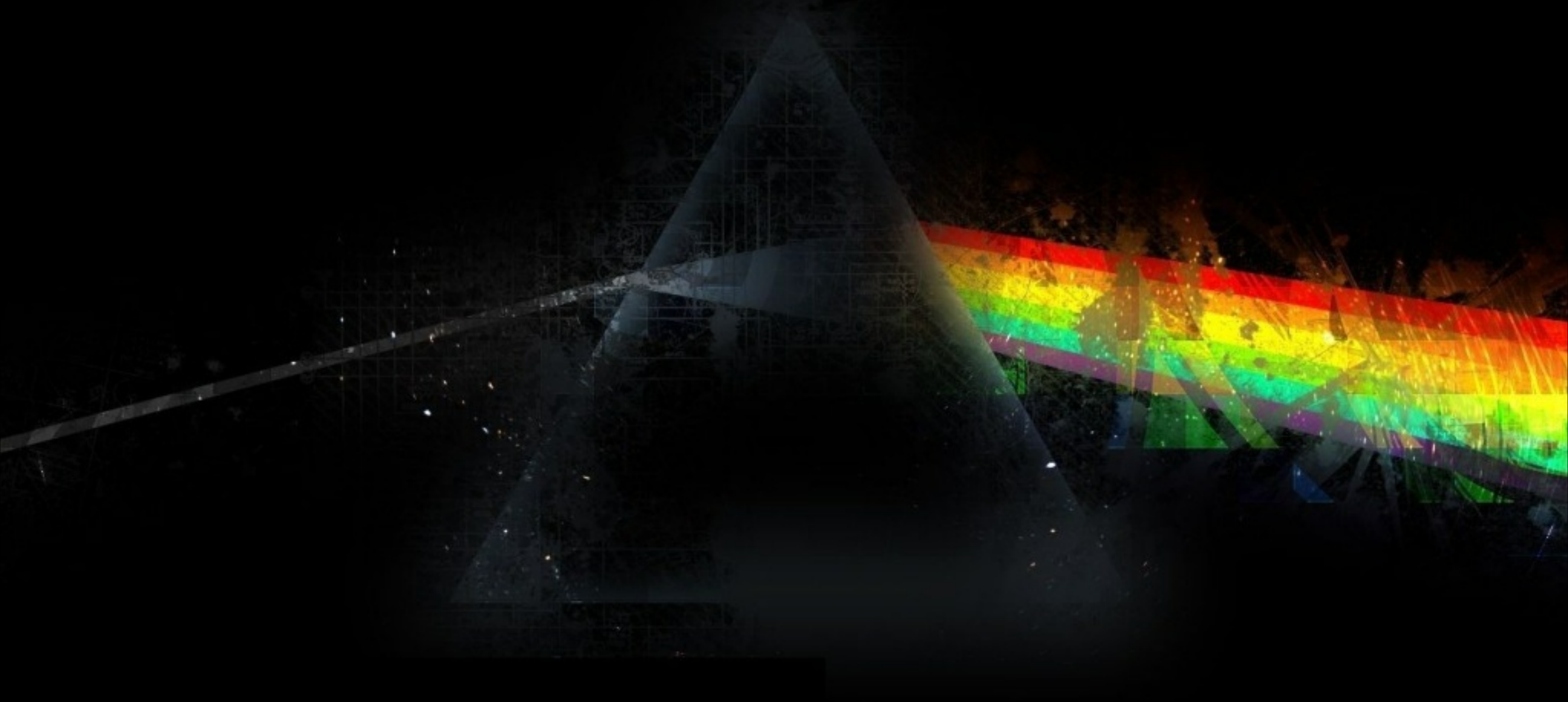
ZHENGFENG JI, IQC, U. WATERLOO

BINARY CONSTRAINT SYSTEM GAMES: CHARACTERIZATION AND REDUCTIONS



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Binary Constraint System Games: Characterization and Reductions

Richard Cleve and Rajat Mittal
arXiv:1209.2729

Zhengfeng Ji
arXiv:1310.3794

Introduction

Binary Constraint System Games

- Two-player one-round games (classical)

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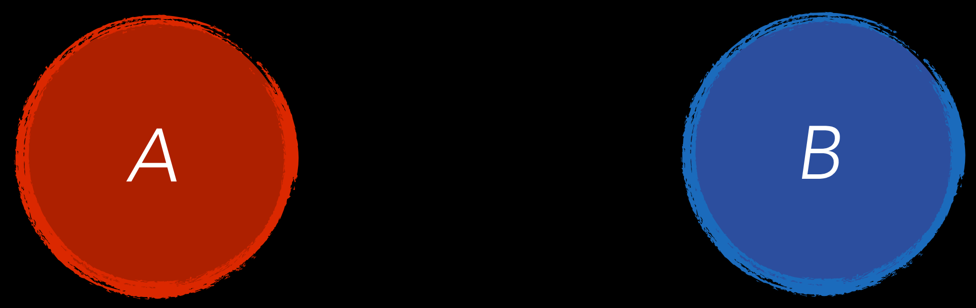

Binary Constraint System Games

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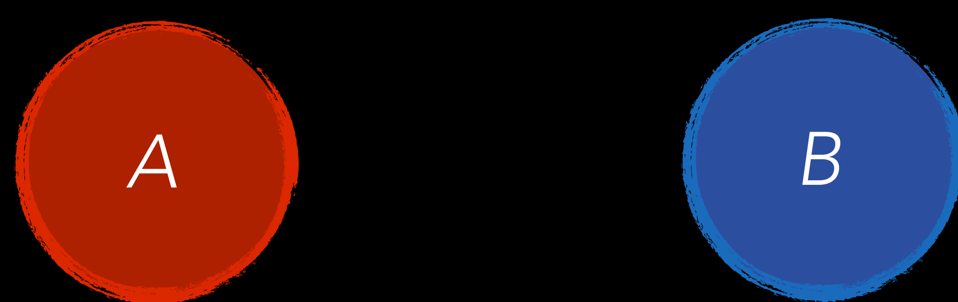

No communication!

Binary Constraint System Games

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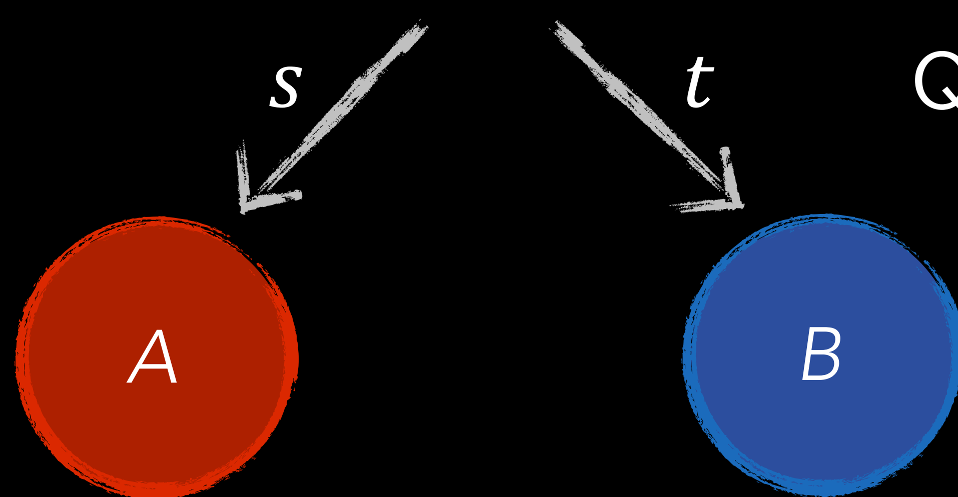

Questions

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Binary Constraint System Games

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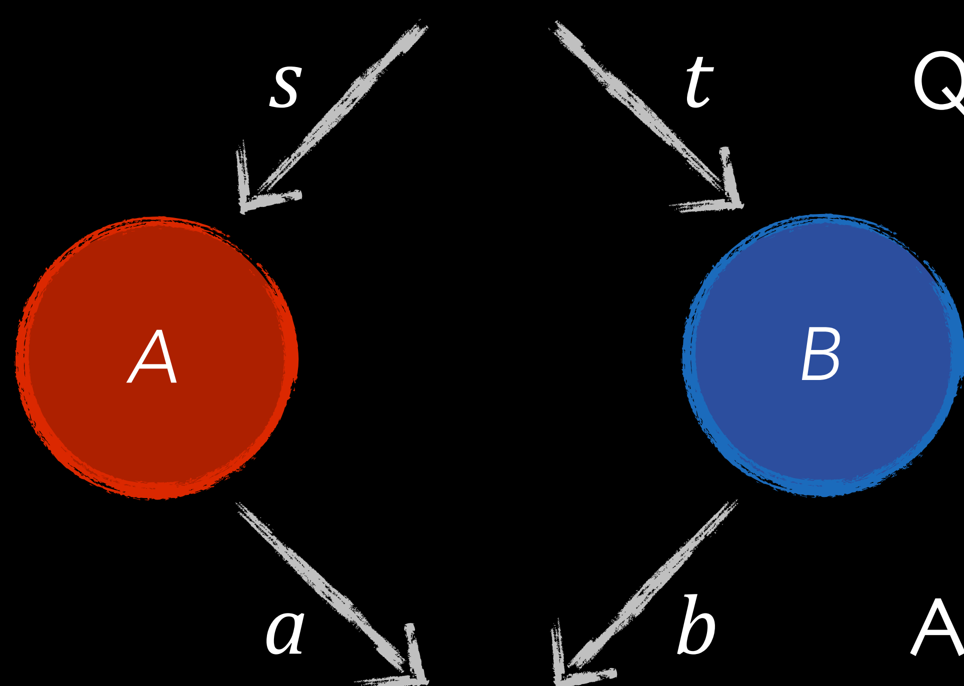


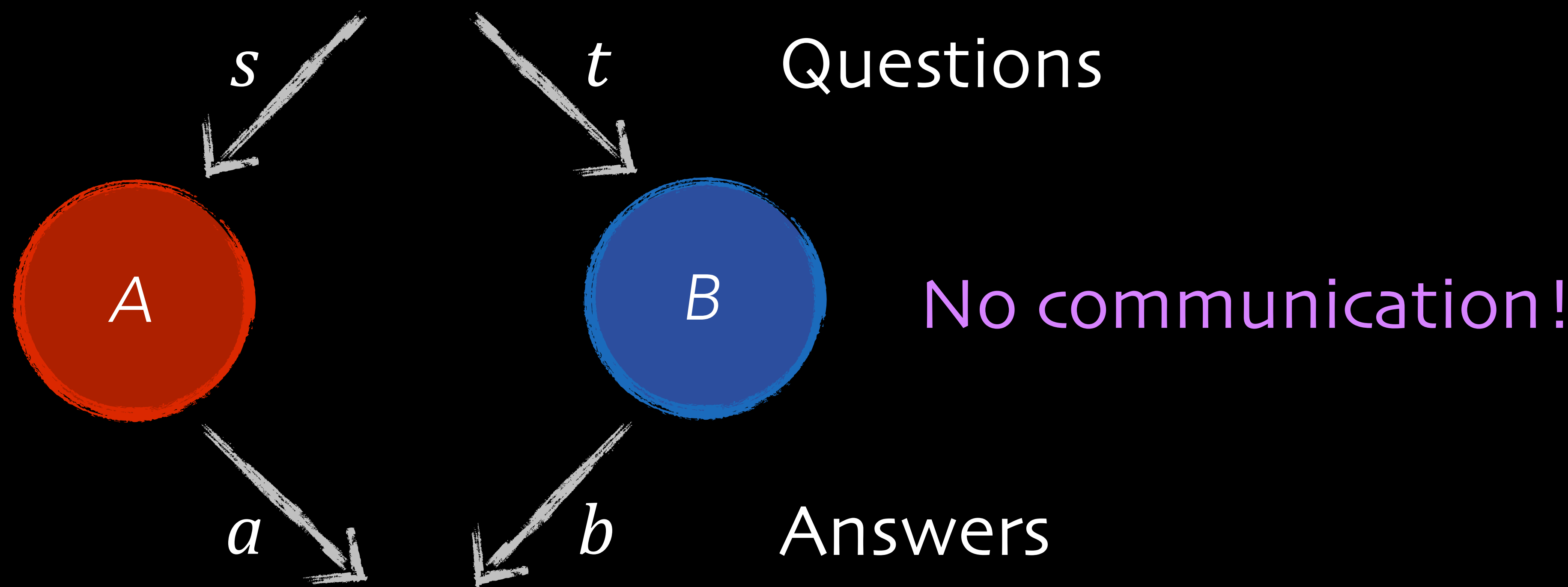
Questions

No communication!

Answers

Binary Constraint System Games

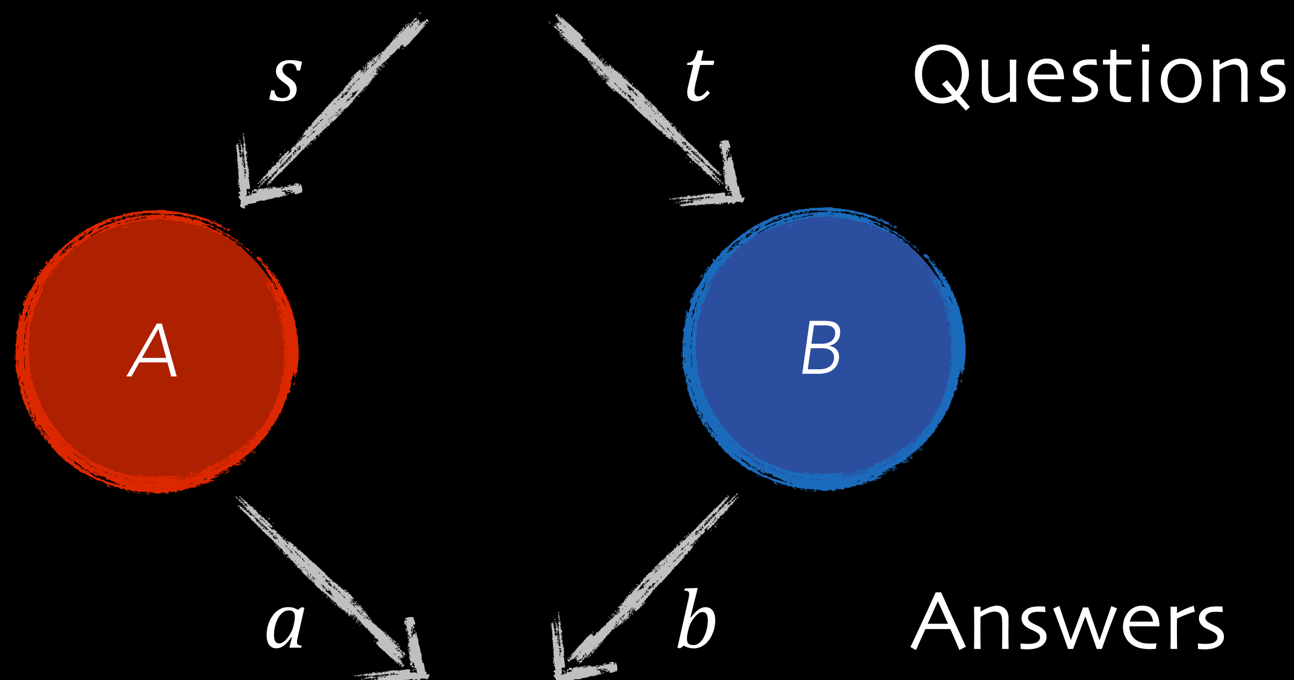
- Two-player one-round games (classical)



