Two Charged Particles in a Box

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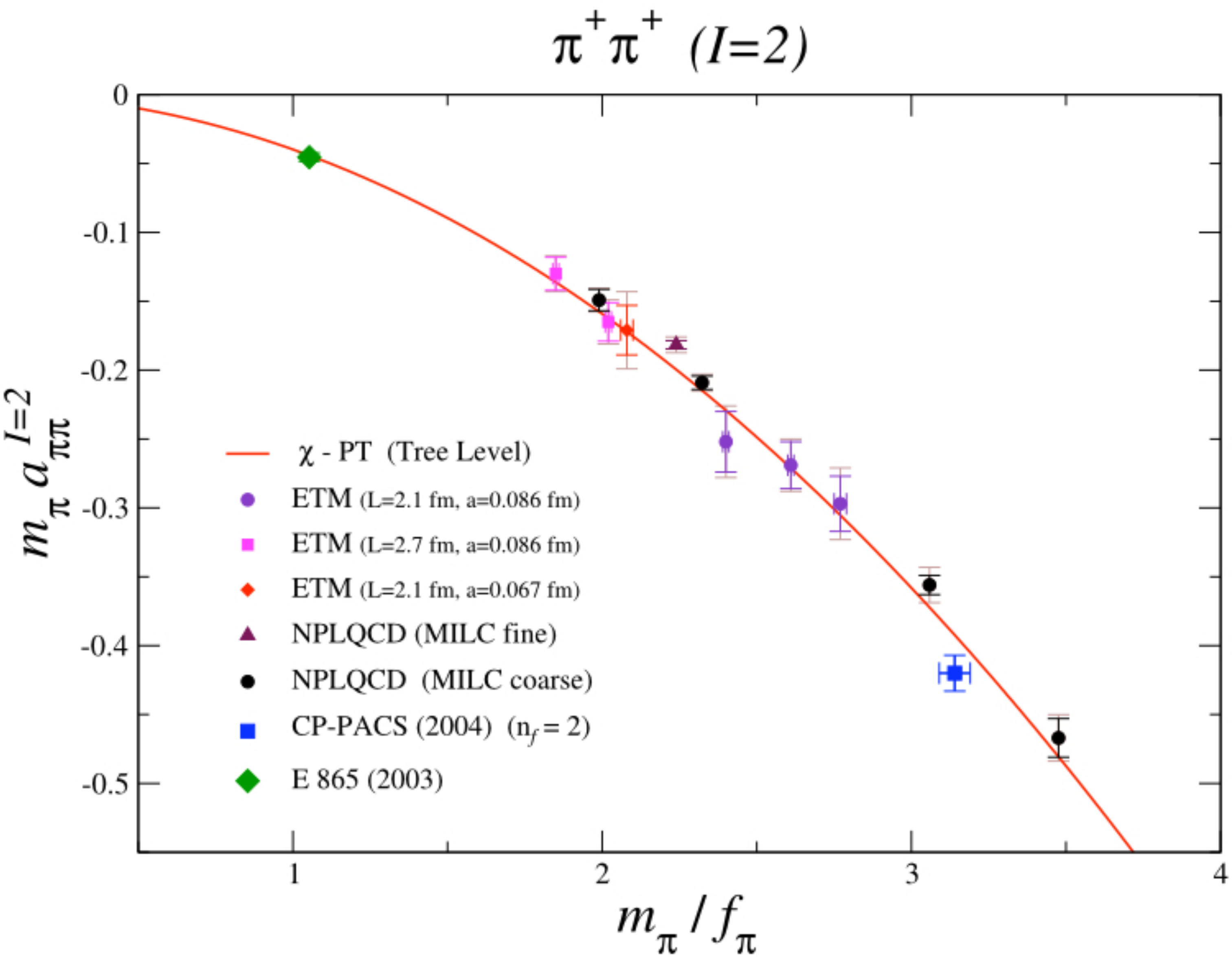
Benasque 7/23/2014

Outline

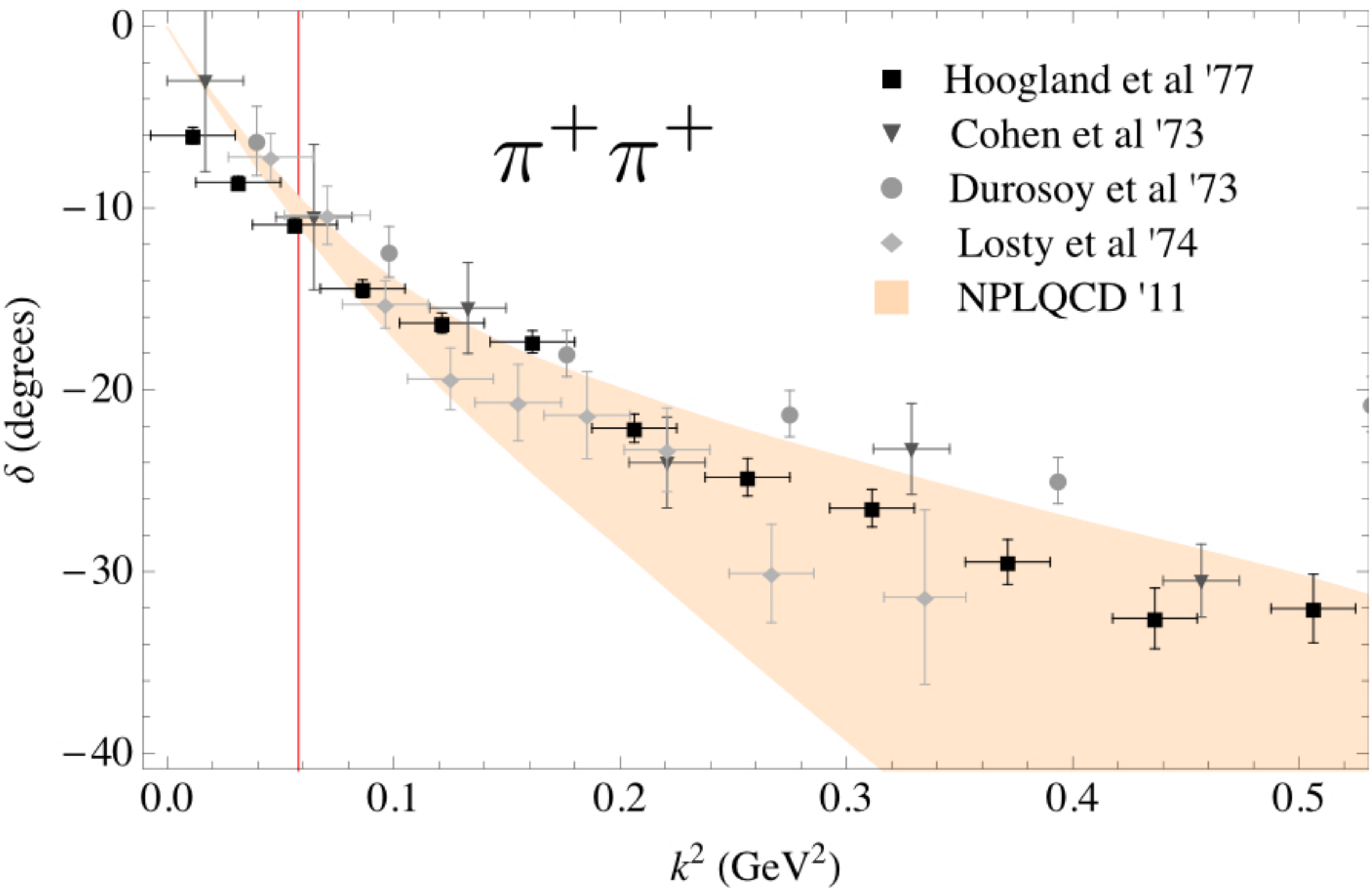
- Motivation
- Effective range theory / with Coulomb
- Review of A_1^+ QC in FV
- Coulomb in a FV arXiv:1407.4846
- Conclusions

✦ Motivation

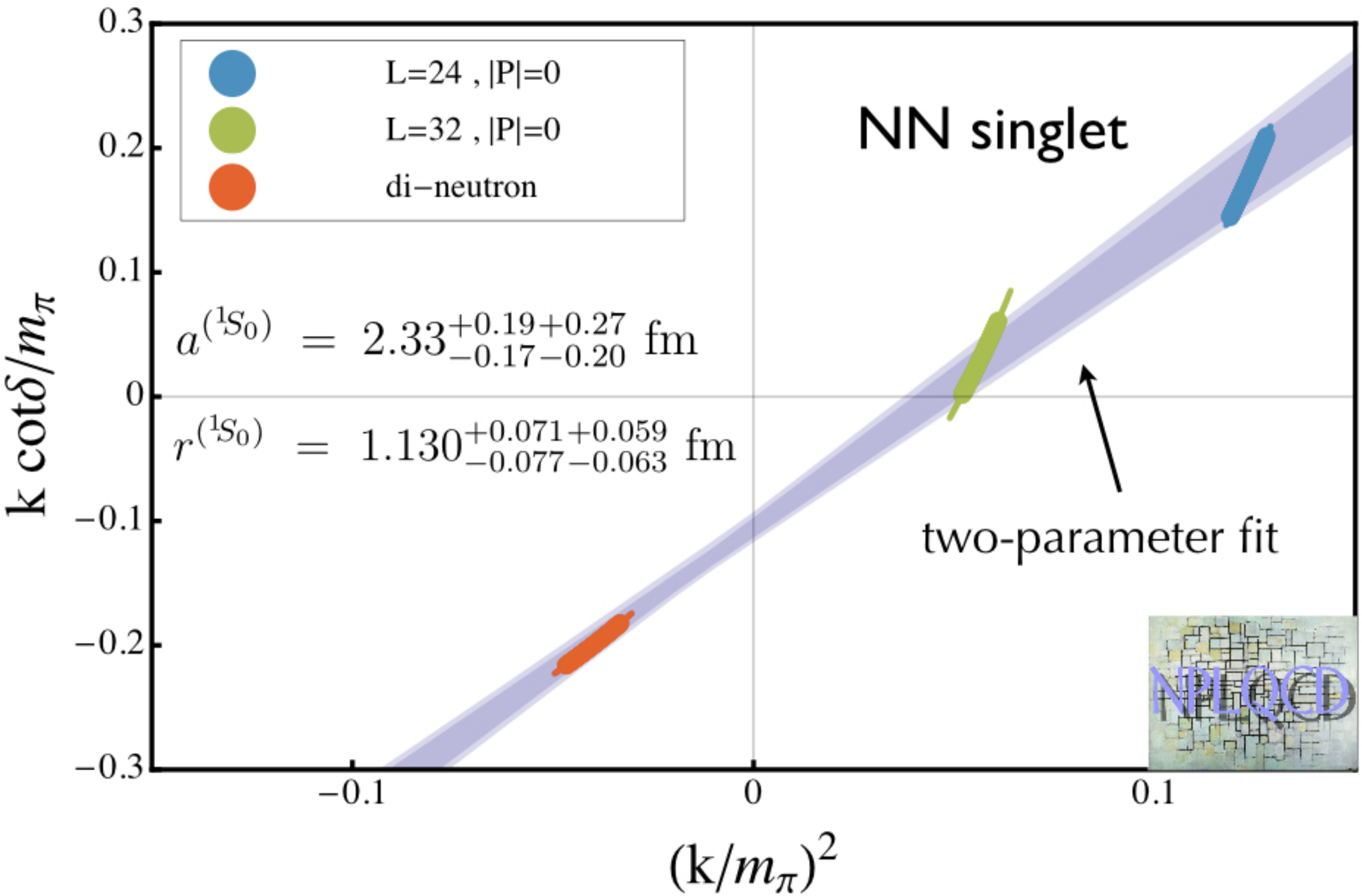
✓ Lattice QCD calculates scattering parameters



