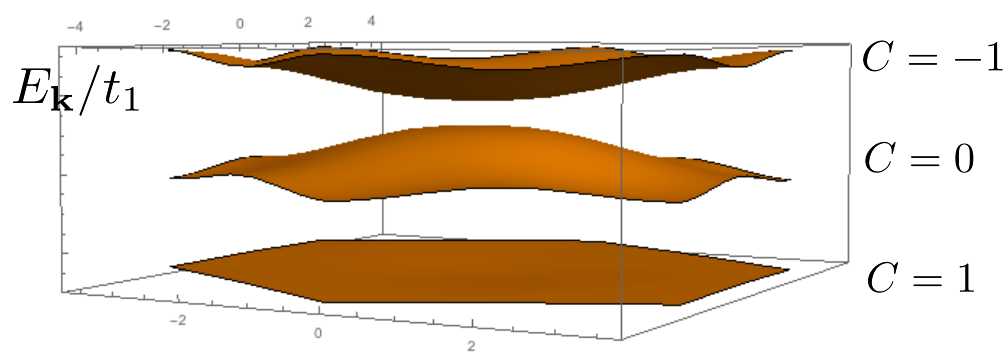
- Important insight:

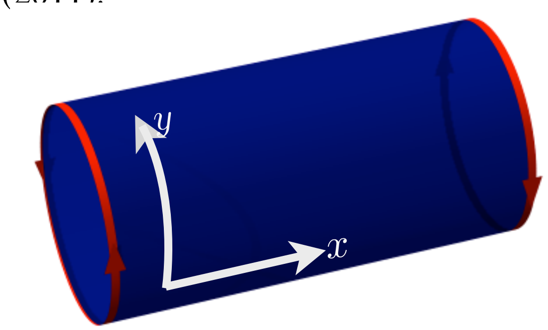
  - Lattice analogues of Landau levels, almost flat topological Chern bands, can form in rather realistic systems with short-range hopping only.

Complex history, recently turned into mainstream since

E. Tang, J.-W. Mei, and X.-G. Wen, Phys. Rev. Lett. **106**, 236802 (2011).
K. Sun, Z. Gu, H. Katsura, and S. Das Sarma, Phys. Rev. Lett. **106**, 236803 (2011).
T. Neupert, L. Santos, C. Chamon, and C. Mudry, Phys. Rev. Lett. **106**, 236804 (2011).







- Topologically protected gapless chiral edge states

- Interaction scale set by lattice spacing $\Rightarrow \Delta E \sim 500K$!?

- Zero external magnetic field!

- Theory: interactions lead to “fractional Chern insulators” (FCIs)

  - Qualitatively new challenges and possibilities arise due to the interplay between (band) topology, interactions and the lattice.

... more on this later...

General remarks

- The energy dispersion of single band can always be “flattened”
  - requires exponentially decaying tail of hopping terms
  - truncations quickly give very flat bands
  - does not change the eigenstates, hence no change in topology!
- The Berry curvature can never be made flat as long as the total number of bands is finite (but it can be exponentially flat in the number of bands)
- The entanglement entropy of Chern insulators obey an area law

$$S(L) = \alpha L + \mathcal{O}(1/L)$$

- But the area law coefficient is arbitrarily tunable as long as it is non-zero

J.C. Budich, J. Eisert and E.J. Bergholtz, *Topological insulators with arbitrarily tunable entanglement* Physical Review B 89, 195120 (2014)

  - There exists very weakly entangled Chern insulators!
  - Intuition; entropy simply related to the edge state velocity, which is non-universal
  - Rigorous proof for all Renyi entropies $p \geq 1$ using Weyl’s perturbation theorem

Two more theorems

- Any two of the following properties can simultaneously be realized, but never all three

  - exactly flat dispersion
  - non-zero Chern number
  - strictly finite-range hopping

L. Chen, T. Mazaheri, A. Seidel, and X. Tang, J. Phys. A: Math. Theor. 47, 152001 (2014).

- In Landau levels, the Wannier functions cannot decay quicker than

$$\sim r^{-2}$$

(Asymmetric choices possible, e.g., $\sim e^{ikx} e^{-(y-k)^2/2}$)

- General statement: exponentially localized Wannier functions if and only if the (total) Chern number vanish.