Accept / Reject (a, b, s, t)

Binary Constraint System Games

- Two-player one-round games (nonlocal)



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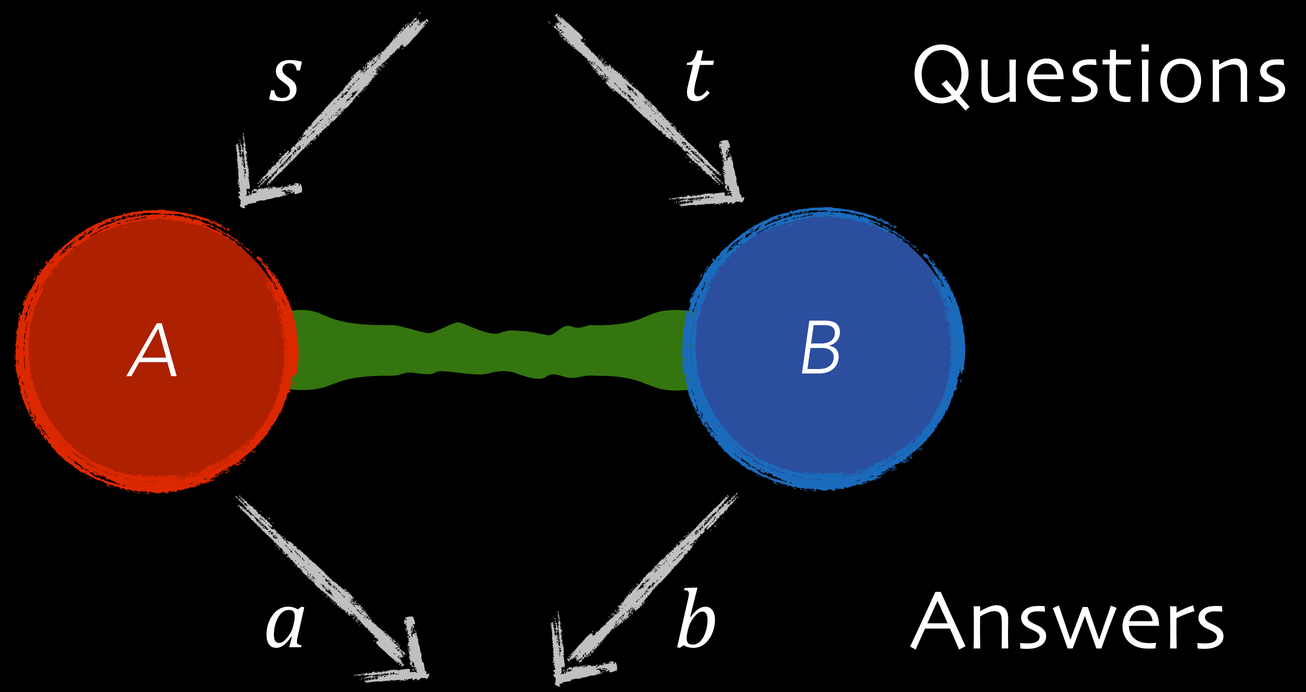

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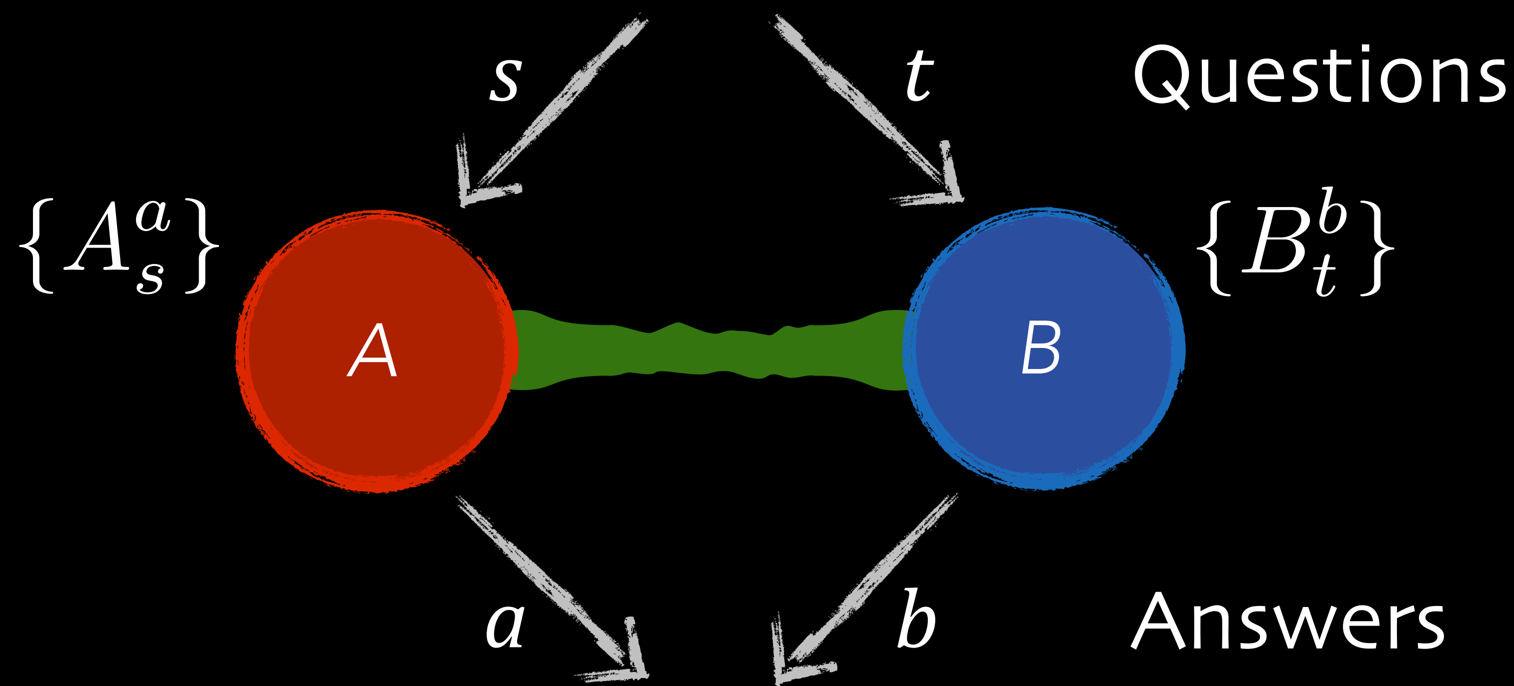

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BINARY CONSTRAINT SYSTEM GAMES

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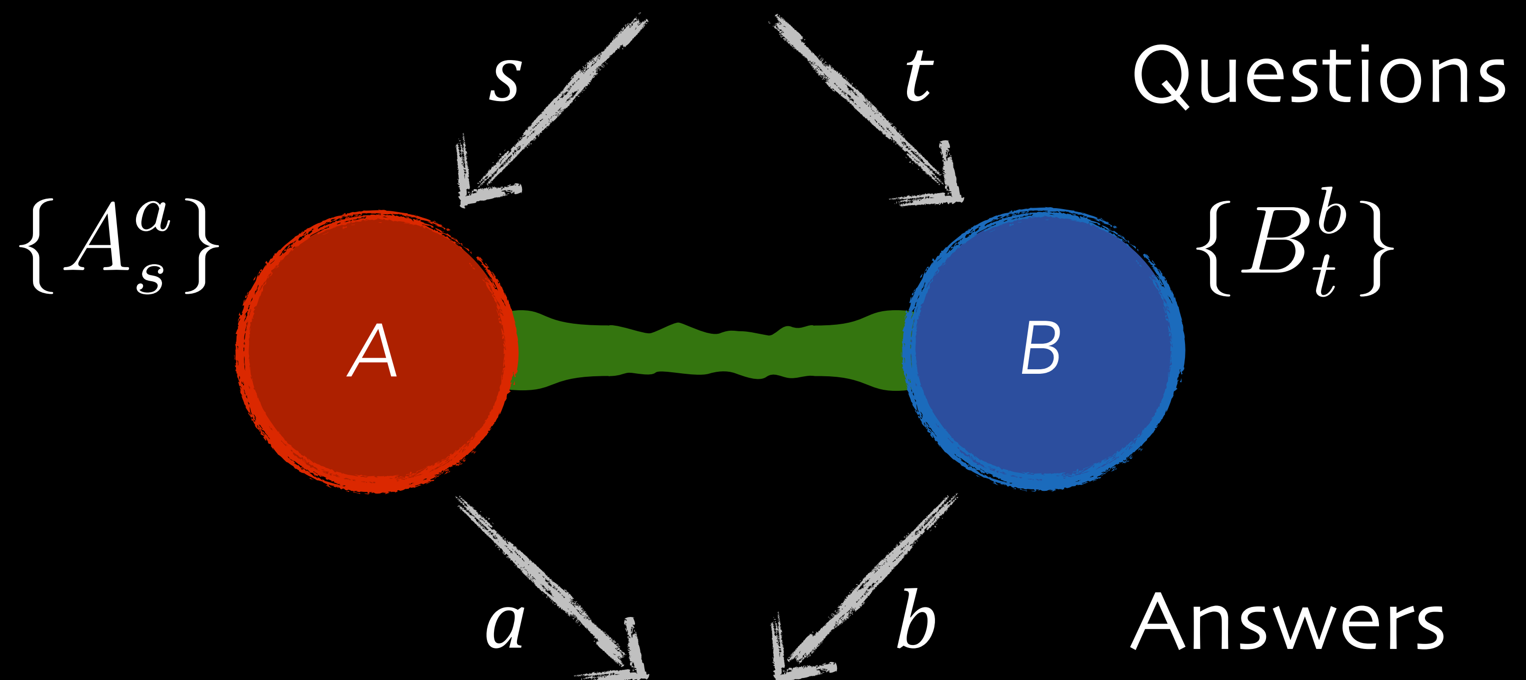

Accept / Reject (a, b, s, t)

Perfect Quantum Strategy

Binary Constraint System Games

Binary Constraint System Games

Variables: $x_1, x_2, \ldots, x_n$

Constraints: $C_1, C_2, \ldots, C_m$

Binary Constraint System Games

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Constraints: $C_1, C_2, \ldots, C_m$

$$x_1 \oplus x_2 = 0,$$
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$$\begin{aligned} x_1 \oplus x_2 \oplus x_3 &= 0, & x_1 \oplus x_4 \oplus x_7 &= 0, \\ x_4 \oplus x_5 \oplus x_6 &= 0, & x_2 \oplus x_5 \oplus x_8 &= 0, \\ x_7 \oplus x_8 \oplus x_9 &= 0, & x_3 \oplus x_6 \oplus x_9 &= 1. \end{aligned}$$

Binary Constraint System Games

Variables: $x_1, x_2, \ldots, x_n \in \{0, 1\}$

Constraints: $C_1, C_2, \ldots, C_m$

$$\begin{array}{ccccc} x_1 & \text{---} & x_2 & \text{---} & x_3 \\ | & & | & & \| \\ x_4 & \text{---} & x_5 & \text{---} & x_6 \\ | & & | & & \| \\ x_7 & \text{---} & x_8 & \text{---} & x_9 \end{array}$$

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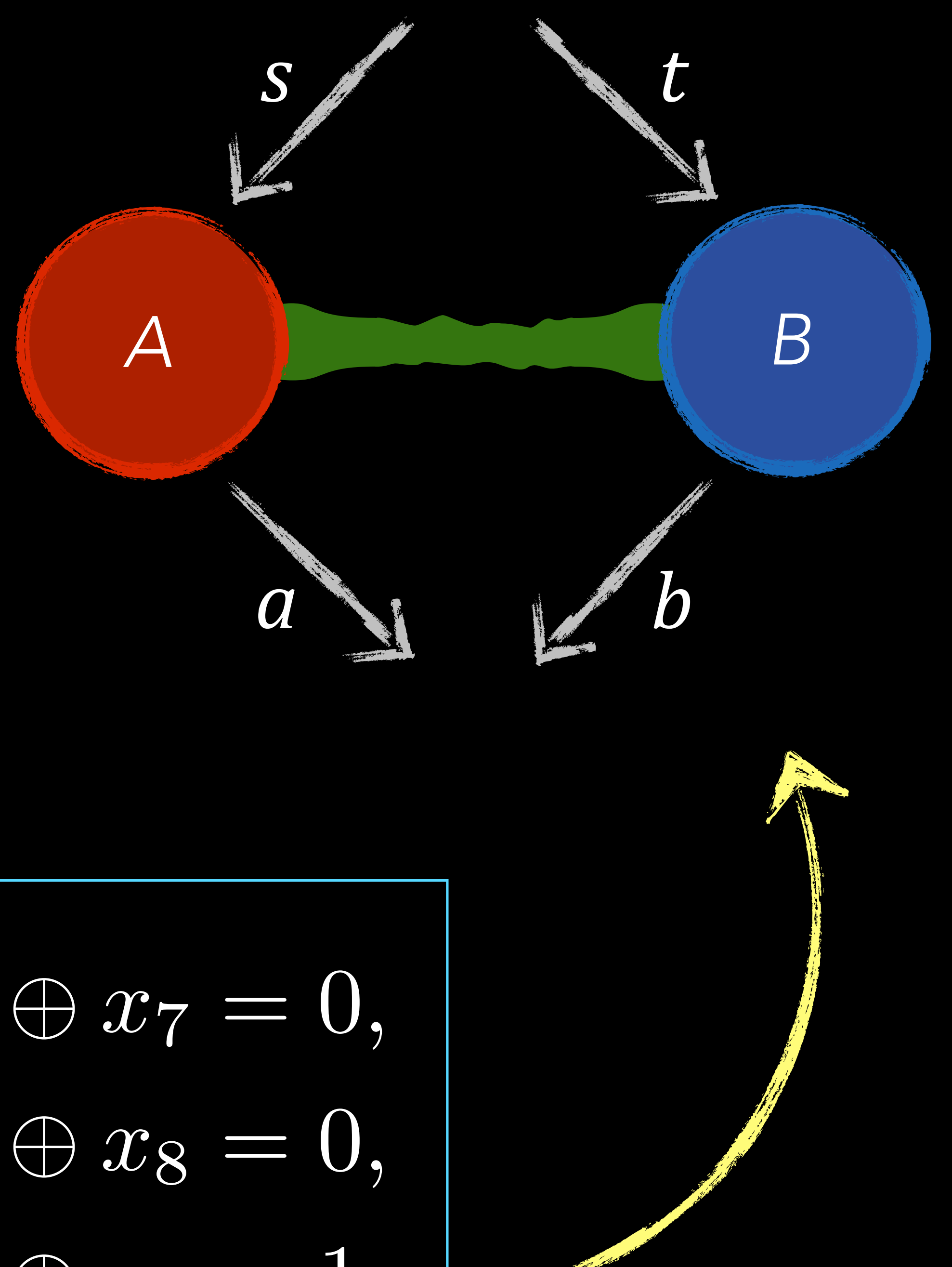