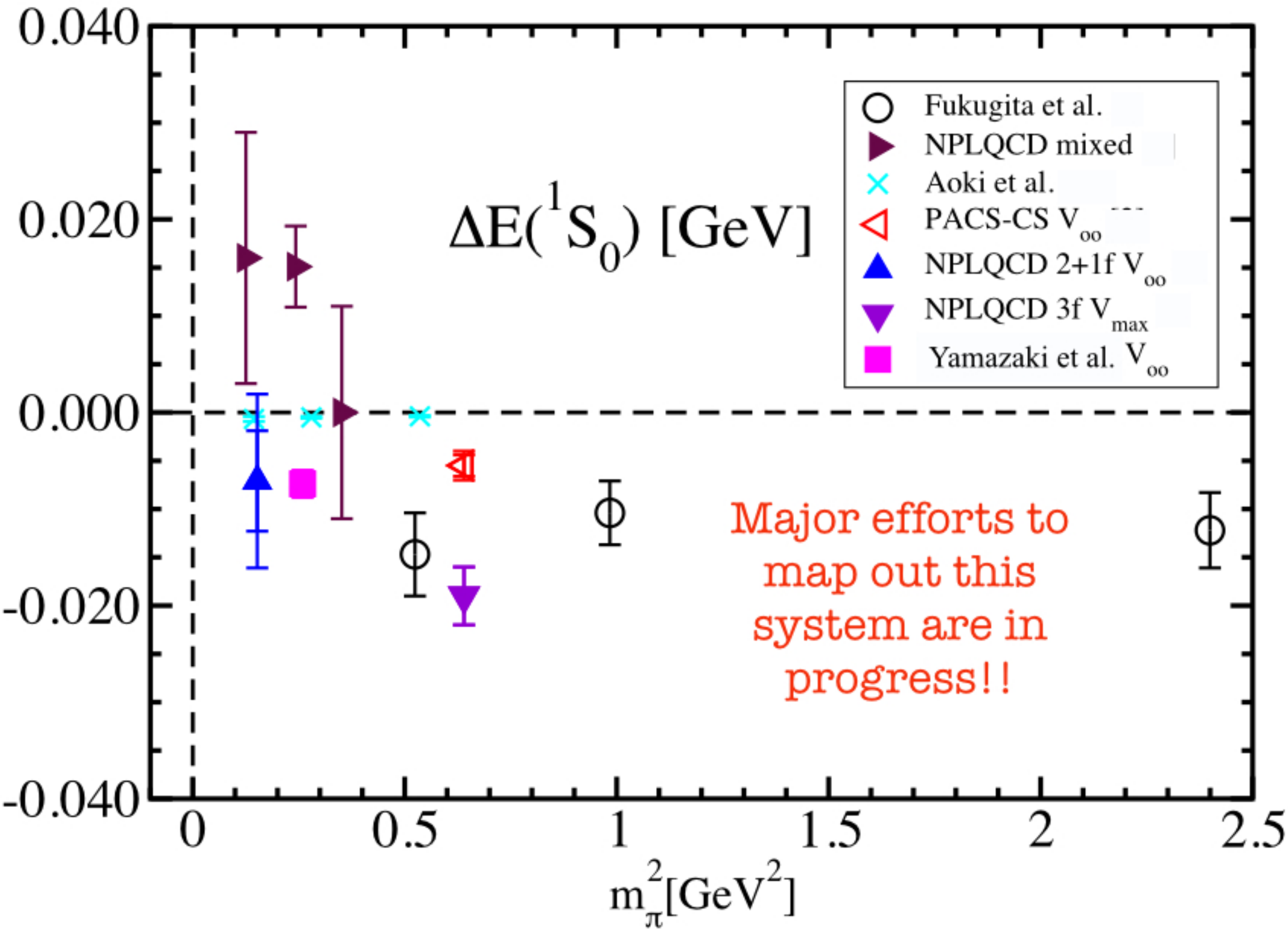
✓ $\alpha = 0$ phase shift prediction



✓ Can one get proton-proton scattering *ab initio* ?



Di-neutron binding energy: world data



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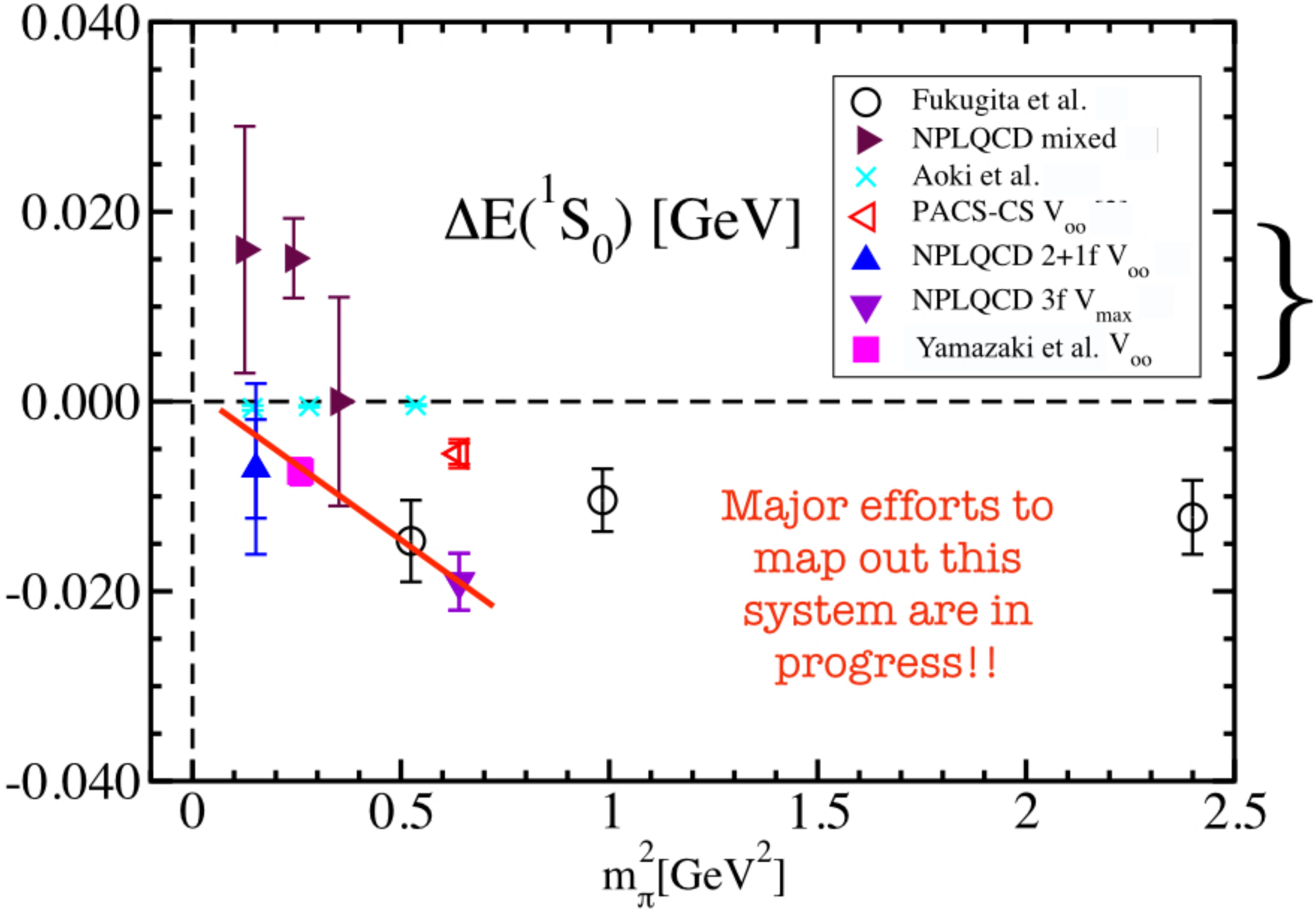


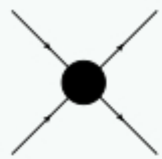
✦ Effective Range Theory (s-wave)

(Bethe, 1949)

✓ R is the range of the interaction

✓ $p \ll R^{-1}$ gives expansion of interaction in local operators:

$$C(E^*) =$$



$$= C_0 + C_2\, M\, T^* + \ldots$$

center-of-mass energy

$$E^* = 2M + \frac{p^2}{M} = 2M + T^*$$

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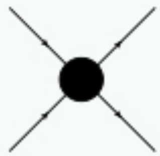

$$= C_0 + C_2\, M\, T^* + \ldots$$

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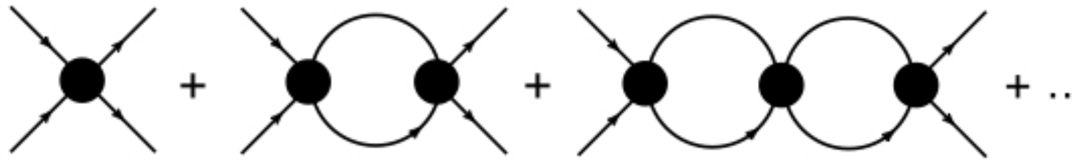
✓ In practice, remove momentum dependence in EFT:

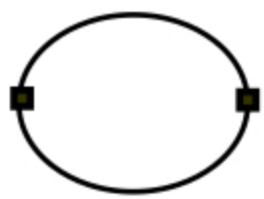
$$-\hat{\theta}\, \psi^T (\overleftarrow{\nabla} - \overrightarrow{\nabla})^2 \psi = 4M\, \hat{\theta} \left[i\partial_0 + \frac{\nabla^2}{4M} \right] \psi^T \psi \equiv 4M\, \hat{\theta}\, \mathcal{O}_{E^*} \psi^T \psi$$

✓ Scattering amplitude: interaction summed to all orders:

$T_S =$ 

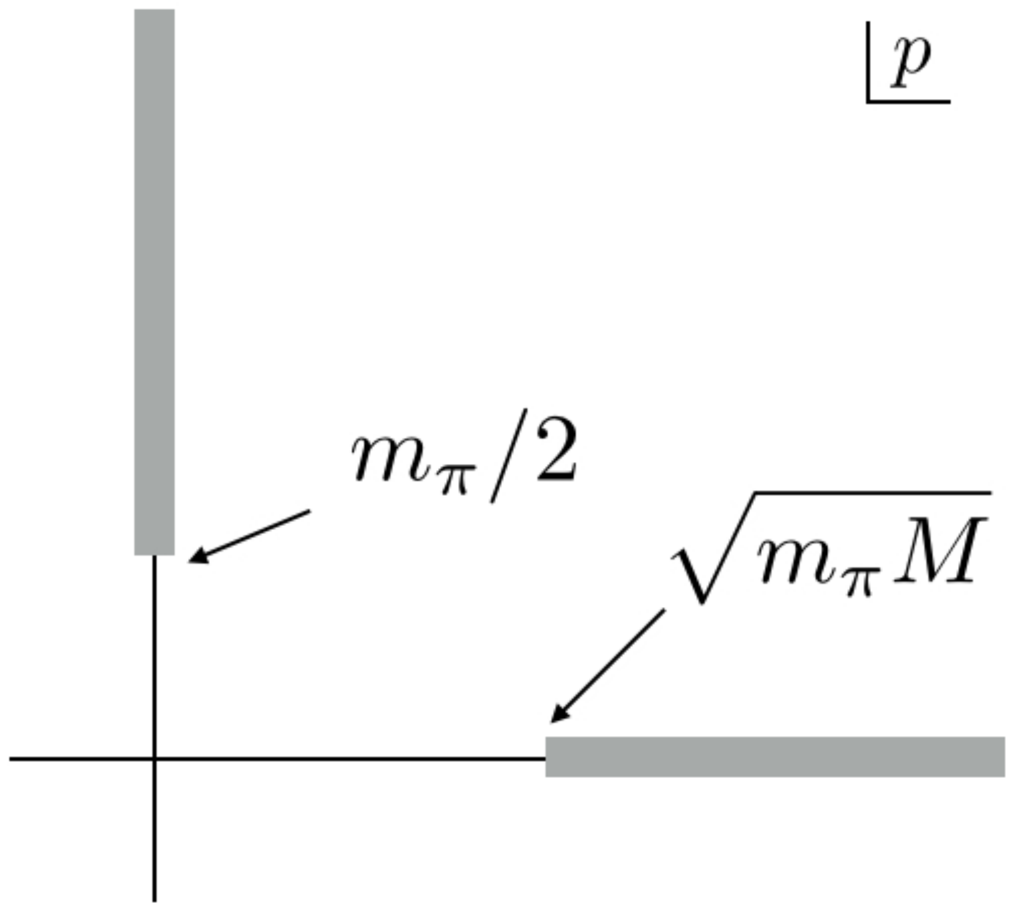
$$\boxed{T_S = \frac{C(E^*)}{1 - C(E^*) J_0^{\infty}(E^*)} = -\frac{4\pi}{M} \frac{1}{p\cot\delta - ip}}$$

✓ Matching achieved through renormalization ($\overline{MS}$):

$J_0^{\infty}(E^*) =$  $= M \int \frac{d^3q}{(2\pi)^3} \frac{1}{p^2 - q^2 + i\epsilon}$ Green's function

$$\boxed{p\cot\delta = -\frac{1}{a} + \frac{1}{2} r_0 p^2 + \ldots}$$

✓ Analytic structure of the scattering amplitude



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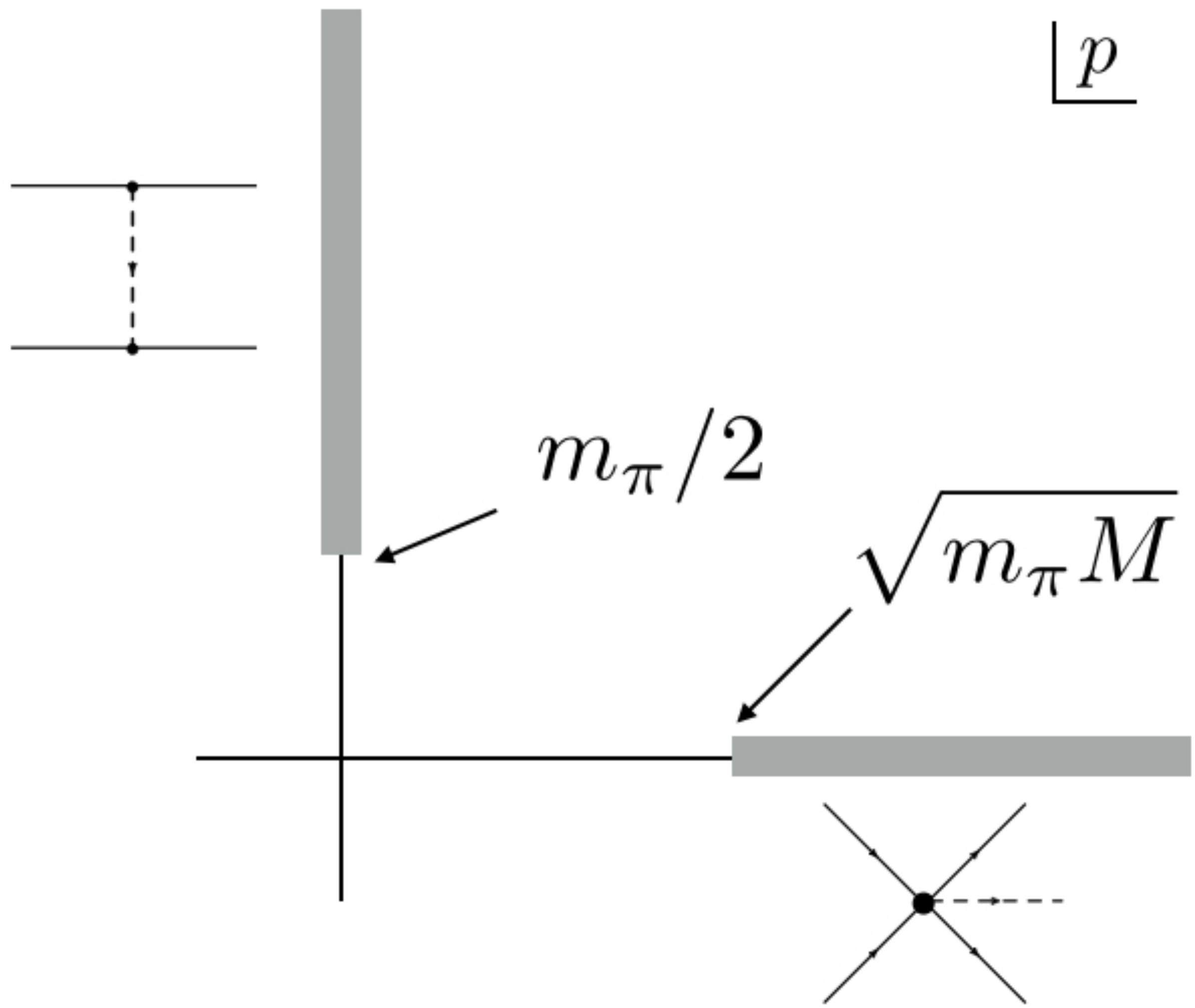

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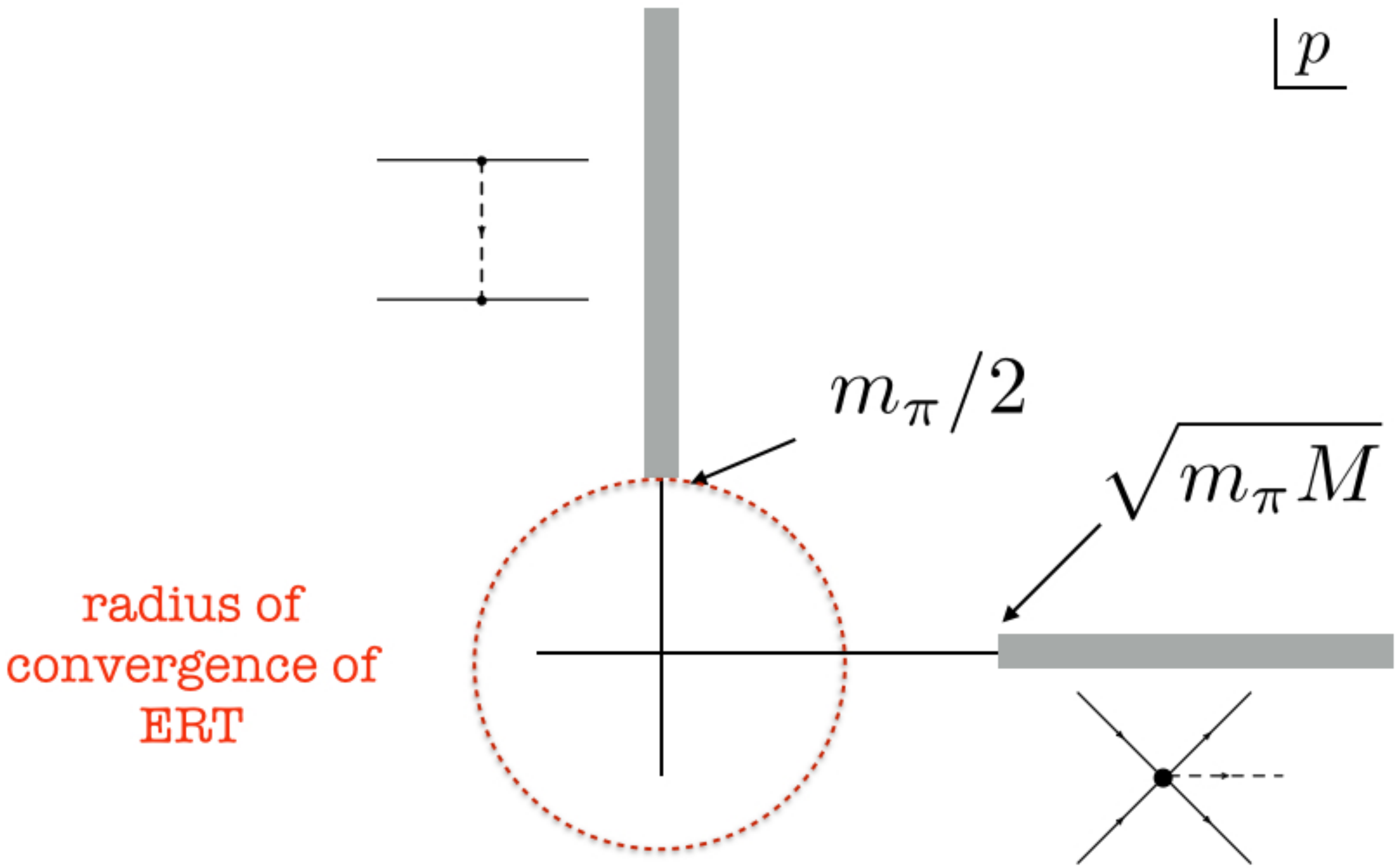
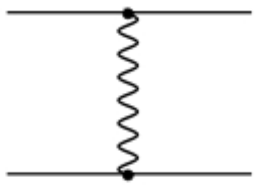