See e.g., Brouder et. al. Phys. Rev. Lett. 98, 046402 (2007)

- Relevant for influence of local disorder and largely prevents the formation of Wigner crystals

Experiments

Quantum Hall effect in zero field
- first Chern insulator

Sciencexpress

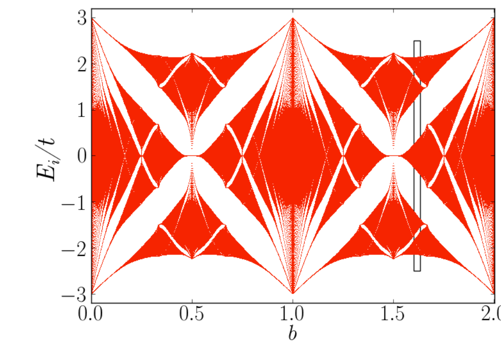
Experimental Observation of the Quantum Anomalous Hall Effect in a Magnetic Topological Insulator

Cui-Zu Chang,[1,2]* Jinsong Zhang,[1]* Xiao Feng,[1,2]* Jie Shen,[2]* Zuocheng Zhang,[1] Minghua Guo,[1] Kang Li,[2] Yunbo Ou,[2] Pang Wei,[2] Li-Li Wang,[2] Zhong-Qing Ji,[2] Yang Feng,[1] Shuaihua Ji,[1] Xi Chen,[1] Jinfeng Jia,[1] Xi Dai,[2] Zhong Fang,[2] Shou-Cheng Zhang,[3] Ke He,[2]† Yayu Wang,[1]† Li Lu,[2] Xu-Cun Ma,[2] Qi-Kun Xue[1,2]†

[1]State Key Laboratory of Low-Dimensional Quantum Physics, Department of Physics, Tsinghua University, Beijing 100084, China. [2]Beijing National Laboratory for Condensed Matter Physics, Institute of Physics, The Chinese Academy of Sciences, Beijing 100190, China. [3]Department of Physics, Stanford University, Stanford, CA 94305–4045, USA.

- Claimed observation in March 2013.
- Followup experiments more convincing, see arXiv: 1406.7450 and http://www.condmatjournalclub.org/?p=2458

Hofstadter Butterfly (partially) observed in several systems (2013)



- First key steps toward the FCI regime
- Claimed in graphene superlattices

LETTER

doi:10.1038/nature12186

Hofstadter's butterfly and the fractal quantum Hall effect in moiré superlattices

C. R. Dean[1], L. Wang[2], P. Maher[3], C. Forsythe[3], F. Ghahari[3], Y. Gao[2], J. Katoch[4], M. Ishigami[4], P. Moon[5], M. Koshino[5], T. Taniguchi[6], K. Watanabe[6], K. L. Shepard[7], J. Hone[2] & P. Kim[3]

LETTER

doi:10.1038/nature12187

Cloning of Dirac fermions in graphene superlattices

L. A. Ponomarenko[1], R. V. Gorbachev[2], G. L. Yu[1], D. C. Elias[1], R. Jalil[2], A. A. Patel[3], A. Mishchenko[1], A. S. Mayorov[1], C. R. Woods[1], J. R. Wallbank[3], M. Mucha-Kruczynski[3], B. A. Piot[4], M. Potemski[4], I. V. Grigorieva[1], K. S. Novoselov[1], F. Guinea[5], V. I. Fal'ko[3] & A. K. Geim[1,2]

Selected for a Viewpoint in *Physics*
PRL **111,** 185302 (2013) PHYSICAL REVIEW LETTERS week ending 1 NOVEMBER 2013

Realizing the Harper Hamiltonian with Laser-Assisted Tunneling in Optical Lattices

Hirokazu Miyake, Georgios A. Siviloglou, Colin J. Kennedy, William Cody Burton, and Wolfgang Ketterle
Research Laboratory of Electronics, MIT-Harvard Center for Ultracold Atoms, Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA
(Received 6 August 2013; published 28 October 2013; publisher error corrected 28 October 2013)

Selected for a Viewpoint in *Physics*
PRL **111,** 185301 (2013) PHYSICAL REVIEW LETTERS week ending 1 NOVEMBER 2013

Realization of the Hofstadter Hamiltonian with Ultracold Atoms in Optical Lattices

M. Aidelsburger,[1,2] M. Atala,[1,2] M. Lohse,[1,2] J. T. Barreiro,[1,2] B. Paredes,[3] and I. Bloch[1,2]
[1]*Fakultät für Physik, Ludwig-Maximilians-Universität, Schellingstrasse 4, 80799 München, Germany*
[2]*Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Strasse 1, 85748 Garching, Germany*
[3]*Instituto de Física Teórica CSIC/UAM C /Nicolás Cabrera, 13-15 Cantoblanco, 28049 Madrid, Spain*
(Received 1 August 2013; published 28 October 2013)