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\end{aligned}
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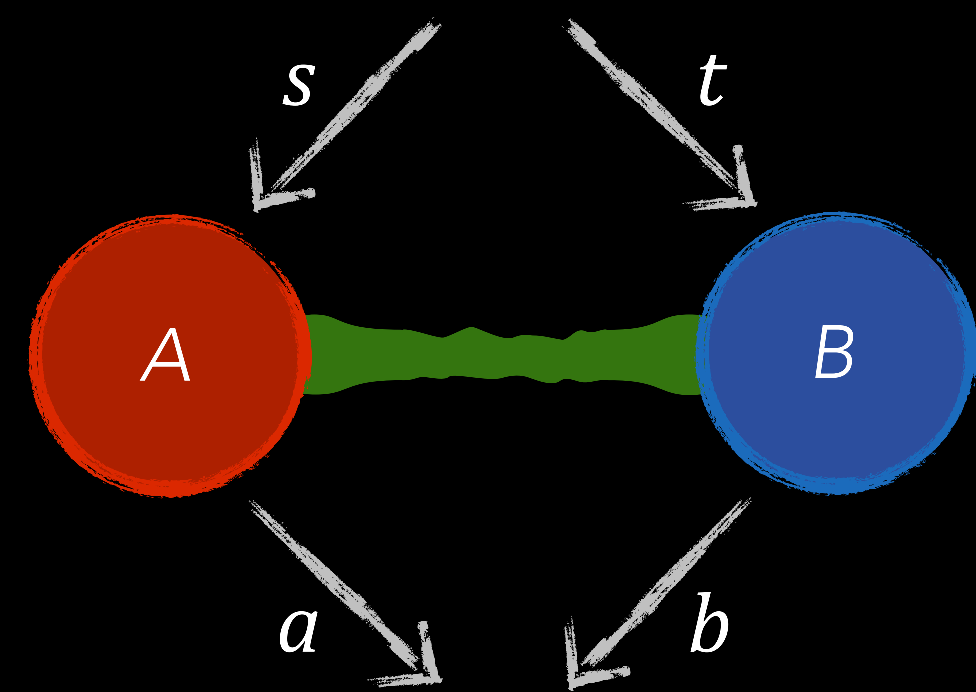
Binary Constraint System Games

I. Choose a random C_s and a random x_t from C_s.

II. Accept if
   1. a satisfies C_s, and
   2. a and b are consistent.



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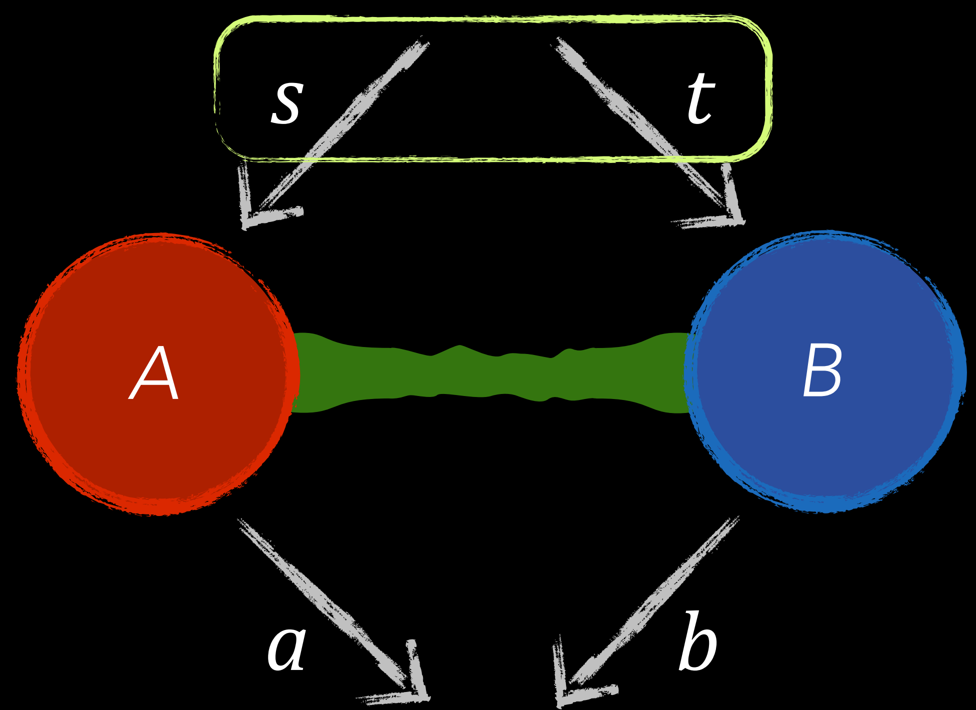

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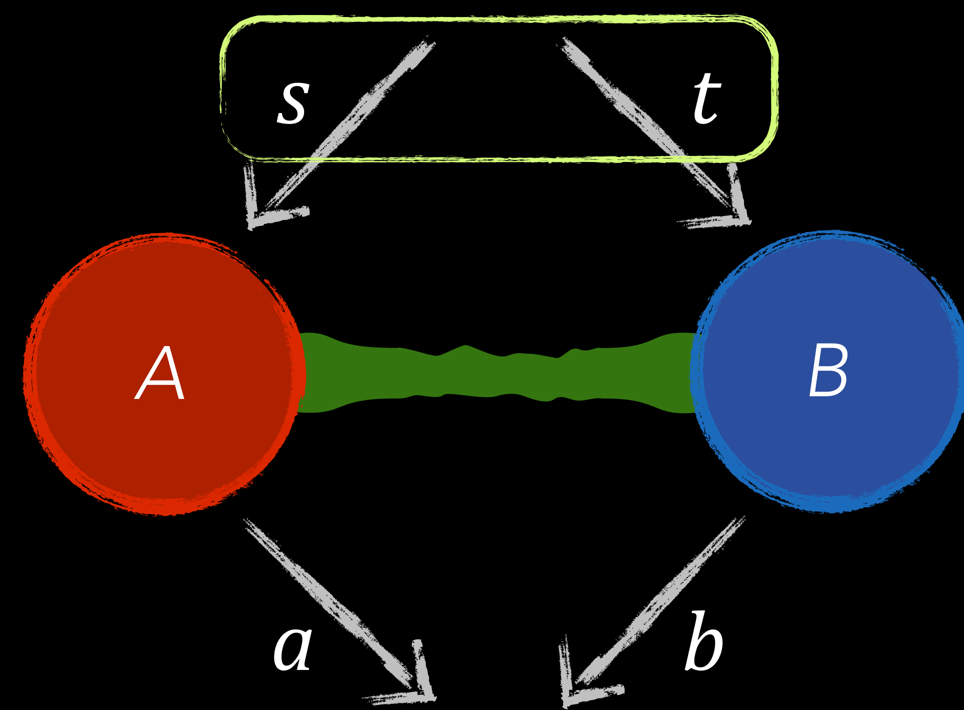

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$s = 5$

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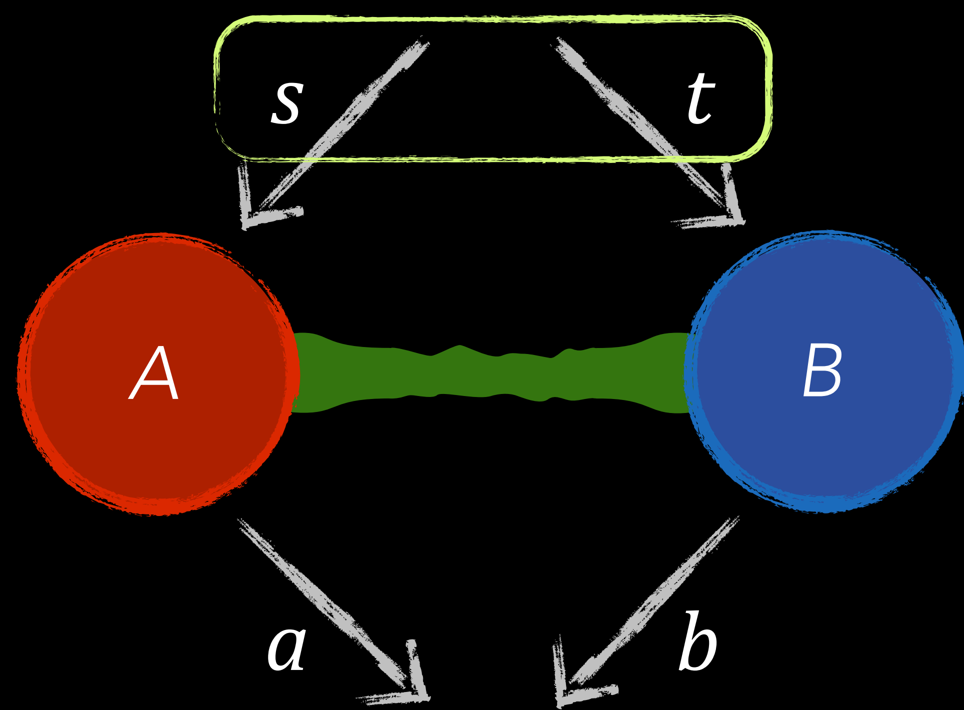

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$s = 5$

$t = 8$

Binary Constraint System Games

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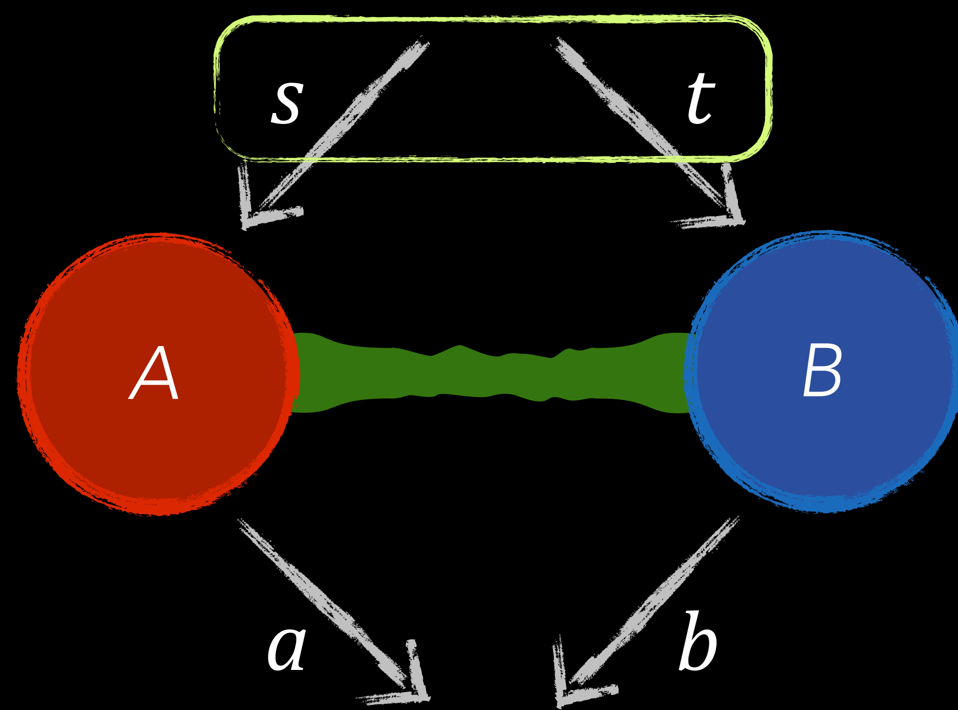

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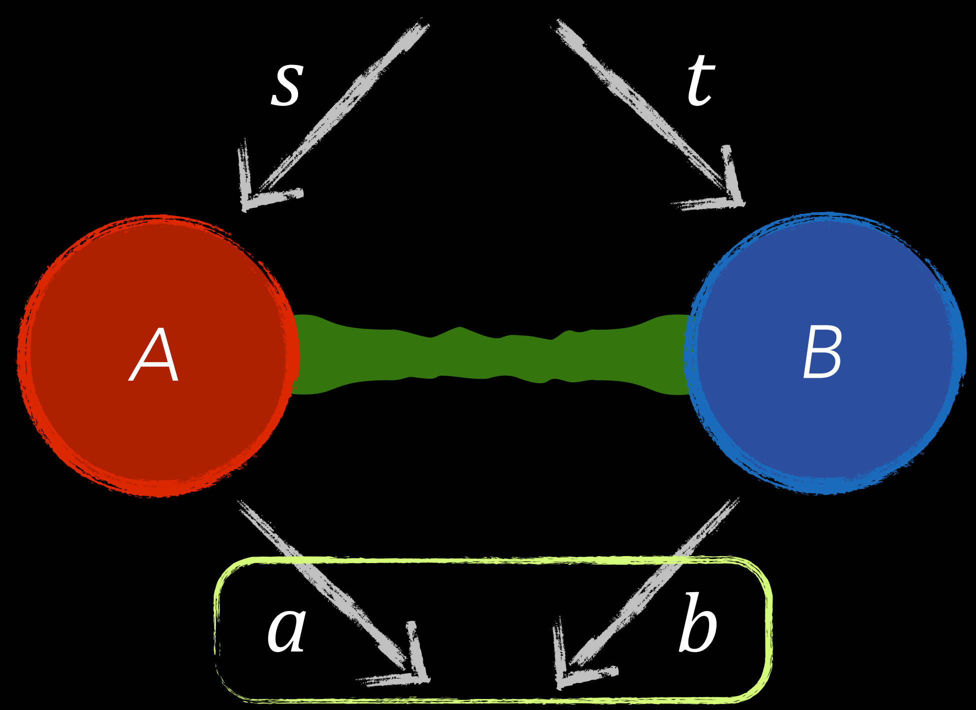

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Binary Constraint System Games

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CHSH game

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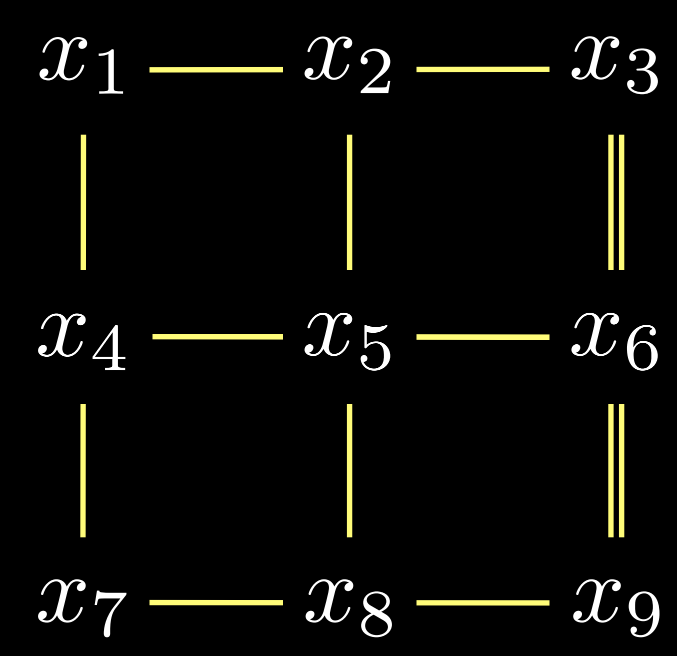

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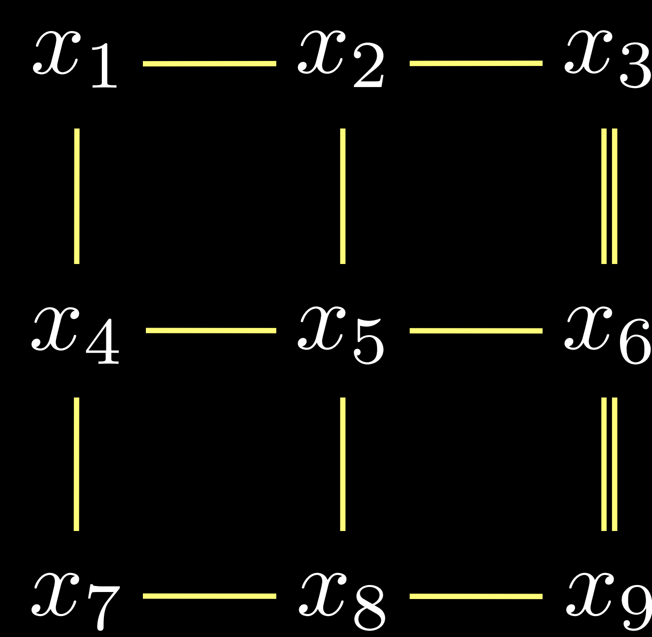