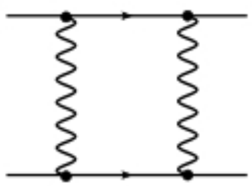
✦ Effective Range Theory with Coulomb

✓ Coulomb is enhanced in the infrared: (Bethe, 1949)

$$\boxed{p \sim \alpha M \sim a_{\mathrm{bohr}}^{-1}}$$



$$\frac{\alpha}{(\mathbf{p}-\mathbf{p}')^2} \sim \mathcal{O}(\alpha^{-1})$$



$$\int d^3\mathbf{k}\, dk_0 \frac{1}{\left(\frac{E}{2}+k_0-\frac{\mathbf{k}^2}{2M}+i\epsilon\right)} \frac{1}{\left(\frac{E}{2}-k_0-\frac{\mathbf{k}^2}{2M}+i\epsilon\right)} \frac{\alpha}{(\mathbf{k}-\mathbf{p})^2} \frac{\alpha}{(\mathbf{k}-\mathbf{p}')^2} \sim \mathcal{O}(\alpha^{-1})$$

Coulomb is non-perturbative

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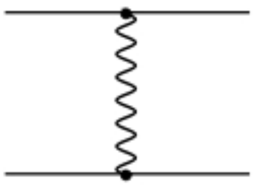

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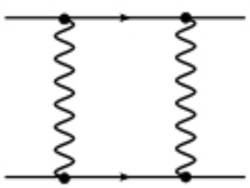
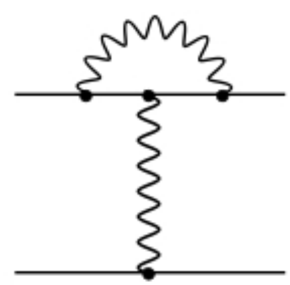


$$\int d^3\mathbf{k}\,dk_0 \frac{1}{\left(\frac{E}{2}+k_0-\frac{\mathbf{k}^2}{2M}+i\epsilon\right)} \frac{1}{\left(\frac{E}{2}-k_0-\frac{\mathbf{k}^2}{2M}+i\epsilon\right)} \frac{\alpha}{(\mathbf{k}-\mathbf{p})^2} \frac{\alpha}{(\mathbf{k}-\mathbf{p}')^2} \sim \mathcal{O}(\alpha^{-1})$$

Coulomb is non-perturbative

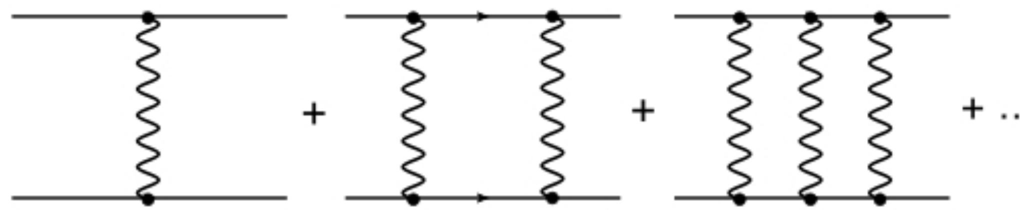
✓ Radiation is suppressed



$$\sim \mathcal{O}(1)$$

✦ Effective Range Theory with Coulomb

✓ Coulomb ladders treated to all orders:



$$\sigma_0 = \arg\Gamma(1+i\eta)$$ is Coulomb phase shift

$$\boxed{T_{SC} = C^2_{\eta(p)} \frac{C(E^*)e^{i2\sigma_0}}{1 - C(E^*)J_0^\infty(E^*)} = -\frac{4\pi}{M}\frac{e^{2i\sigma_0}}{p\cot\delta - ip}}$$

$$C^2_{\eta(p)} = \frac{2\pi\eta(p)}{e^{2\pi\eta(p)}-1}$$

$$\eta \equiv \frac{\alpha M}{2p} \sim \frac{\alpha}{v} \sim 1$$

$$J_0^{\infty}(E^*) \;=\; M\int \frac{d^3q}{(2\pi)^3}\frac{C^2_{\eta(q)}}{p^2-q^2+i\epsilon}$$

QED "dressed" Green's function

✓ Match to effective range parameters with $\overline{MS}$:

$$\boxed{C^2_{\eta(p)}\, p\cot\delta + \alpha M h(\eta) \;=\; -\frac{1}{a_C} \;+\; \frac{1}{2}r_0 p^2 \;+\; \ldots}$$

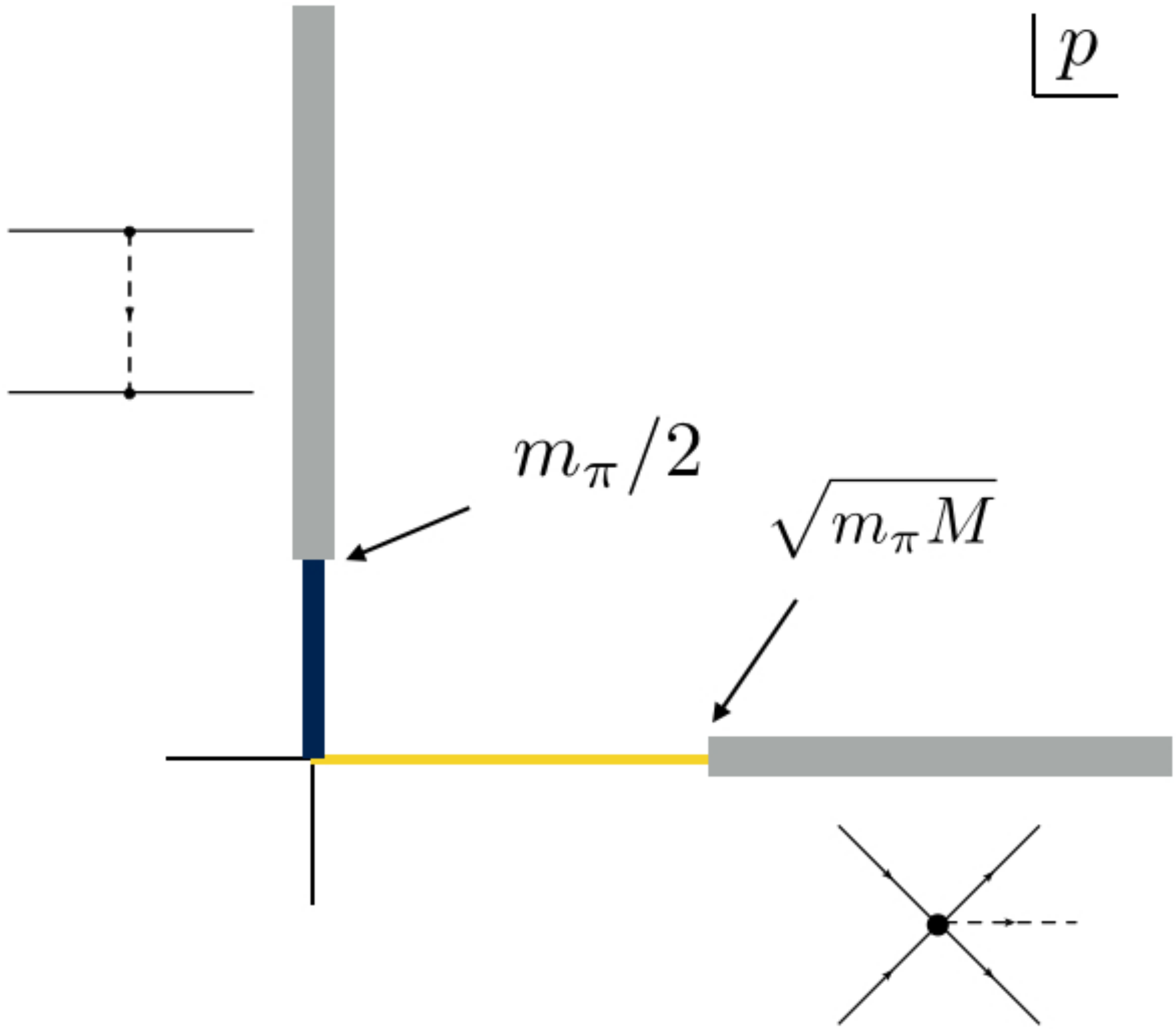
$$\Big\downarrow$$

$$\ln\left(\frac{M\alpha}{p}\right) \;+\; \mathcal{O}(\alpha)$$

Removes non-analytic piece!

$$(pp: \quad a_c \;=\; -7.82\ \mathrm{fm})$$

✓ Analytic structure of the scattering amplitude



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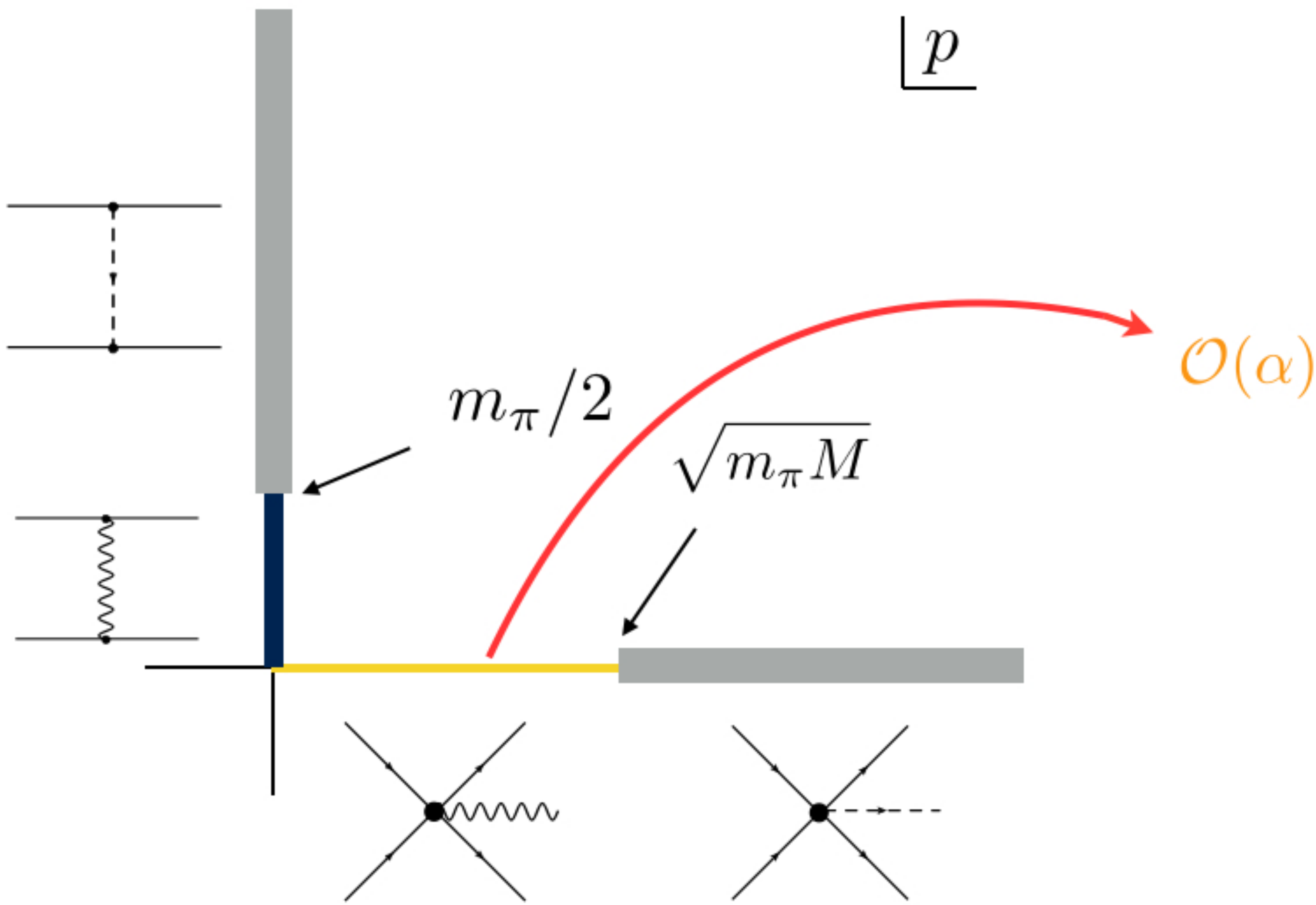