- ...and in optical lattices

Haldane model engineered in cold atom systems

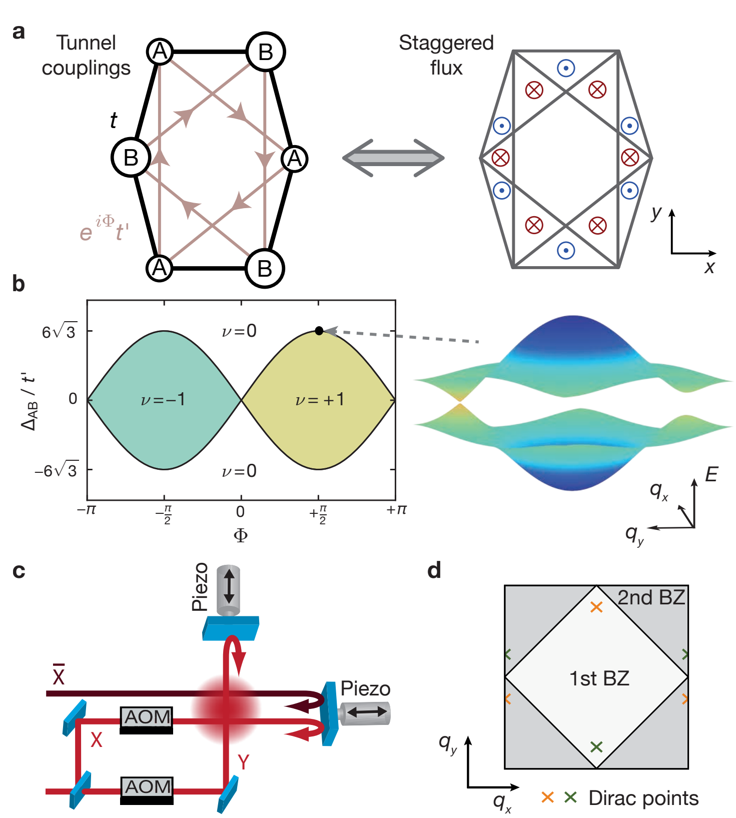
arXiv:1406.7874v1 [cond-mat.quant-gas] 30 Jun 2014

Experimental realisation of the topological Haldane model

Gregor Jotzu, Michael Messer, Rémi Desbuquois, Martin Lebrat,
Thomas Uehlinger, Daniel Greif & Tilman Esslinger
Institute for Quantum Electronics, ETH Zurich, 8093 Zurich, Switzerland
(Dated: July 1, 2014)

PACS numbers: 03.75.Ss, 67.85.Lm, 03.65.Vf, 73.43.-f, 73.43.Nq, 71.10.Fd, 73.22.Pr

- Huge experimental progress -- topological flat bands to come?



Fractional Chern insulators

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S. A. Parameswaran, R. Roy & S. L. Sondhi
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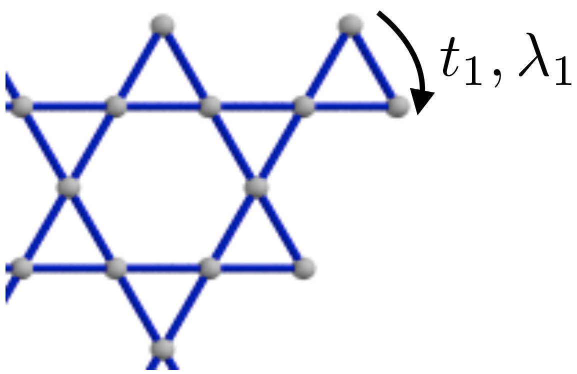
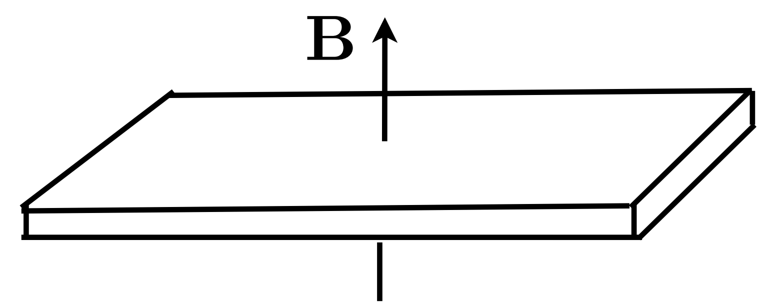
C. R. Physique 14, 816 (2013) [arXiv:1302.6606]

E. J. Bergholtz & Z. Liu
Topological Flat Band Models and Fractional Chern Insulators
Int. J. Mod. Phys. B 27, 1330017 (2013) [arXiv:1308.0343]