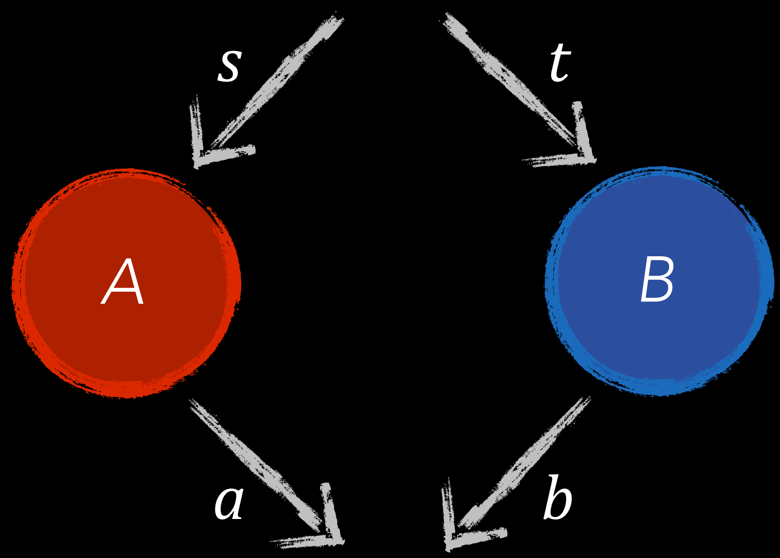
Magic square game

$$\begin{aligned} x_1 \oplus x_2 \oplus x_3 &= 0, & x_1 \oplus x_4 \oplus x_7 &= 0, \\ x_4 \oplus x_5 \oplus x_6 &= 0, & x_2 \oplus x_5 \oplus x_8 &= 0, \\ x_7 \oplus x_8 \oplus x_9 &= 0, & x_3 \oplus x_6 \oplus x_9 &= 1. \end{aligned}$$

Characterization

CHARACTERIZATION OF BCS GAMES

- Classical version



A BCS game has a perfect classical strategy

if and only if

the corresponding BCS has a satisfying assignment

$$\boxed{\begin{aligned} x_1 \oplus x_2 &= 0, \\ x_1 \oplus x_2 &= 1. \end{aligned}}$$

$$x_i \mapsto \nu(x_i) \in \{0, 1\}$$

CHARACTERIZATION OF BCS GAMES

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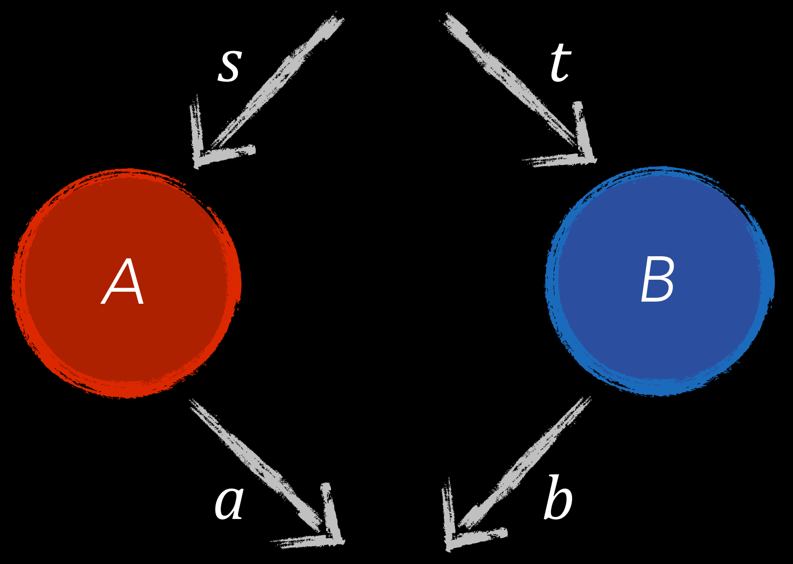

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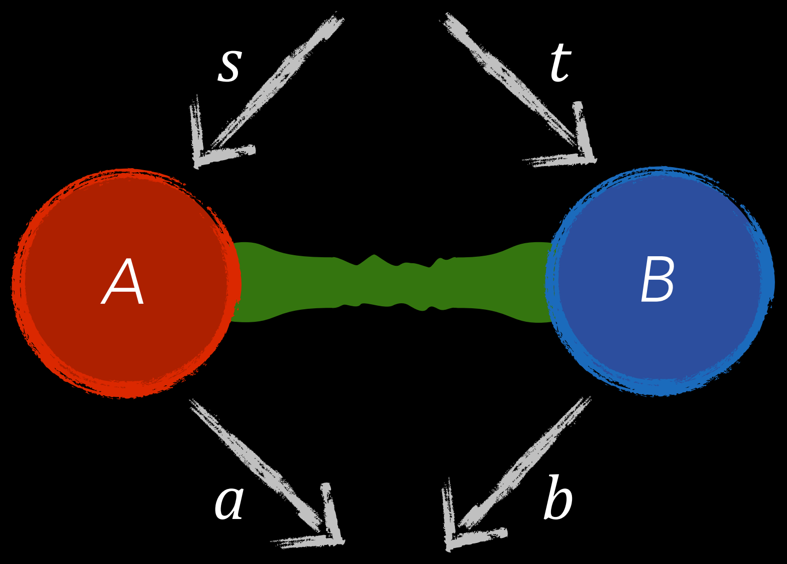
Characterization of BCS Games

- Quantum version



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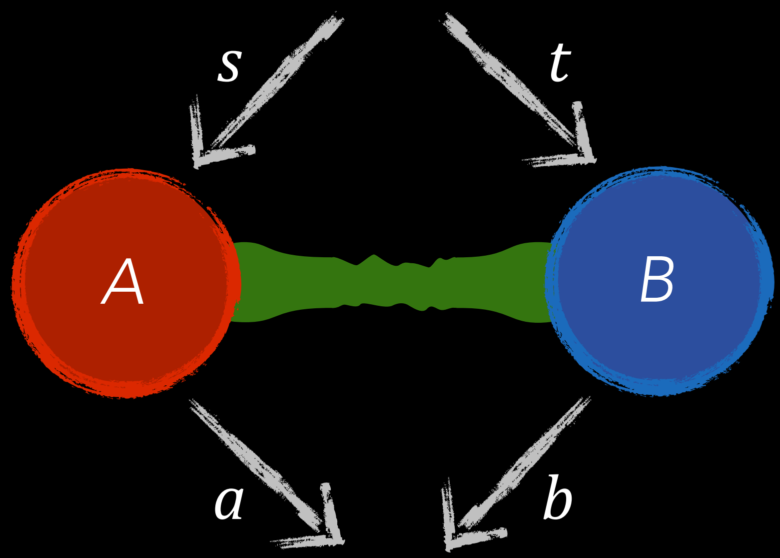
CHARACTERIZATION OF BCS GAMES

- Quantum version



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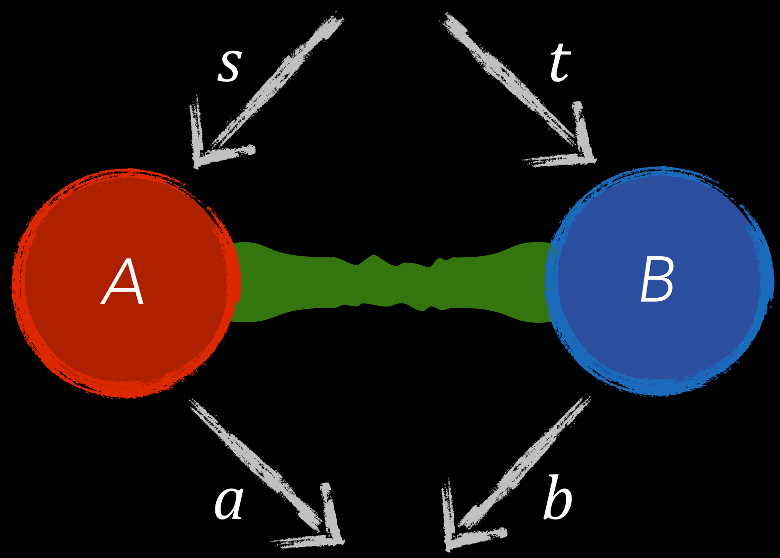
???

CHARACTERIZATION OF BCS GAMES

- Quantum version



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[CLEVE AND MITTAL, ARXIV:1209.2729]

Quantum Satisfying Assignment

Quantum Satisfying Assignment

- Rewrite constraints as polynomials over reals

$$\boxed{\begin{aligned} x_1 \oplus x_2 &= 0, \\ x_1 \oplus x_2 &= 1. \end{aligned}} \longrightarrow \boxed{\begin{aligned} x_1 + x_2 - 2x_1x_2 &= 0, \\ x_1 + x_2 - 1 &= 0. \end{aligned}}$$

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Quantum Satisfying Assignment $x_j \mapsto X_j$

Quantum Satisfying Assignment

- Rewrite constraints as polynomials over reals

$$x_1 \oplus x_2 = 0,$$
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$\rightarrow$

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Quantum Satisfying Assignment $x_j \mapsto X_j$

(a) Satisfy every polynomial constraints.

(b) For all j, $X_j^2 = X_j$.

(c) Each pair of operators X_j, X_k appearing in the same constraint commute.

Quantum Satisfying Assignment

- Rewrite constraints as polynomials over reals

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Locally Commutative Condition

Quantum Satisfying Assignment

- Rewrite constraints as polynomials over reals

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Quantum Satisfiability

Quantum Satisfying Assignment $x_j \mapsto X_j$

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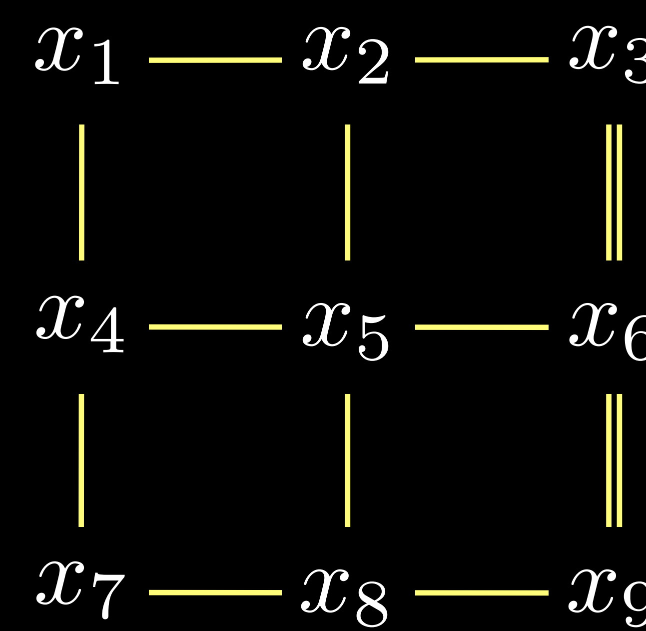
Locally Commutative Condition

Characterization of BCS Games

$$
\begin{array}{ccccc}
x_1 & \text{---} & x_2 & \text{---} & x_3 \\
| & & | & & \| \\
x_4 & \text{---} & x_5 & \text{---} & x_6 \\
| & & | & & \| \\
x_7 & \text{---} & x_8 & \text{---} & x_9
\end{array}
$$

Characterization of BCS Games

- Proof sketch



Characterization of BCS Games

- Proof sketch

  The structure of A's measurement

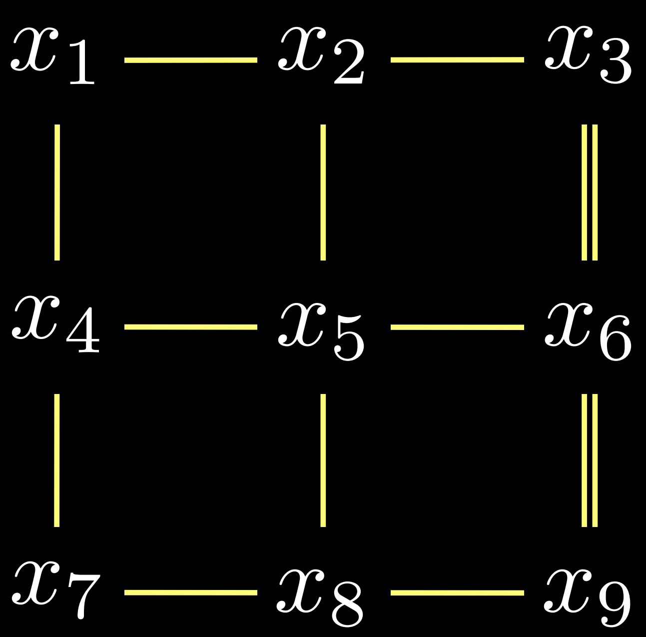
$$\begin{tikzcd}
x_1 \arrow[r, dash] \arrow[d, dash] & x_2 \arrow[r, dash] \arrow[d, dash] & x_3 \arrow[d, equal] \\
x_4 \arrow[r, dash] \arrow[d, dash] & x_5 \arrow[r, dash] \arrow[d, dash] & x_6 \arrow[d, equal] \\
x_7 \arrow[r, dash] & x_8 \arrow[r, dash] & x_9
\end{tikzcd}$$

Characterization of BCS Games

- Proof sketch

  The structure of A's measurement

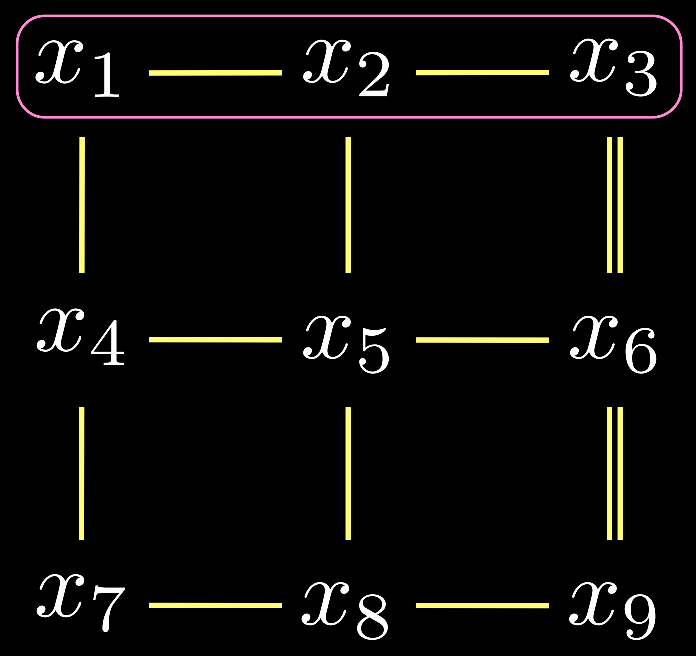
  Assume that A uses projective measurements