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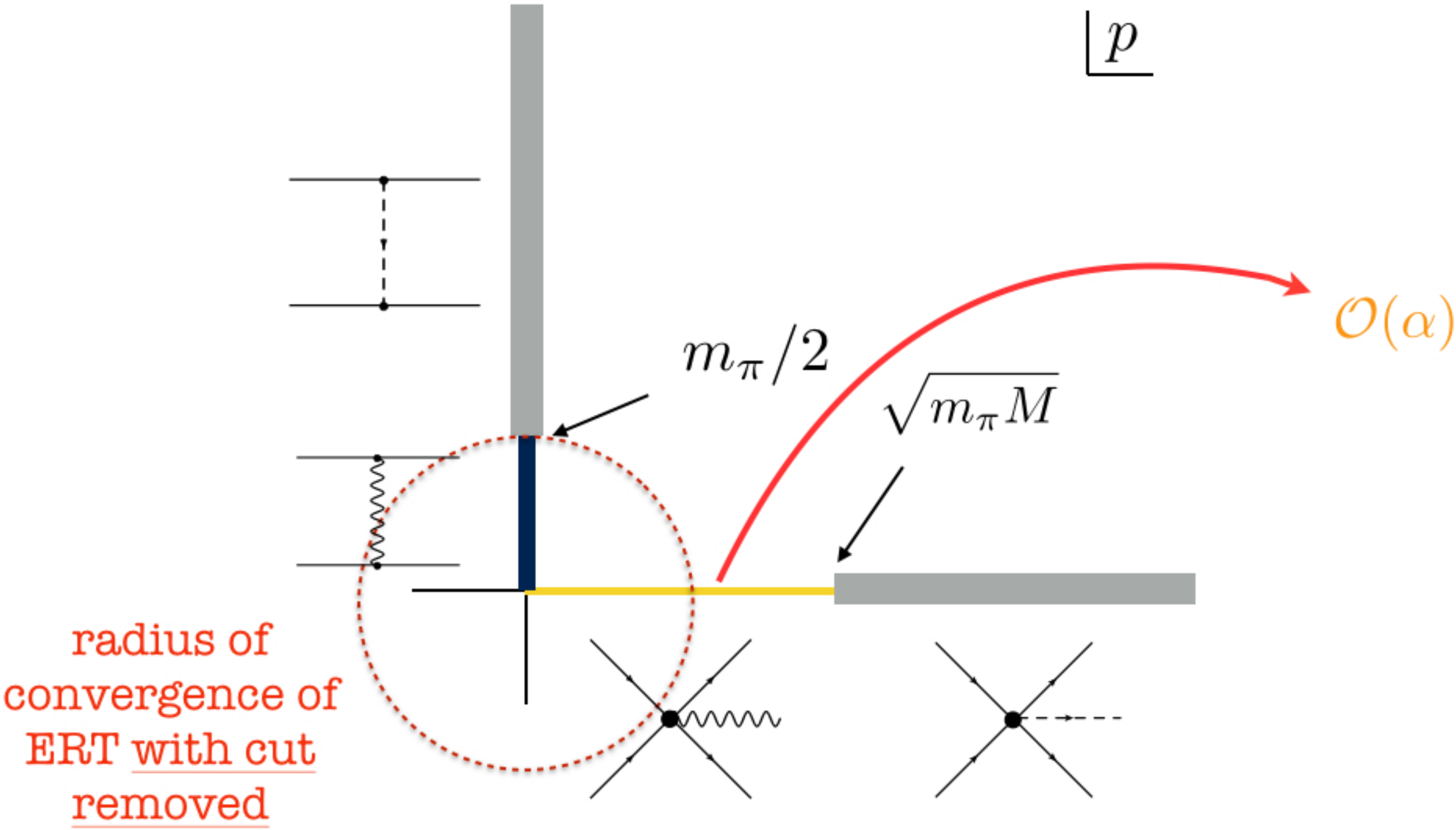
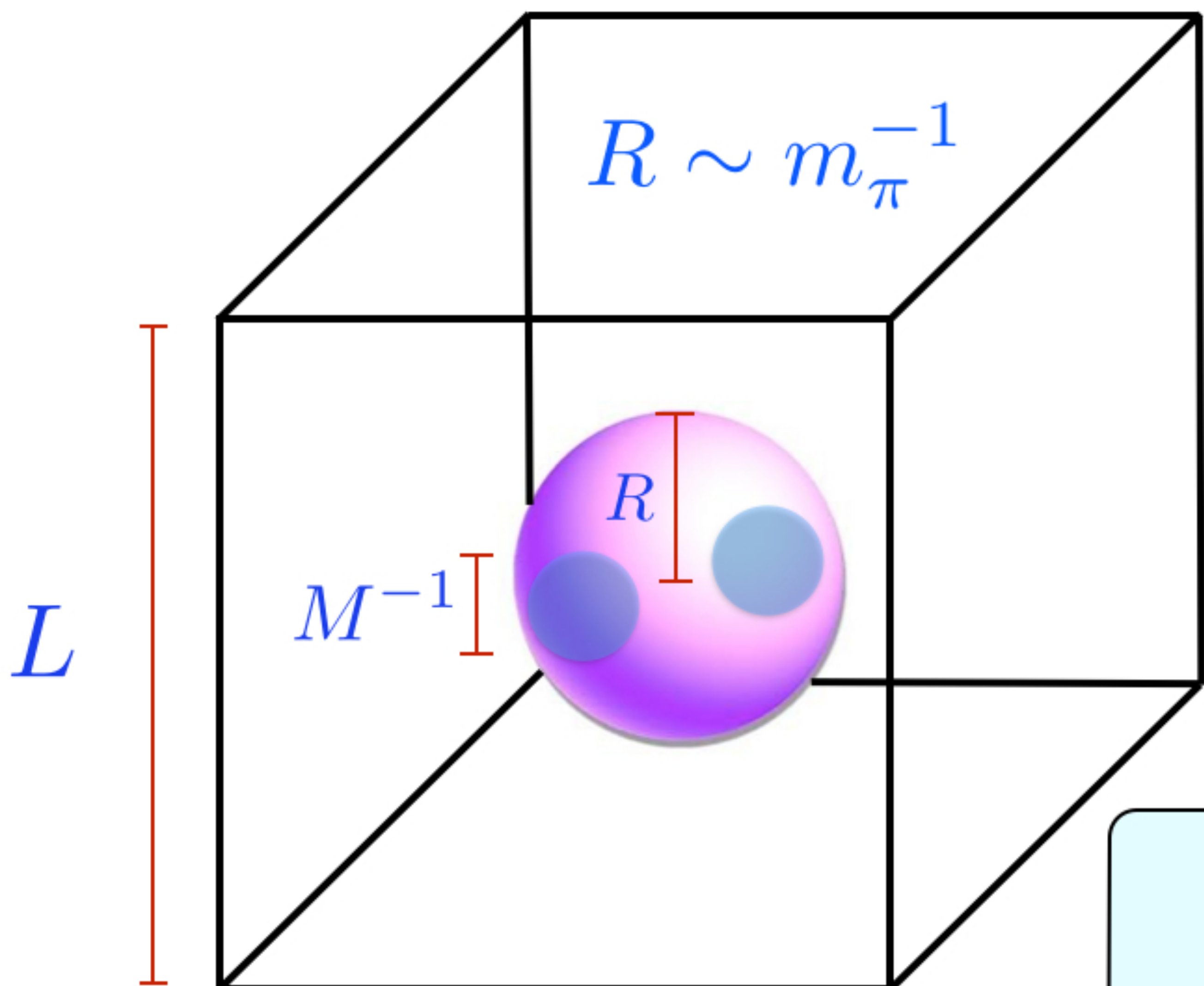


✦ Review of A_1^+ QC in FV

$b = 0,\, T = \infty,\, s - wave$

(Luscher, 1990)



$R < L/2$

✓ Two-particle correlation function in FV

Source

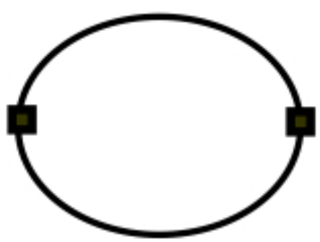
Sink

$$S^\dagger \left[J_0^L(E^*) \;+\; C^L(E^*)\left(J_0^L(E^*)\right)^2 \;+\; C^L(E^*)^2\left(J_0^L(E^*)\right)^3 \;+\; \ldots \right] S$$

$$= \; S^\dagger \frac{1}{1/J_0^L(E^*) - C^L(E^*)} S$$

Singularities of the correlation function

$$\boxed{\frac{1}{C^L(E^*)} \;=\; J_0^L(E^*)}$$



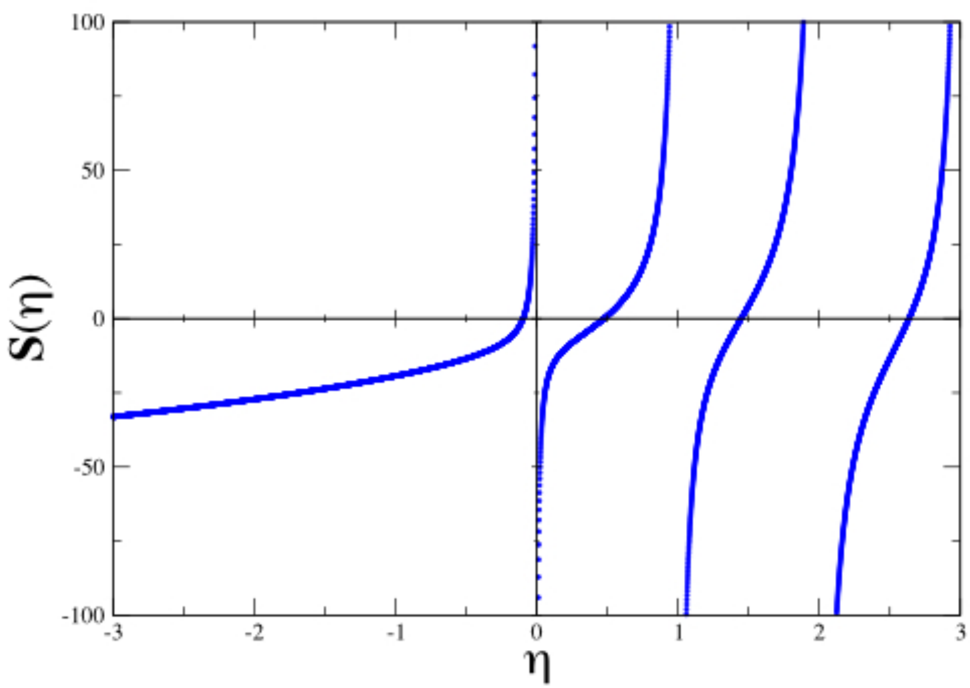
$$J_0^L(E^*) \;=\; -\,\frac{M}{4\pi^2 L}\sum_{\mathbf{n}}^{\Lambda_n}\frac{1}{|\mathbf{n}|^2 - \tilde{p}^2}$$