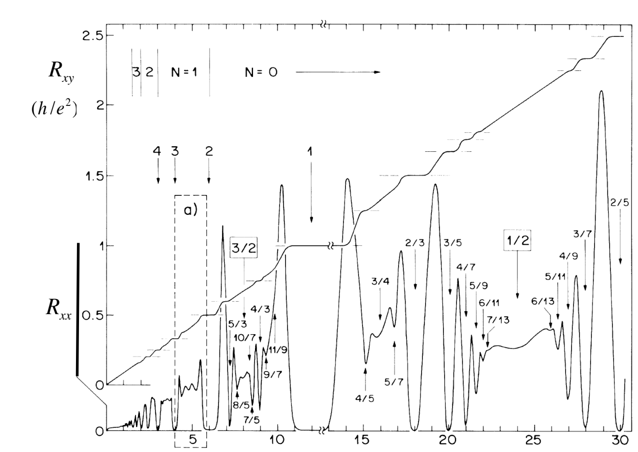
Quick recap: Chern bands vs. Landau levels



- Lattice models with complex hopping parameters: e.g. spin-orbit coupled systems. Time-reversal broken explicitly or spontaneously.
- Flat bands with any Chern number, C=N, possible (N chiral edge states).
- Interaction scale set by lattice spacing $\Rightarrow \Delta E \sim 500K$!? (very optimistic estimate... :))
- Experiments hopefully to come
  - Solid state, oxide interfaces?
  - Cold atoms?
- No need for strong magnetic fields!

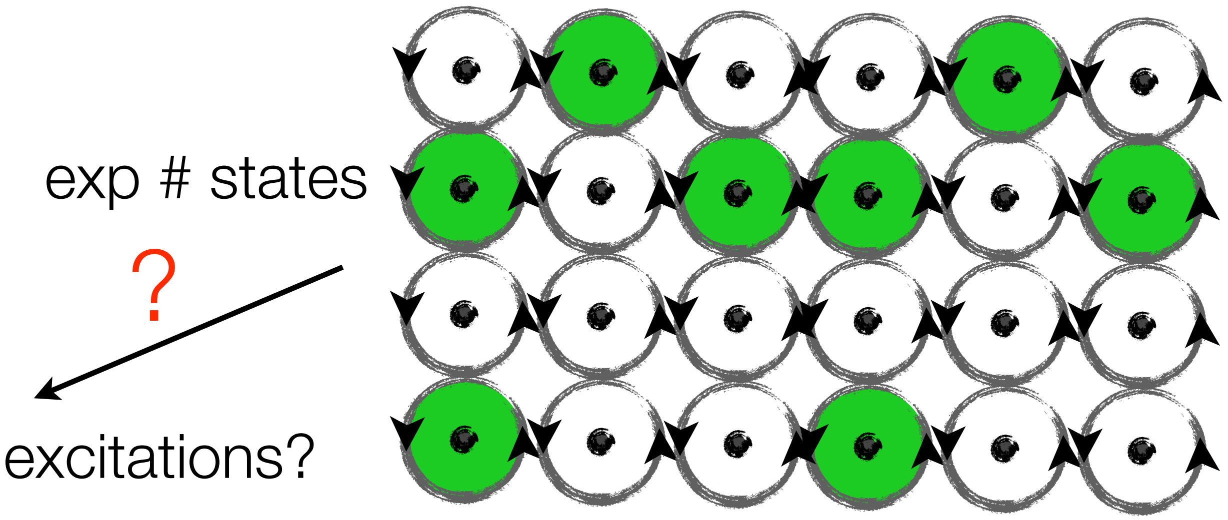


- Cold 2D electrons in a strong magnetic field.
- Flat Landau bands with C=1 (one chiral edge state).
- Interaction scale set by the magnetic length $\Rightarrow \Delta E \sim 1K$
- Experiments + theory!



Methods: how to attack the problem of a partially flat Chern band?

- Extremely hard, non-perturbative problem with no generically applicable cure.



- But, luckily FQH and FCI states have very short correlation lengths
  - Exact diagonalization is often appropriate in combination with analytical insights
- Analytical approaches, CFT, wave functions, low-energy Chern Simons theory etc provide useful reference points
  - Typically not enough to gain insights beyond the continuum quantum Hall regime
- The problem is also very well suited for entanglement based methods
  - Finite-dimensional Hamiltonian without band projection, hence a local Hamiltonian is a good starting point.
  - DMRG/MPS methods works very well

    See e.g., Frank’s talk
  - Ultimately also 2d tensor network approaches may be applicable

End of first lecture