CHARACTERIZATION OF BCS GAMES

- Proof sketch

$\Pi_{000}, \Pi_{001}, \cdots, \Pi_{111}$



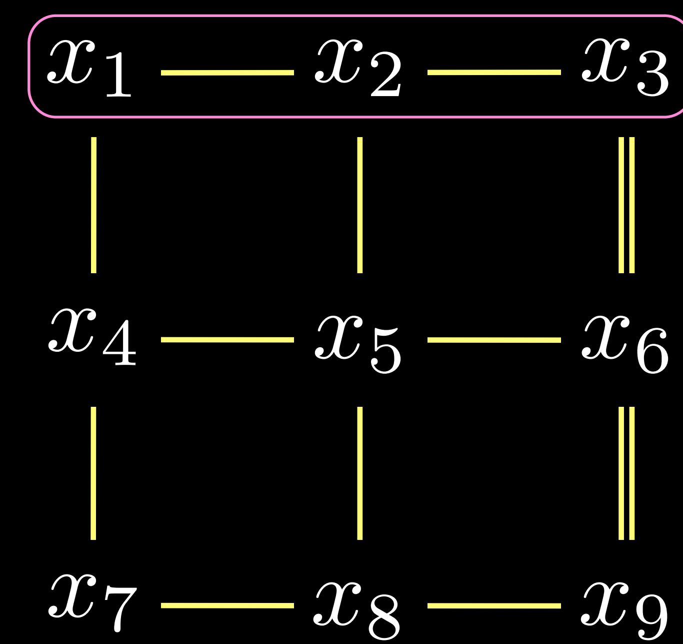
The structure of A's measurement

Assume that A uses projective measurements

CHARACTERIZATION OF BCS GAMES

- Proof sketch

$$\Pi_{000}, \Pi_{001}, \cdots, \Pi_{111}$$



The structure of A's measurement

Assume that A uses projective measurements

$$A_1 = \Pi_{100} + \Pi_{101} + \Pi_{110} + \Pi_{111},$$

$$A_2 = \Pi_{010} + \Pi_{011} + \Pi_{110} + \Pi_{111},$$

$$A_3 = \Pi_{001} + \Pi_{011} + \Pi_{101} + \Pi_{111}.$$

Characterization of BCS Games

- Proof sketch

$$\Pi_{000}, \Pi_{001}, \cdots, \Pi_{111}$$

The structure of A's measurement

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$$\begin{array}{ccccc} \boxed{x_1 & — & x_2 & — & x_3} \\ | & & | & & \| \\ x_4 & — & x_5 & — & x_6 \\ | & & | & & \| \\ x_7 & — & x_8 & — & x_9 \end{array}$$

$$\boxed{A_1 \quad A_2 \quad A_3}$$

$$\boxed{A_4 \quad A_5 \quad A_6}$$

$$\boxed{A_7 \quad A_8 \quad A_9}$$

CHARACTERIZATION OF BCS GAMES

- Proof sketch

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The structure of A's measurement

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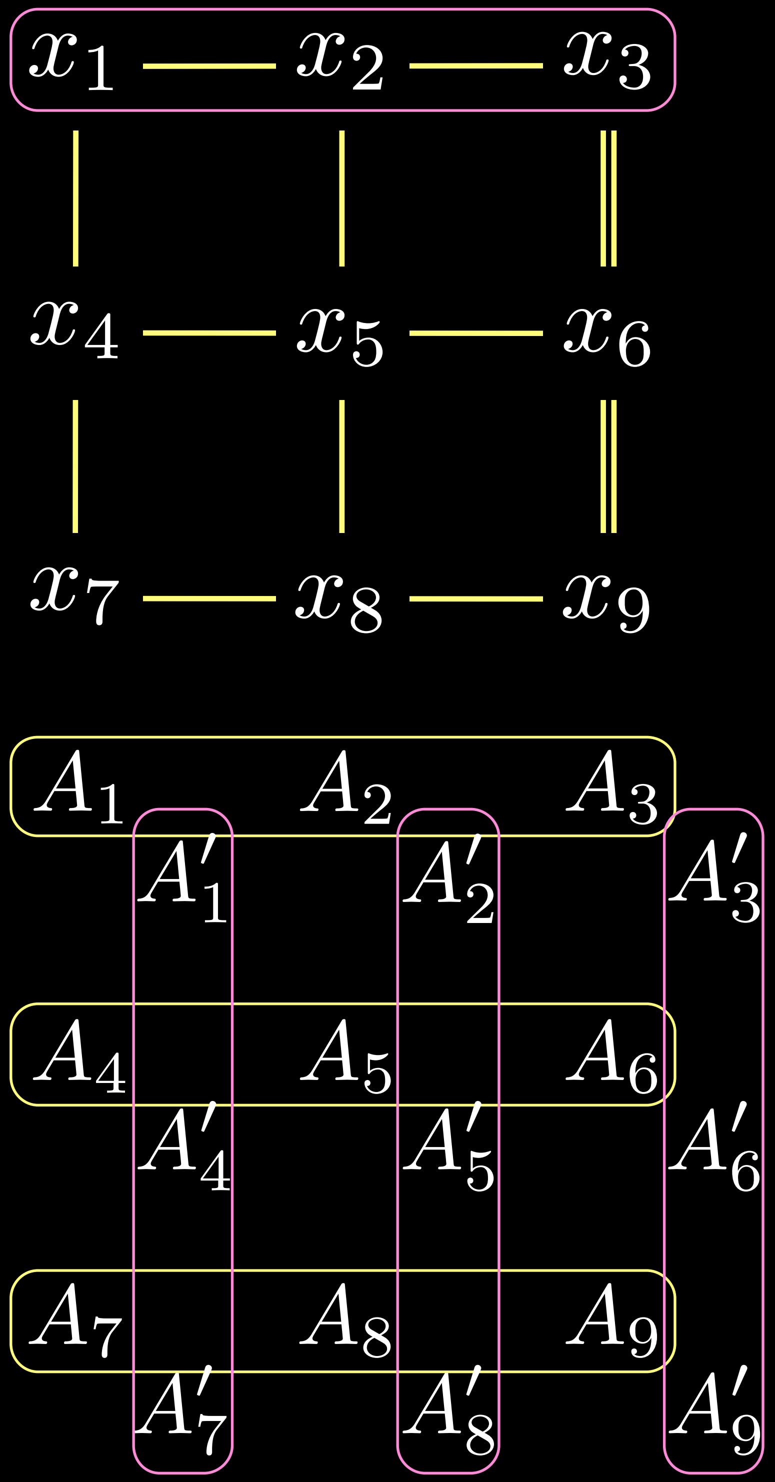

Characterization of BCS Games

- Proof sketch $\Pi_{000}, \Pi_{001}, \cdots, \Pi_{111}$

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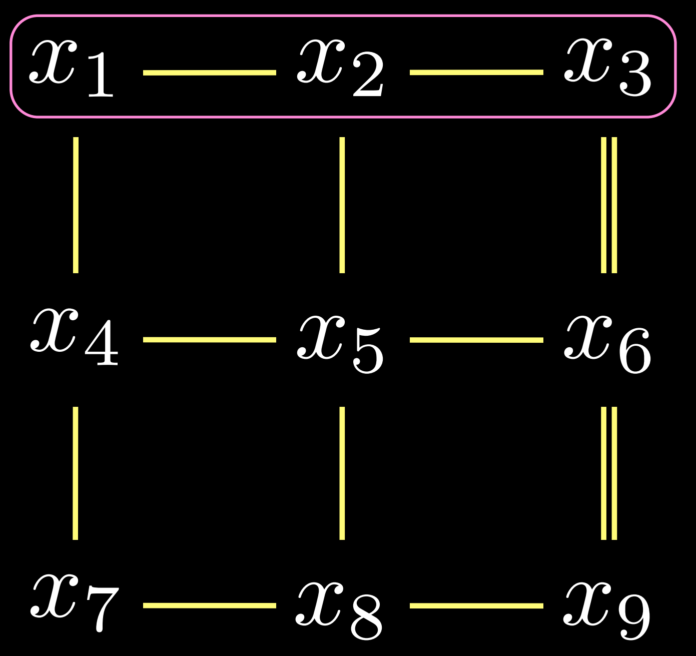

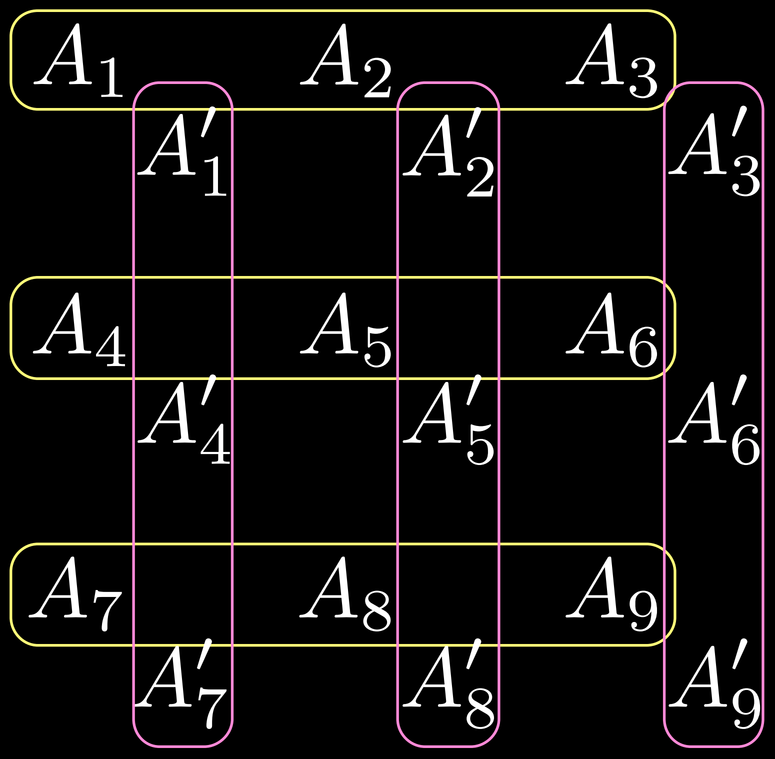

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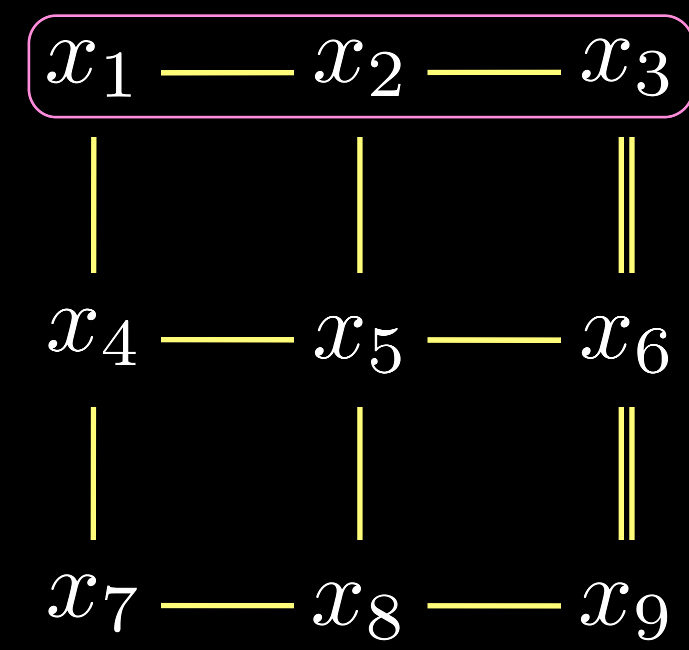

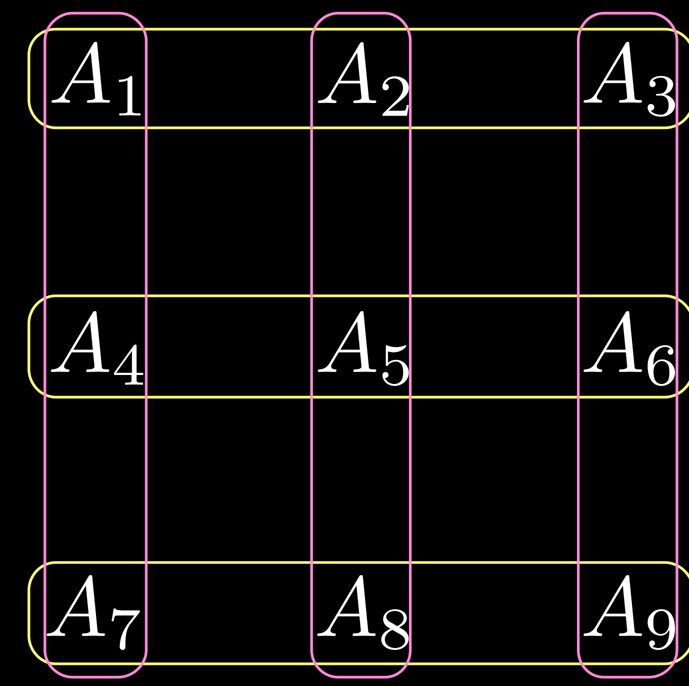

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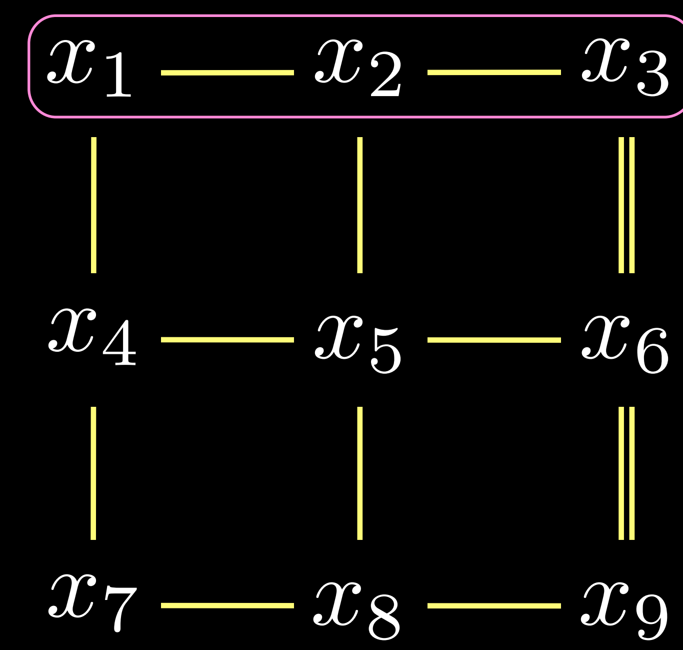