UV same as the continuum

✓ Quantization condition:

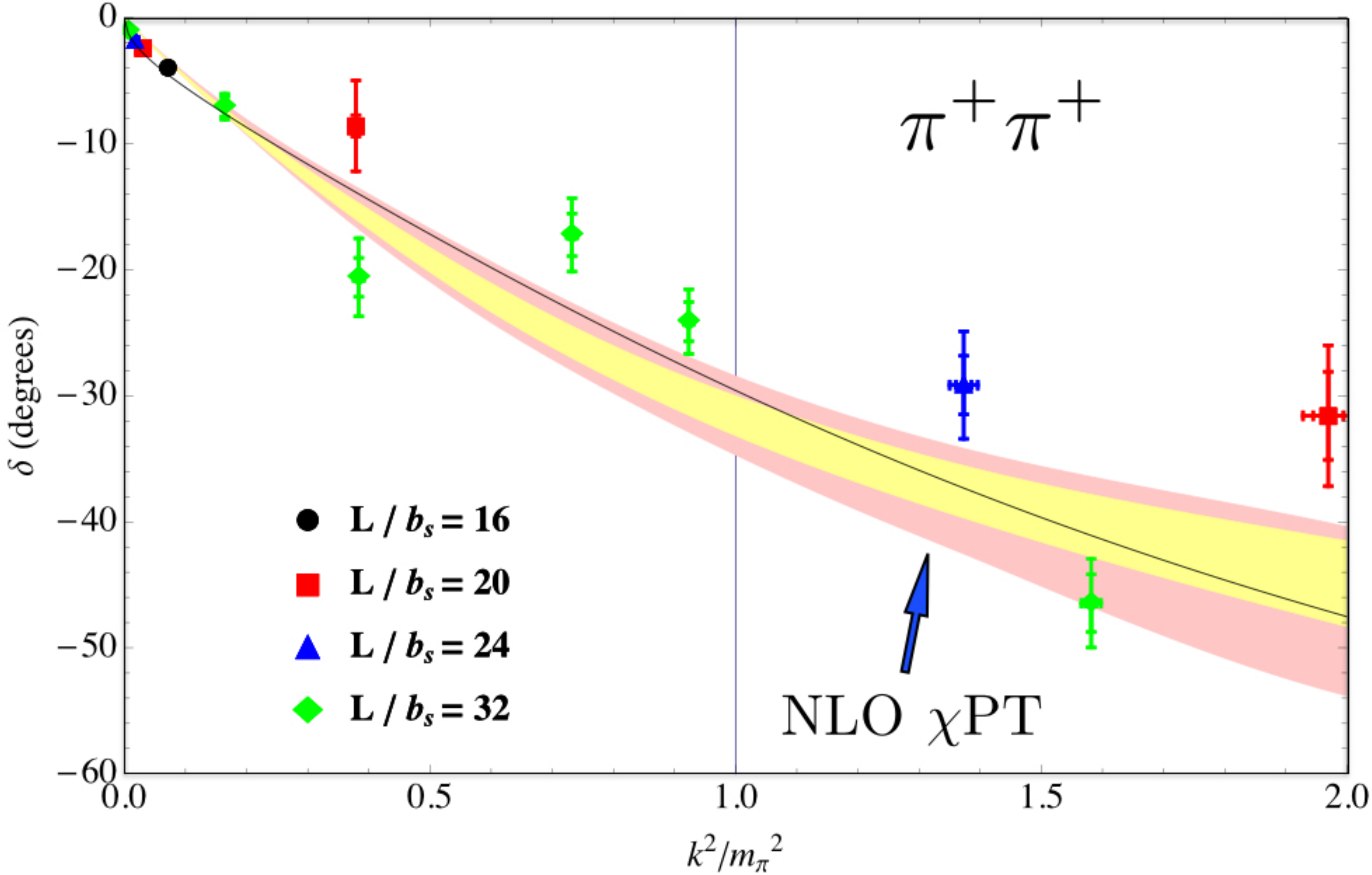
$$\boxed{p \cot \delta \;=\; \frac{1}{\pi L}\mathcal{S}(\tilde{p})} \;+\; \mathcal{O}\left(e^{-m_\pi L}\right)$$

Valid in QFT up to inelastic threshold



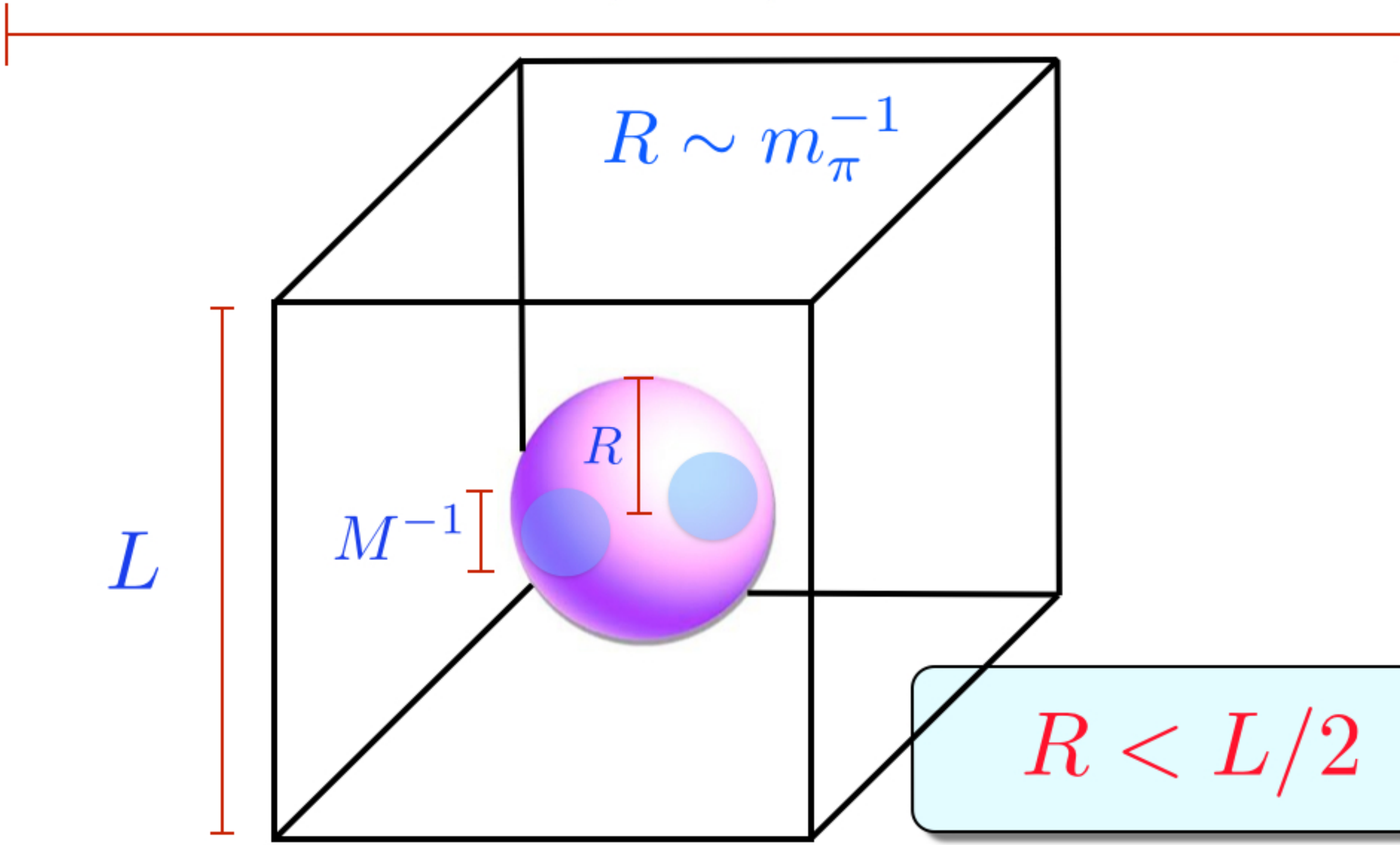
$$\mathcal{S}(x) \;\equiv\; \sum_{\mathbf{n}}^{\Lambda_n} \frac{1}{|\mathbf{n}|^2 - x^2} \;-\; 4\pi\Lambda_n$$

✓ Quantization condition in practice:



✦ A_1^+ QC in FV with Coulomb

$$R_C \sim (\alpha M)^{-1}$$



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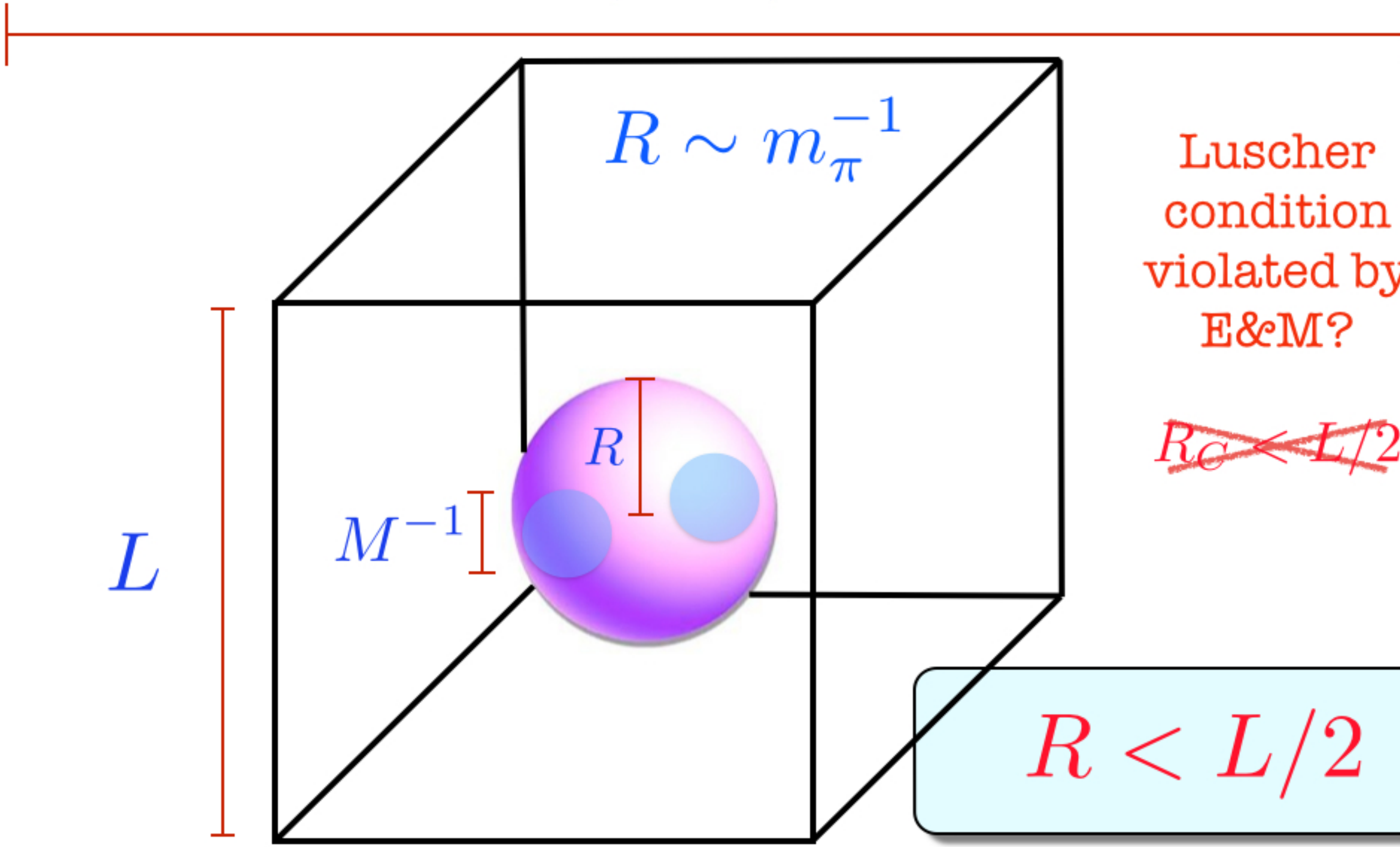


Luscher condition violated by E&M?

$\cancel{R_C < L/2}$

$$R < L/2$$

A prior there are many issues at FV…

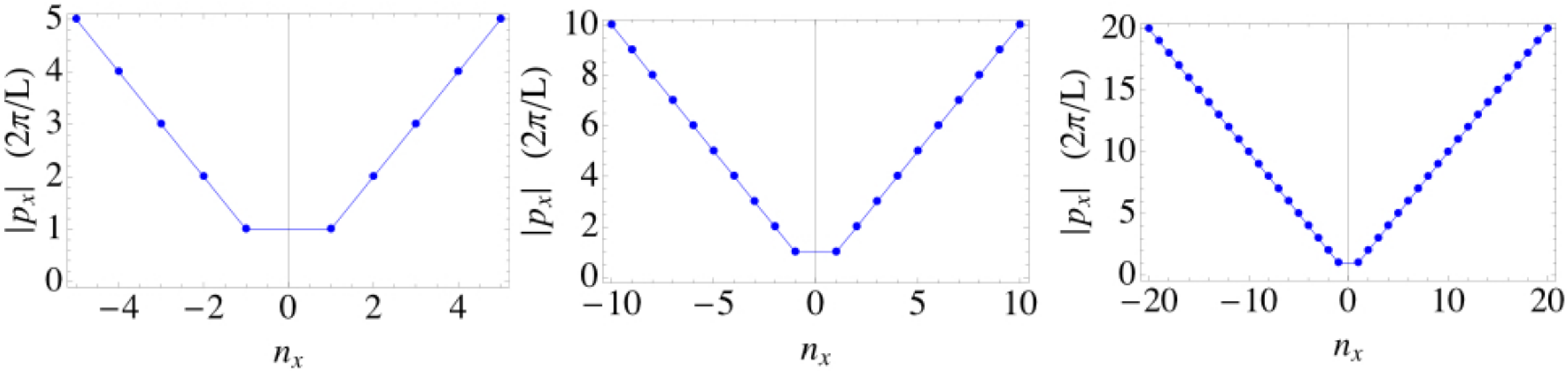
- QED ill defined with PBCs
- Increasingly complex integer sums .. many infrared problems
- Inelastic thresholds suppressed relative to Coulomb?
- Kinematics altered by power law effects
- Lack of universality

✓ Absence of zero mode: gap in momentum operator

$$p_{min} = \frac{2\pi}{L}$$

(See MJS Talk)

$$L \to \infty$$



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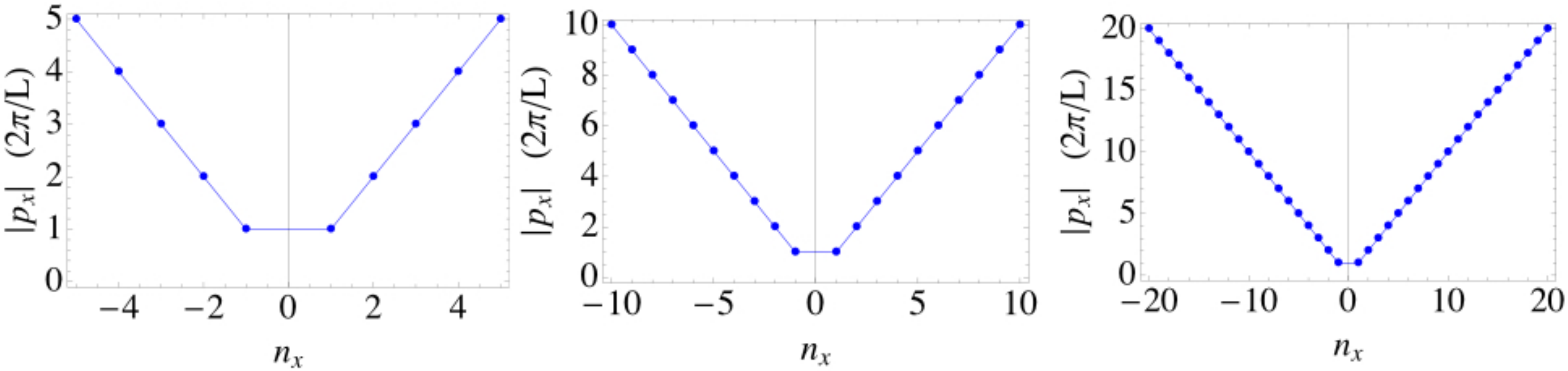
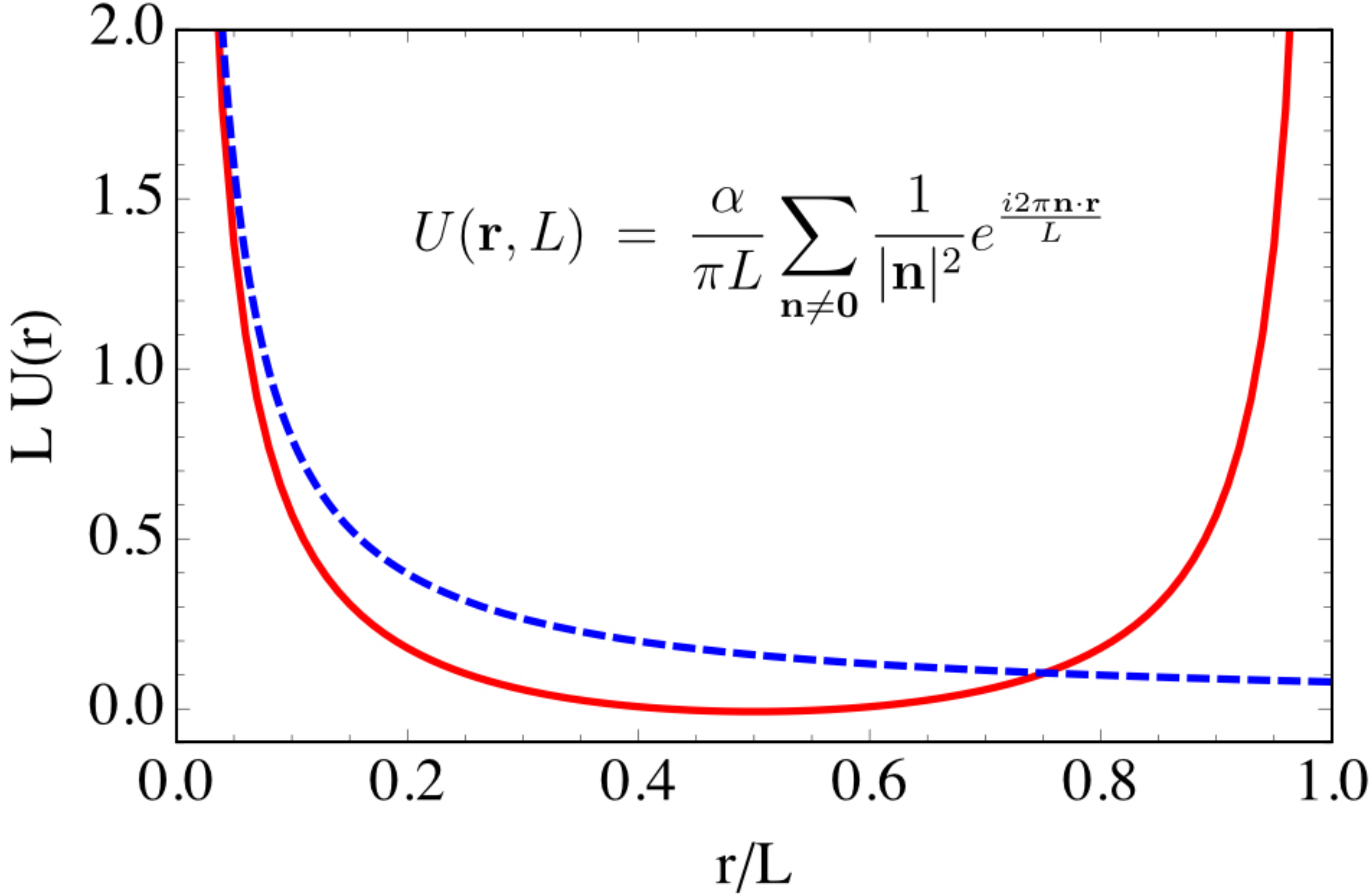


Gap is a good thing: $\eta \equiv \frac{\alpha M}{2p} \sim \alpha M L$

$\eta \ll 1$ if $1 \ll ML \ll 1/\alpha$

Coulomb is perturbative!