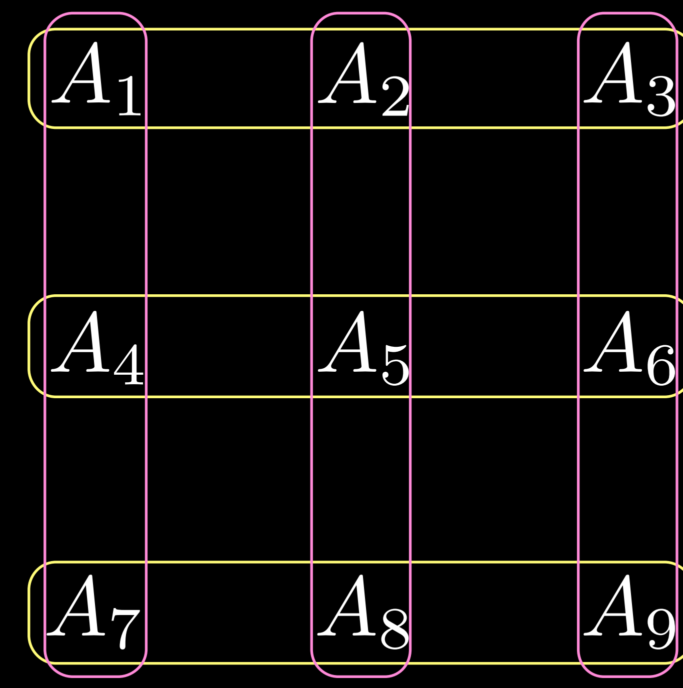

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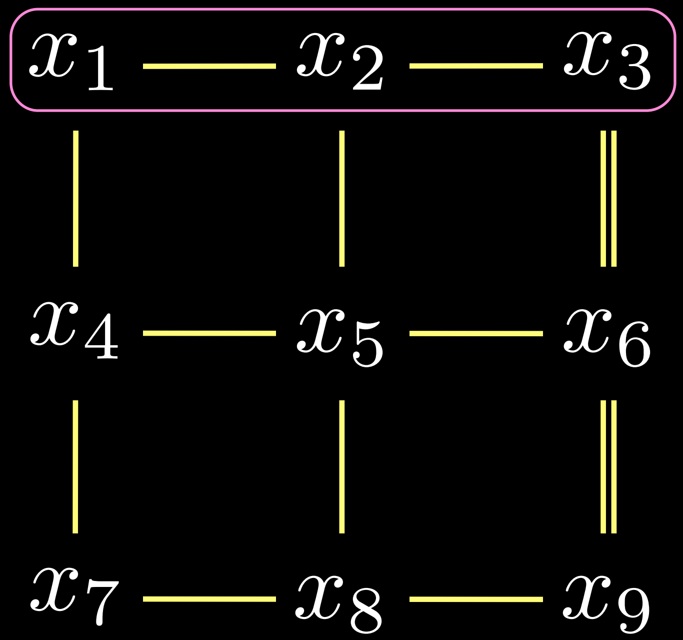
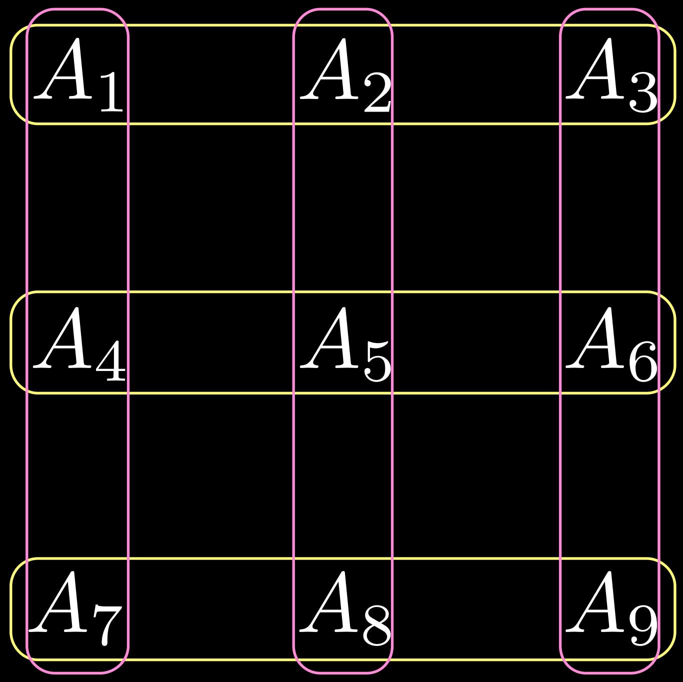


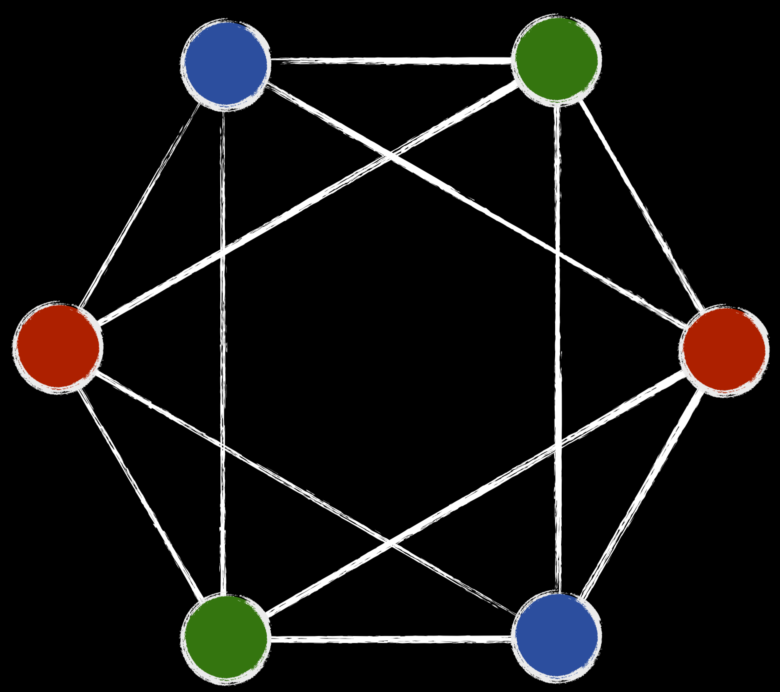


MORE EXAMPLES

Quantum Coloring Game

Quantum Coloring Game

- Graph G=(V,E), Number of colors *k*



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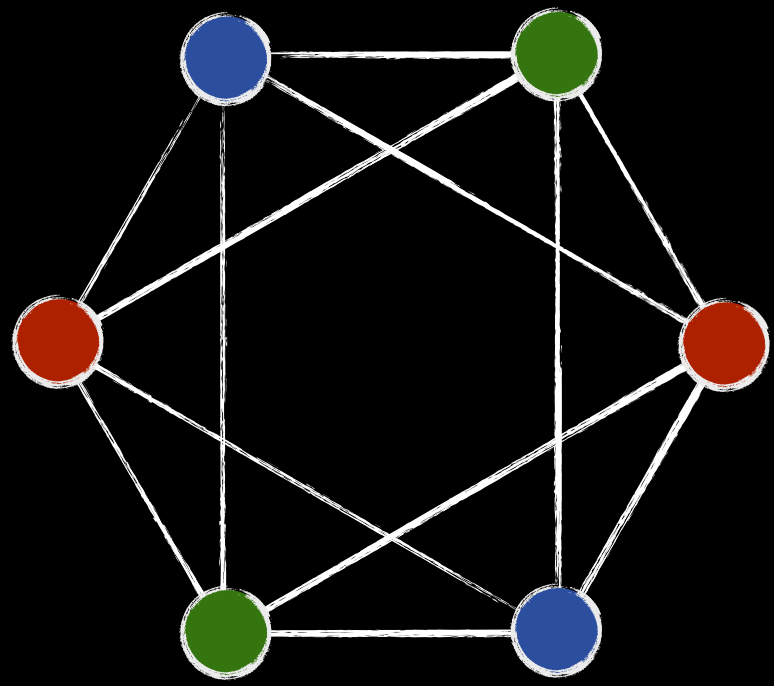

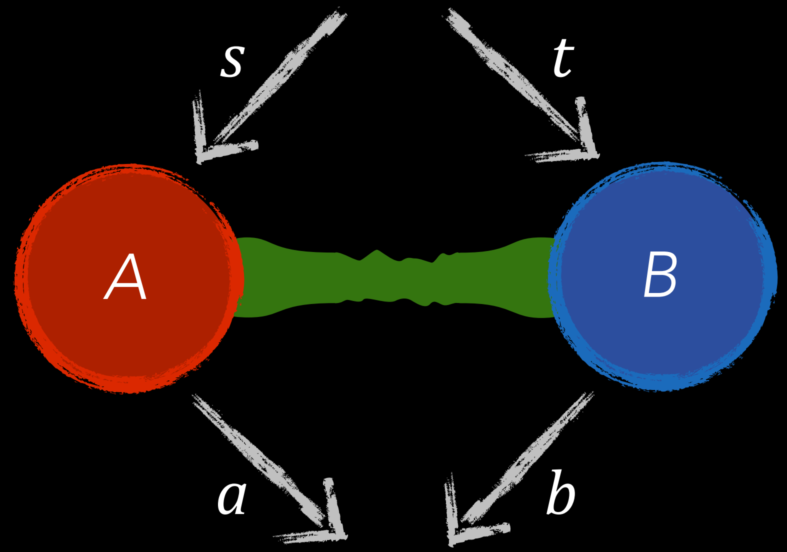

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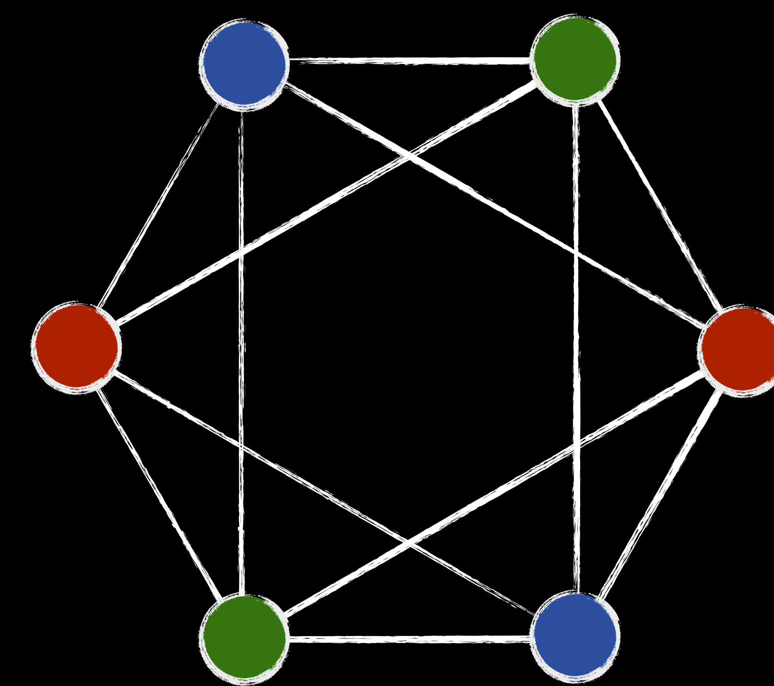

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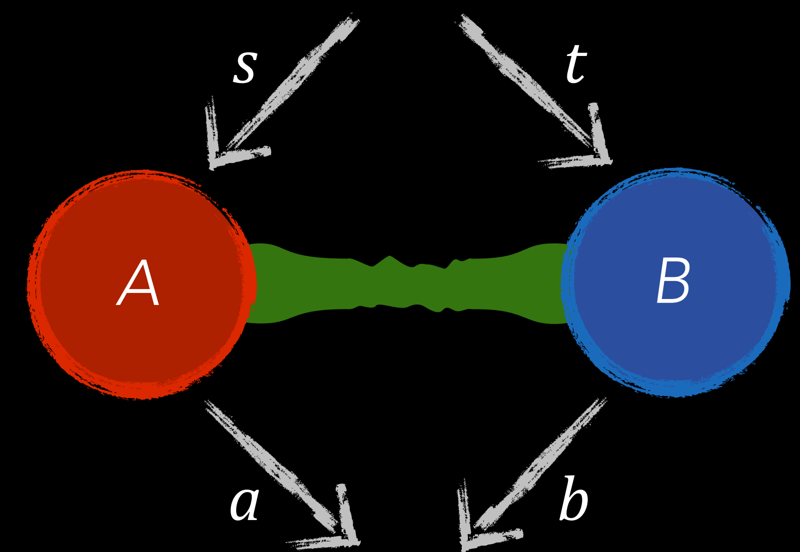

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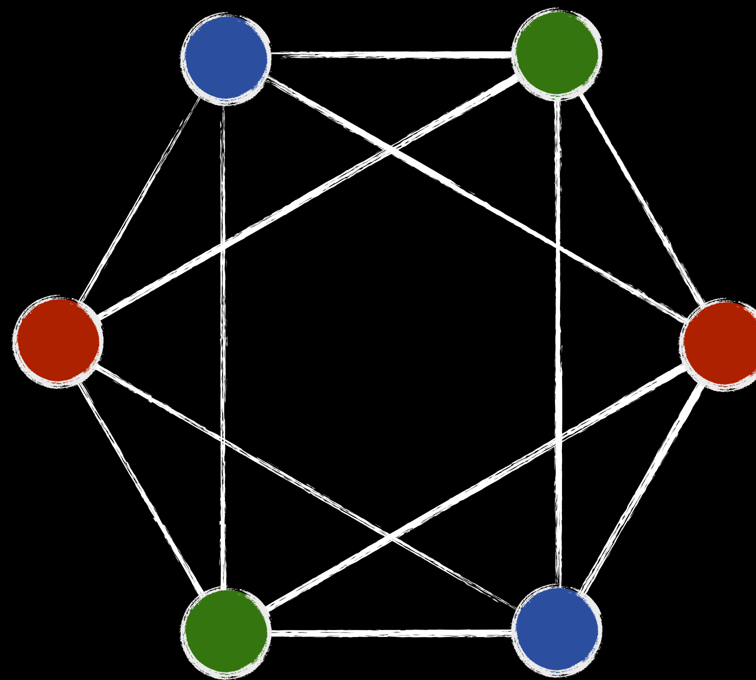

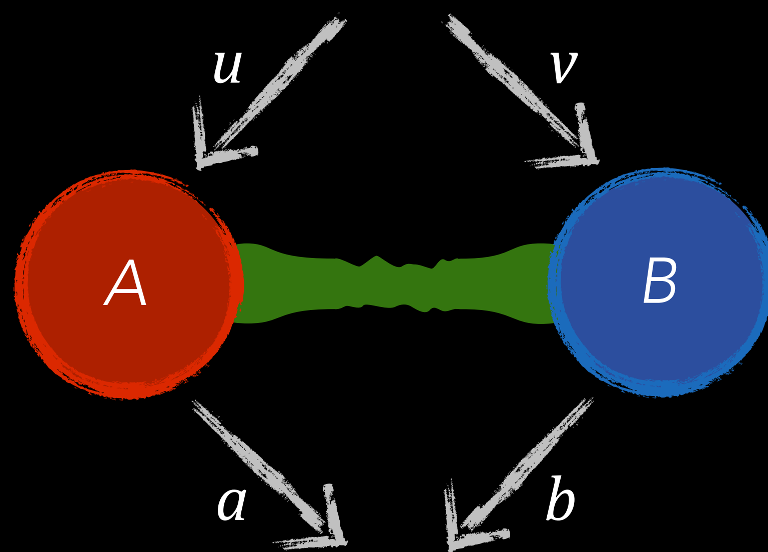

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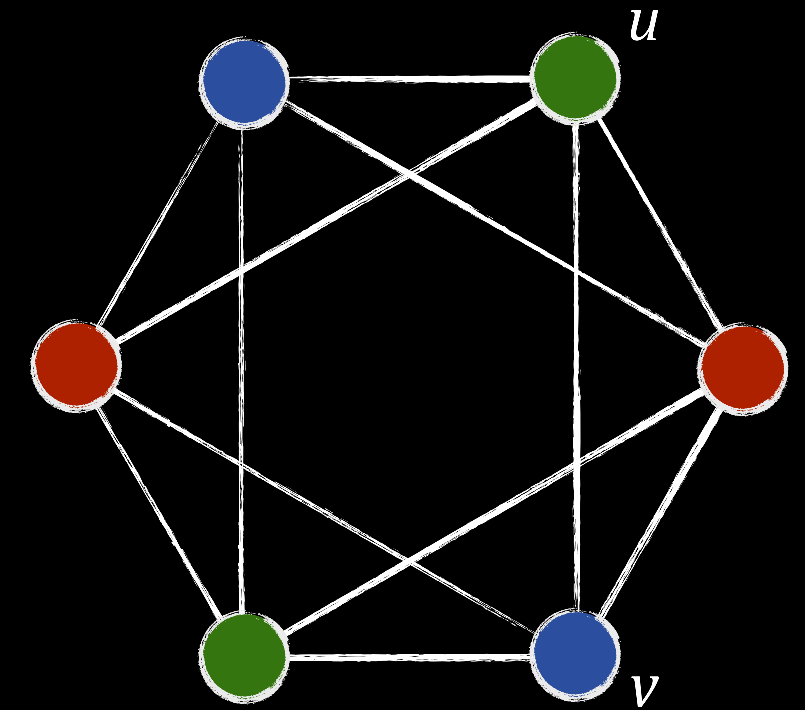