✓ Zero mode removed



✓ Kinematics altered by power-law corrections:

$$E^* = 2M^L + T^{*L}$$

NRQED

$$\boxed{p\cot\delta \quad \rightarrow -\frac{1}{a'_C} + \frac{1}{2} r'_0 M T^{*L} + r'_1 M^2 (T^{*L})^2 \; + \; \ldots}$$

$$\frac{1}{a'_C} = \frac{1}{a_C} - \frac{\alpha\ r_0 M\,\mathcal{I}}{2\pi L} \; + \; \mathcal{O}(\alpha^2;\alpha/L^2)$$

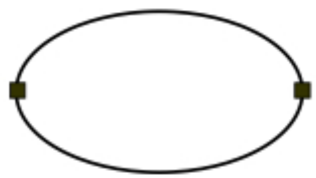
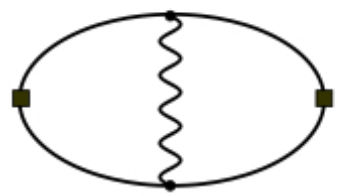
$$r'_0 = r_0 + \frac{4\ \alpha\ r_1 M\ \mathcal{I}}{\pi L} \; + \; \mathcal{O}(\alpha^2;\alpha/L^2)$$

Kinematically shifted effective range parameters

$$(\mathcal{I} = \pi c_1)$$

Source  Sink

$$= S^\dagger \frac{1}{1/J_0^L(E^*) - C^L(E^*)} S \quad \longrightarrow \quad \boxed{\frac{1}{C^L(E^*)} = J_0^L(E^*)}$$

$J_0^\infty(E^*) =$  $+$ 

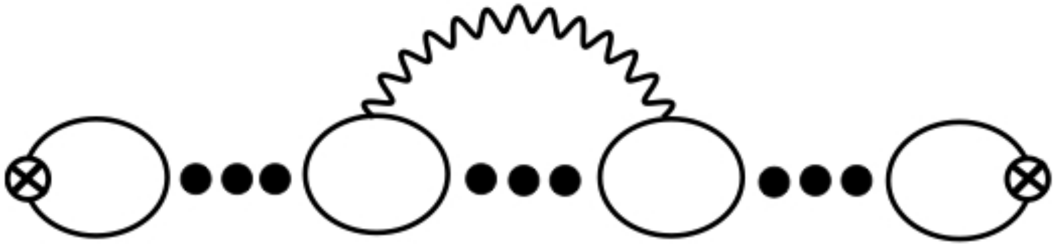
UV same as the continuum

$$-M \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2 - p^2} + 4\pi\alpha M^2 \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \frac{1}{q^2 - p^2} \frac{1}{k^2 - p^2} \frac{1}{|\mathbf{q} - \mathbf{k}|^2}$$

FV with PBCs

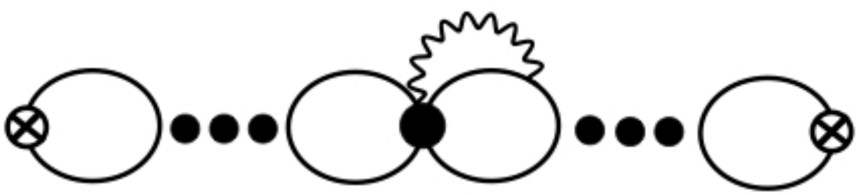
$$-\frac{M}{4\pi^2 L} \sum_{\mathbf{n}}^{\Lambda_n} \frac{1}{|\mathbf{n}|^2 - \tilde{p}^2} + \frac{\alpha M^2}{16\pi^5} \sum_{\mathbf{n}}^{\Lambda_n} \sum_{\mathbf{m} \neq \mathbf{n}}^{\infty} \frac{1}{|\mathbf{n}|^2 - \tilde{p}^2} \frac{1}{|\mathbf{m}|^2 - \tilde{p}^2} \frac{1}{|\mathbf{n} - \mathbf{m}|^2} \qquad \tilde{p} = Lp/2\pi$$

✓ Radiation *not* suppressed by α



$$\sim \ (\sqrt{M/L})^n$$

Infrared suppressed!



$$\delta C^{(FV)}(E^*) \ = \ -\alpha \left(\frac{4a_C^2 r_0}{L}\,\mathcal{I} \ + \ .... \right) \qquad J_0^L \ \rightarrow \ J_0^L \ + \ (J_0^L)^2\,\delta C^{(FV)}$$

$$(\mathcal{I} = \pi c_1)$$

Infrared suppressed but let's keep it..

✓ Quantization condition:

$$\boxed{-\frac{1}{a'_C} + \frac{1}{2} r'_0 p^2 + \ldots = \frac{1}{\pi L} \mathcal{S}^C(\tilde{p}) + \alpha M \left[\ln\left(\frac{4\pi}{\alpha M \mathrm{L}} \right) - \gamma_E \right]} + \ldots$$

Valid up to inelastic threshold!

$$\mathcal{S}^C(x) \equiv \mathcal{S}(x) - \frac{\alpha M L}{4\pi^3} \mathcal{S}_2(x) + \frac{\alpha\, M a_C^2 r_0}{\pi^2 L^2} \mathcal{I}\left[\mathcal{S}(x)\right]^2$$

$$\mathcal{S}_2(x) \equiv \sum_{\mathbf{n}}^{\Lambda_n} \sum_{\mathbf{m} \neq \mathbf{n}}^{\infty} \frac{1}{|\mathbf{n}|^2 - x^2} \frac{1}{|\mathbf{m}|^2 - x^2} \frac{1}{|\mathbf{n} - \mathbf{m}|^2} - 4\pi^4 \ln \Lambda_n$$

✓ Note the breakdown of universality

+ ... contains infrared suppressed, subleading QED, and 1/M effects

✓ Define a scale-dependent scattering length:

$$\boxed{\frac{1}{a(\mu)} \equiv \frac{4\pi}{MC(0;\mu)} = \frac{1}{a_C} + \alpha M \left[\ln\left(\frac{2\mu}{\alpha M}\right) - \gamma_E \right]}$$

$\overline{MS}_{FV}$

exhibits the impossibility of separating QCD and QED contributions

$$p\cot\bar{\delta}(p,\mu) \equiv -\frac{1}{a(\mu)} + \frac{1}{2} r_0 p^2 + \ldots$$

✓ Alternate form of QC:

$$\boxed{p\cot\bar{\delta}'(p,2\pi/L) \equiv -\frac{1}{a'(2\pi/L)} + \frac{1}{2} r_0' p^2 + \ldots = \frac{1}{\pi L} \mathcal{S}^C(\tilde{p})}$$

leading QED effects from distance scales $> 2\pi/L$ are removed

Approximate formulas

✓ Ground state energy level:

$$\Delta E_0^C = \frac{4\pi\, a_C}{M\, L^3}\left\{1 - \left(\frac{a_C}{\pi\, L}\right)\mathcal{I} + \left(\frac{a_C}{\pi\, L}\right)^2 \left[\mathcal{I}^2 - \mathcal{J}\right] + \ldots\right\}$$

$$- \frac{2\alpha\, a_C}{L^2\pi^2}\left\{\mathcal{J} + \left(\frac{a_C}{\pi\, L}\right)\left[\mathcal{K} - \mathcal{I}\mathcal{J} - \tilde{\mathcal{R}}/2\right] + \left(\frac{a_C}{\pi\, L}\right)^2\left[\tilde{\mathcal{R}}\mathcal{I} + \mathcal{I}^2\mathcal{J} - 2\mathcal{J}^2 - 2\mathcal{I}\mathcal{K} + \mathcal{L} - \mathcal{R}_{24}\right] + \frac{a_C r_0 \pi^2}{L^2}\mathcal{I} + \ldots\right\}$$

geometric constants: $\mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{L}, \mathcal{R}, \mathcal{R}_{24}$

$$\tilde{\mathcal{R}} \equiv \mathcal{R} - 4\pi^4\left[\ln\left(\frac{4\pi}{\alpha M L}\right) - \gamma_E\right]$$

First excited energy level:

$$\Delta E_1^C = \frac{4\pi^2}{ML^2} - \frac{12\tan\overline{\delta}'}{ML^2}\left(1 + c_1'\tan\overline{\delta}' + c_2'\tan^2\overline{\delta}' + \ldots\right)$$
$$\boxed{+\frac{9\alpha}{4\pi \mathrm{L}}\left(1 + c_{1\alpha}'\tan\overline{\delta}' + \left(c_{2\alpha}' + \frac{8}{3}\log\left(\alpha ML\right)\right)\tan^2\overline{\delta}' + \ldots\right)}$$

geometric constants: $c_1', c_2', c_{1\alpha}', c_{2\alpha}'$

$$\frac{9\alpha}{4\pi \mathrm{L}} \xrightarrow[\mathrm{L}\,=\,10\ \mathrm{fm}]{} \sim 100\ \mathrm{keV}$$

Coulomb effects are very small!

Bound state

✓ Bound state QC with binding momentum κ :

$$-\frac{1}{a_C} - \frac{1}{2} r_0 \kappa^2 = -\kappa - \alpha M \left(\gamma_E + \log \left(\frac{\alpha M}{4\kappa} \right) \right) - \frac{\alpha M}{2\pi \kappa L} (1 - \kappa r_0) \mathcal{I}$$

$$\boxed{B^C = \frac{\kappa_0^2}{M} - \frac{2\alpha\kappa_0}{1 - \kappa_0 r_0} \left[\gamma_E + \log \left(\frac{\alpha M}{4\kappa_0} \right) \right] - \frac{\alpha}{\pi L} \mathcal{I} + \ldots}$$

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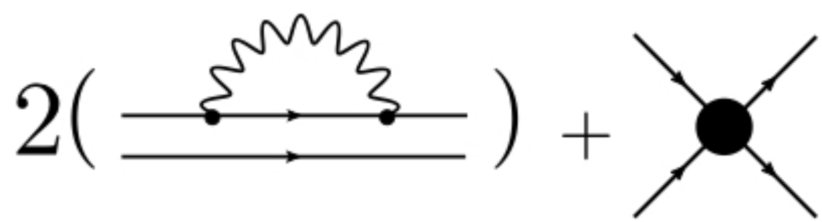
$$\boxed{B^C = \frac{\kappa_0^2}{M} - \frac{2\alpha\kappa_0}{1-\kappa_0 r_0}\left[\gamma_E + \log\left(\frac{\alpha M}{4\kappa_0} \right) \right] - \frac{\alpha}{\pi L}\mathcal{I} + \ldots}$$

✓ Two contributions in the *compact* limit:

$$\delta M_{BS}^{(FV)} = 2\delta M^{(FV)} - \delta B^C = 2\left(\frac{\alpha}{2\pi L}\mathcal{I} \right) + \frac{\alpha}{\pi L}\mathcal{I} + \ldots = \boxed{\frac{2\alpha}{\pi L}\mathcal{I}} + \ldots$$



Summary

- Electromagnetism essential for *ab initio* nuclear physics
- QED in a FV with PBCs is consistent without zero mode
- Two particle QC is tricky: violates Luscher condition
- There is hierarchy of scales that allows perturbative treatment of Coulomb
- It would be very interesting to look at a simple system in lattice QCD.. $\pi^+\pi^+$?