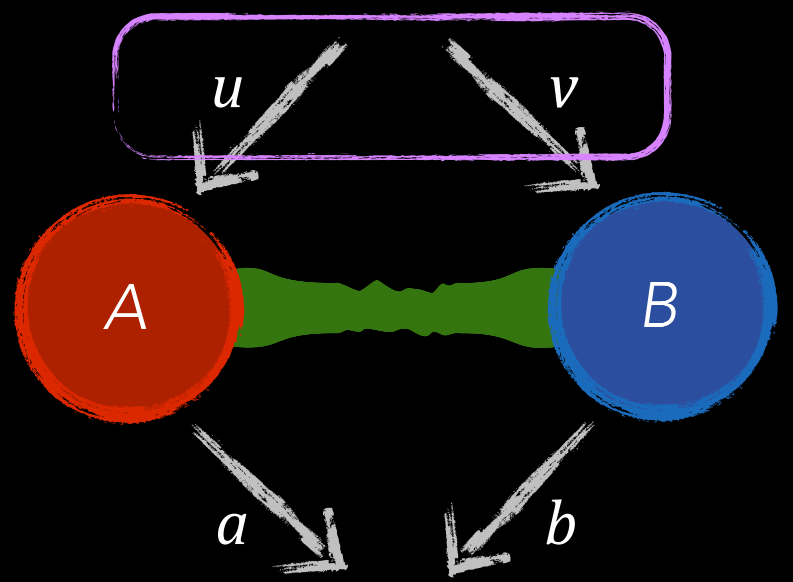

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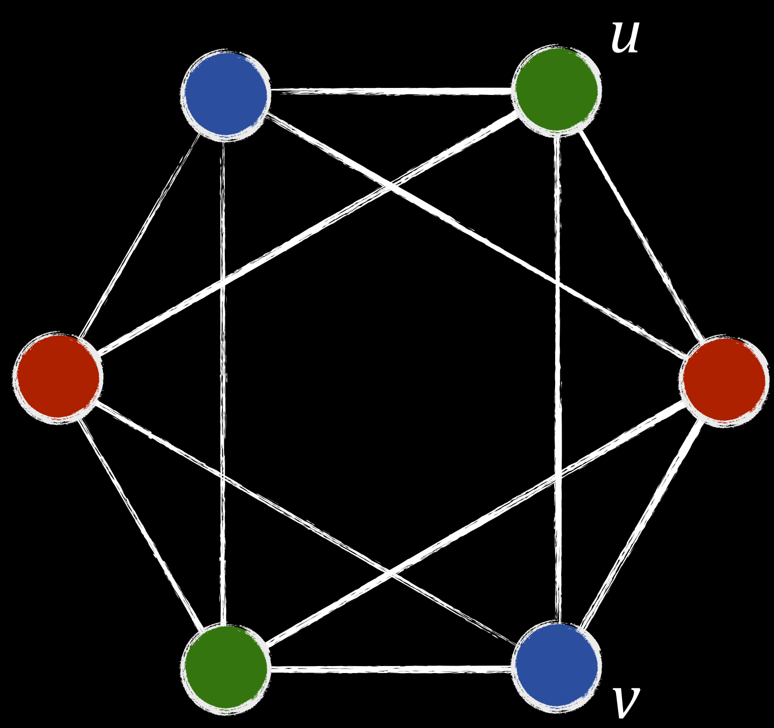

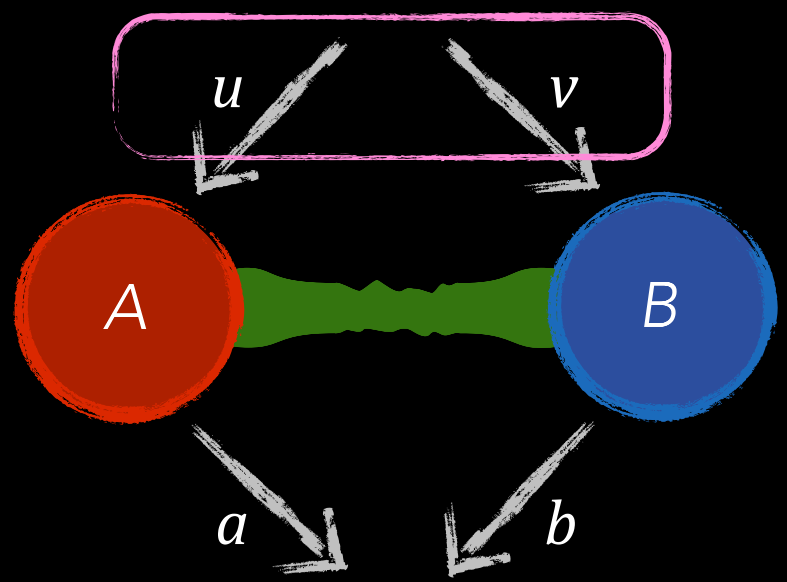

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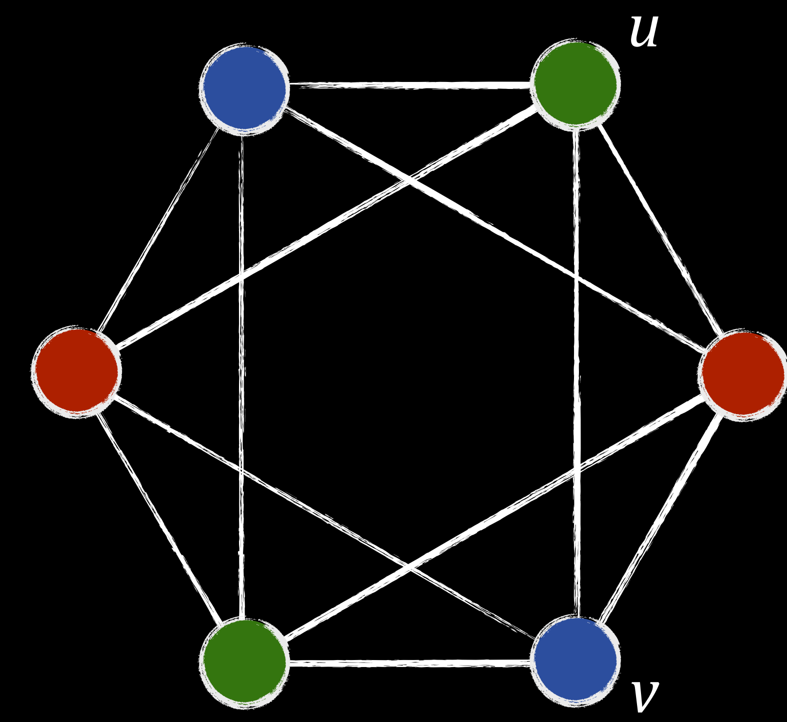

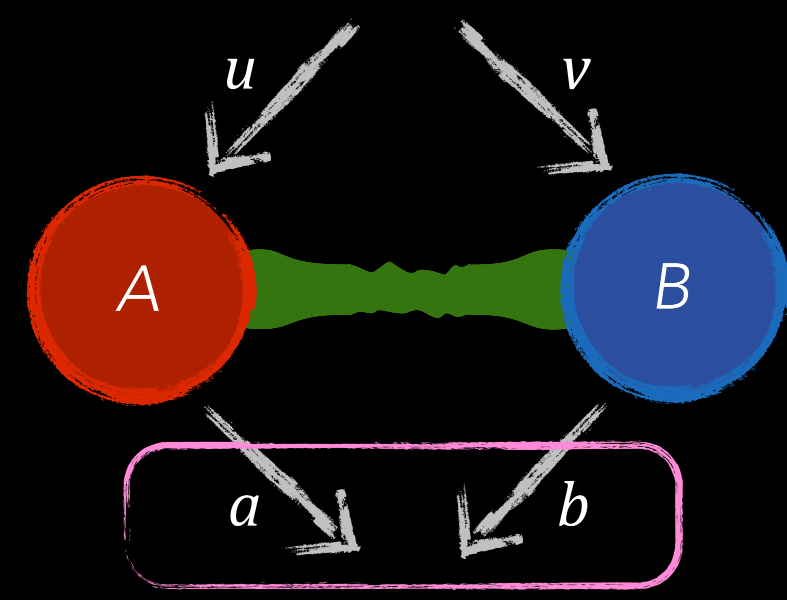

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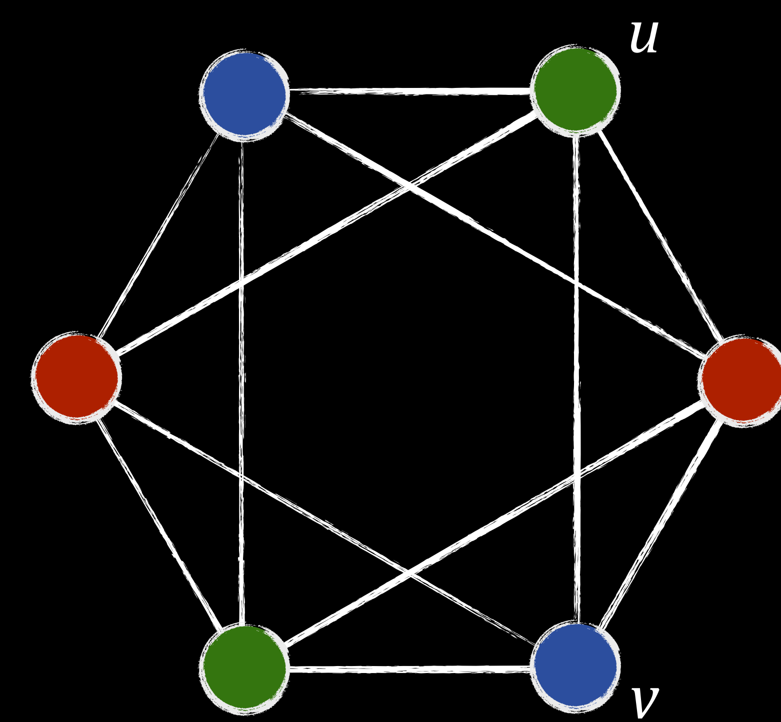

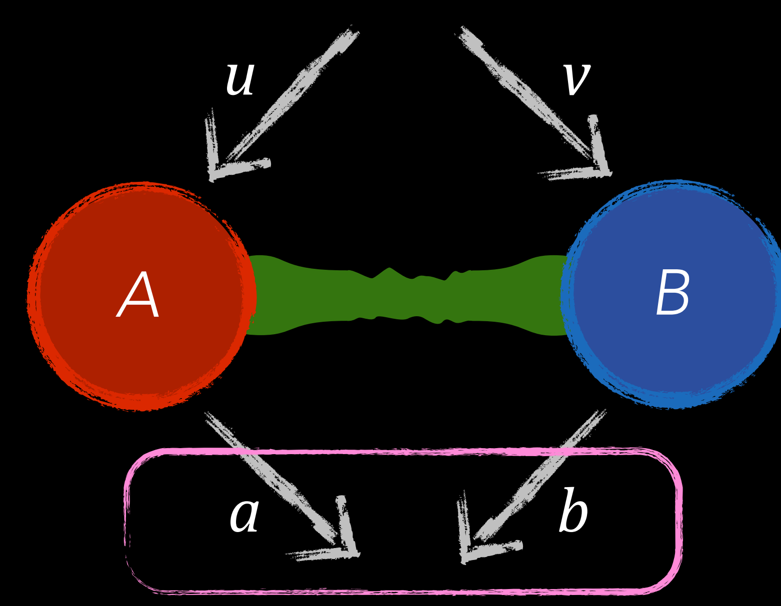

k-COLORING*

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For each vertex *v*, define *k* binary variables: $x_{v,0}, x_{v,1}, \ldots, x_{v,k-1}$.

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[Cameron, Montanaro, Newman, Severini and Winter, arXiv:quant-ph/0608016]

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[CAMERON, MONTANARO, NEWMAN, SEVERINI AND WINTER, ARXIV:QUANT-PH/0608016]

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for any subset B such that $\sum_{P_j \in B} P_j = I$.

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First construction: 117 variables,
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[ROBERSON AND MANCINSKA, ARXIV:1212.1724]

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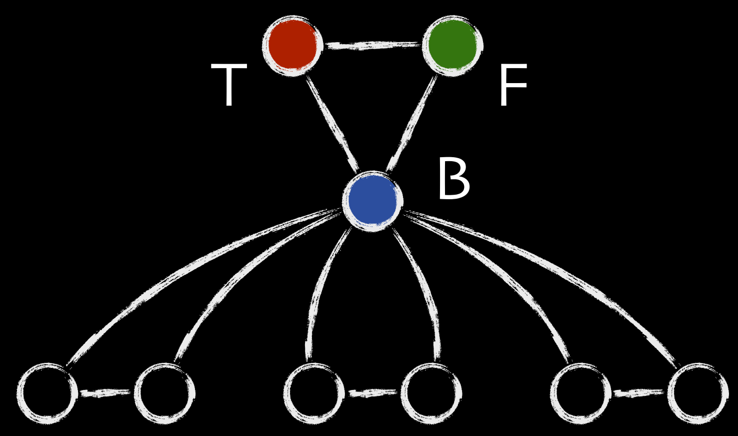

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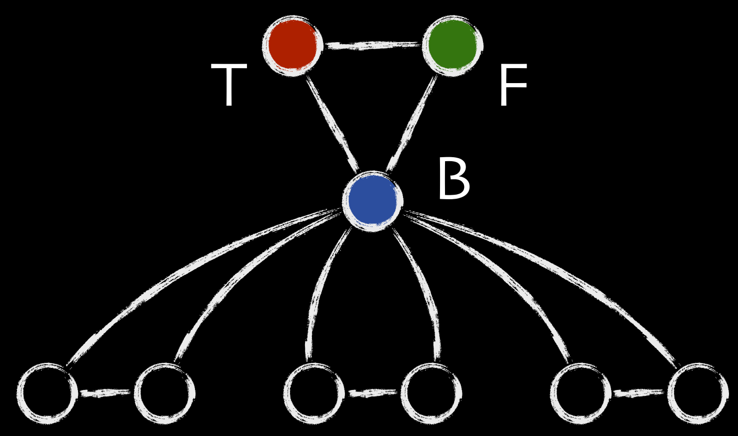

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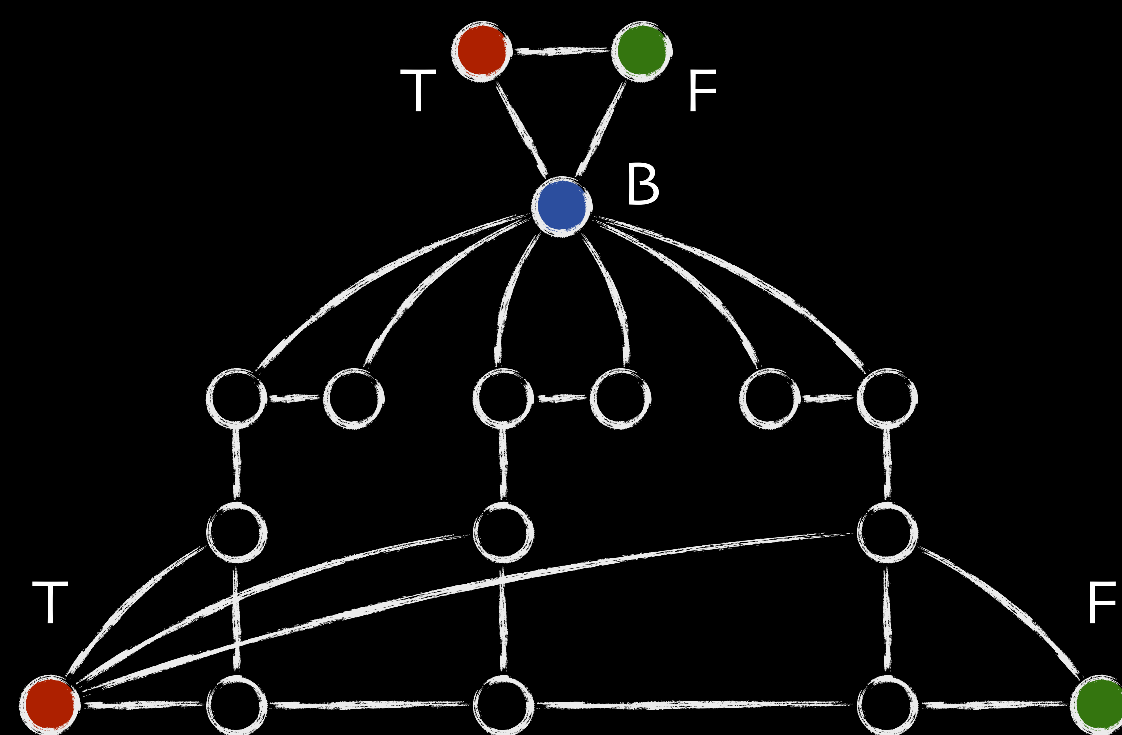

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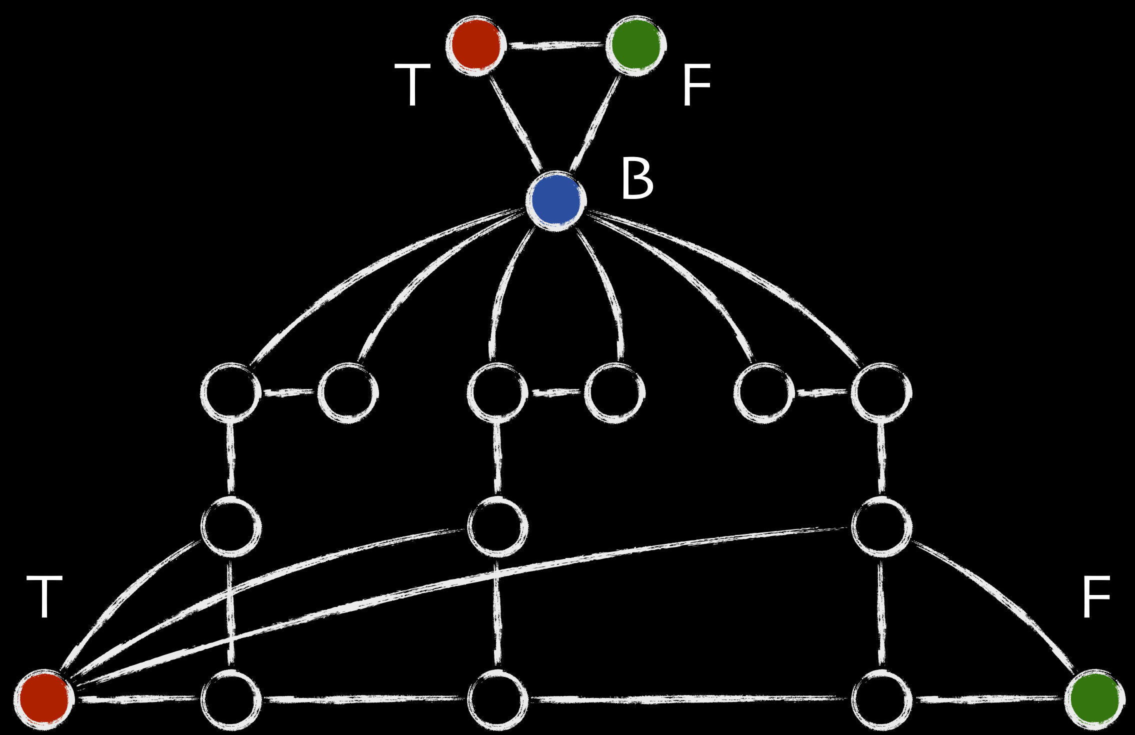

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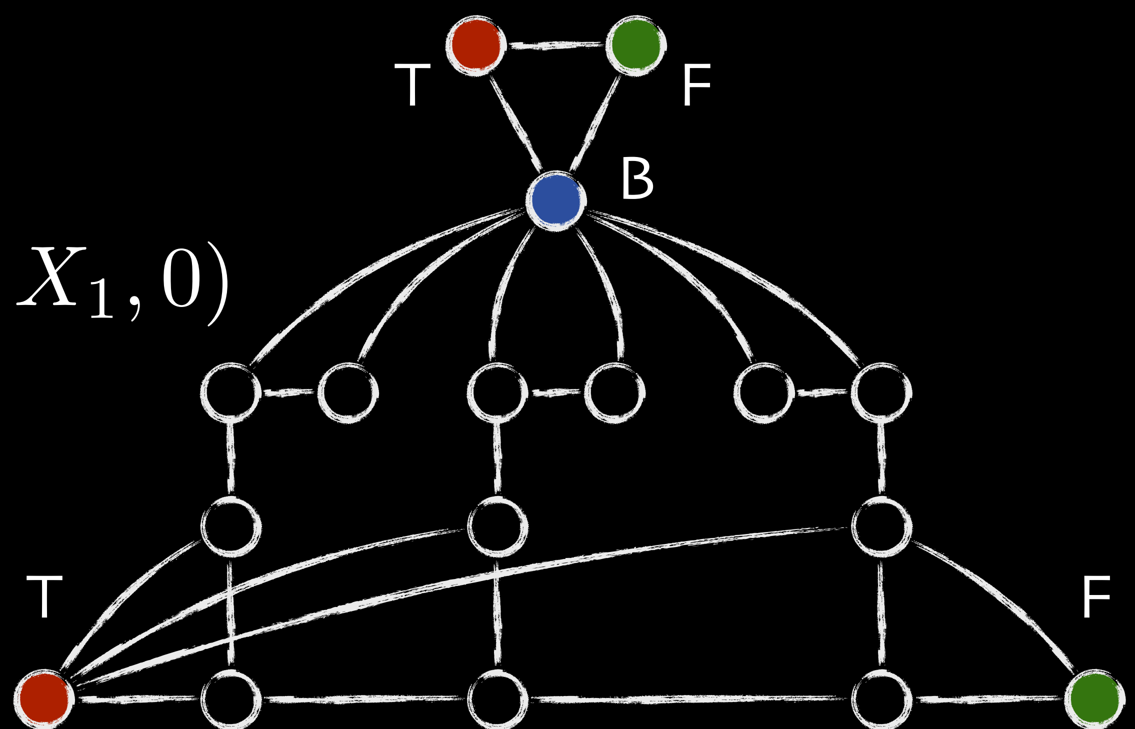

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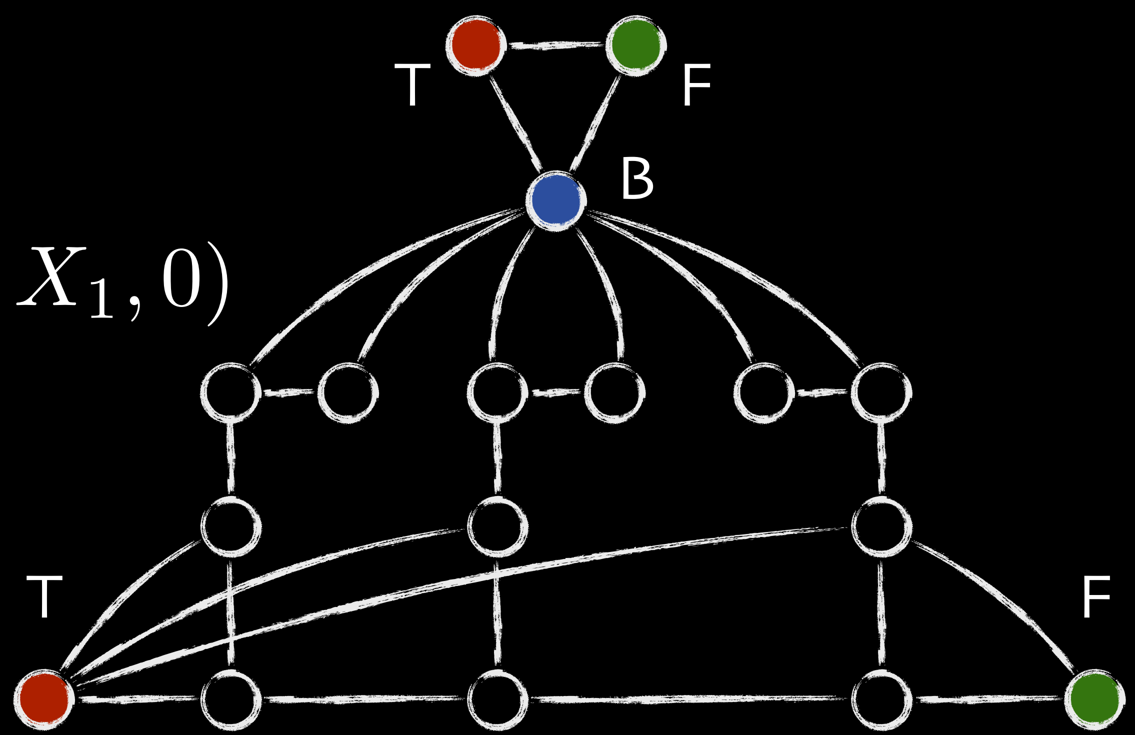

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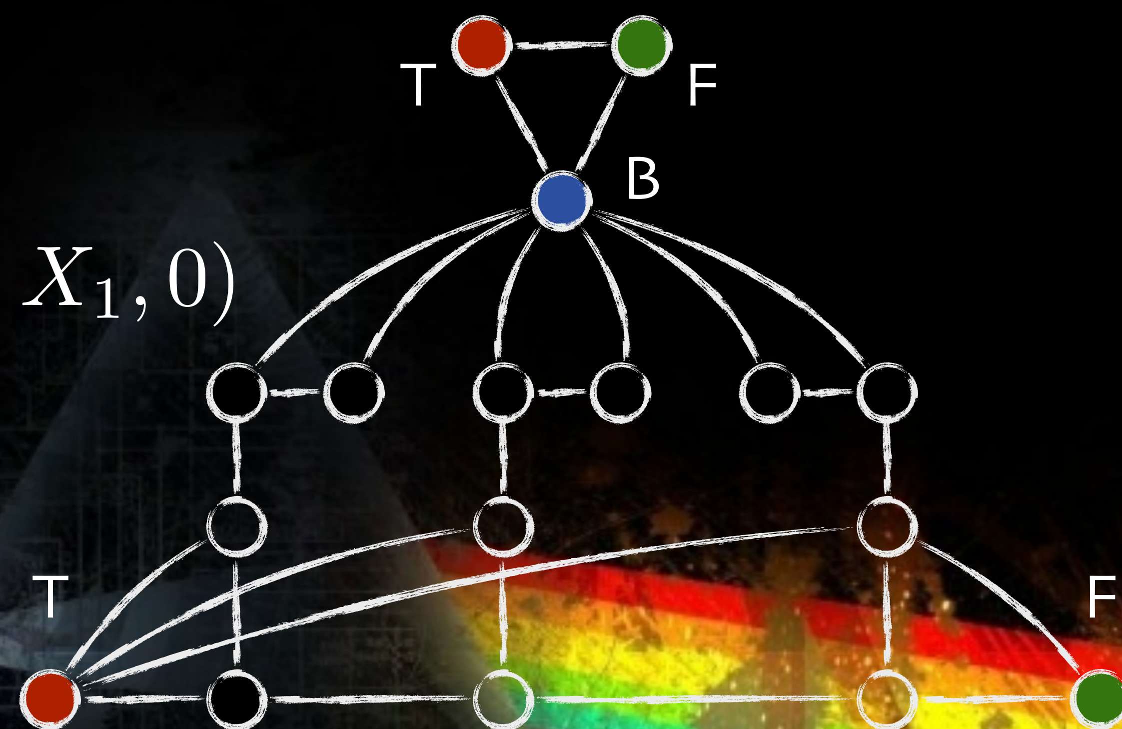

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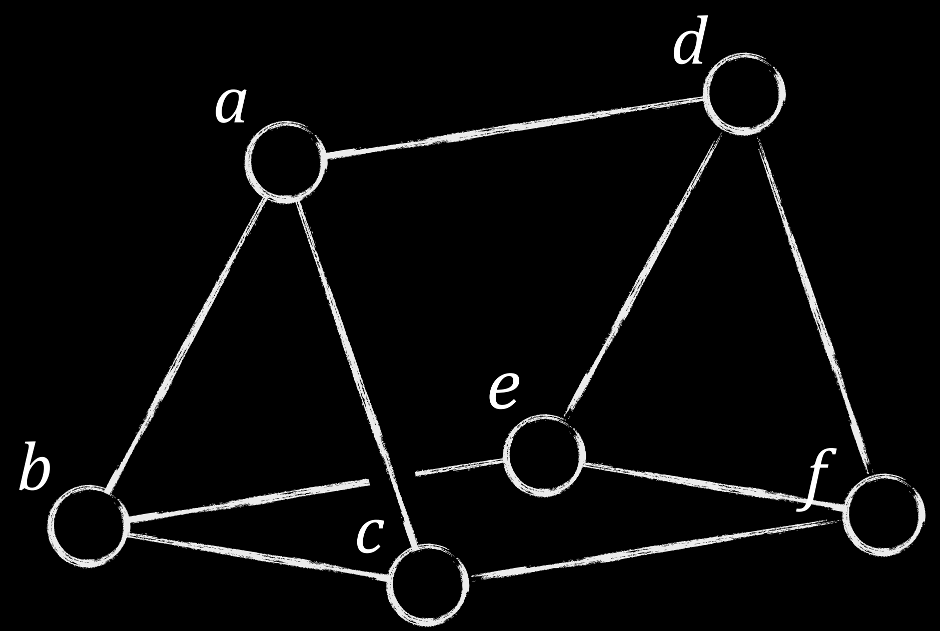
Triangular Prism Gadget

Lemma. The only constraint on the coloring operators of vertices a and e in the gadget is that they commute.



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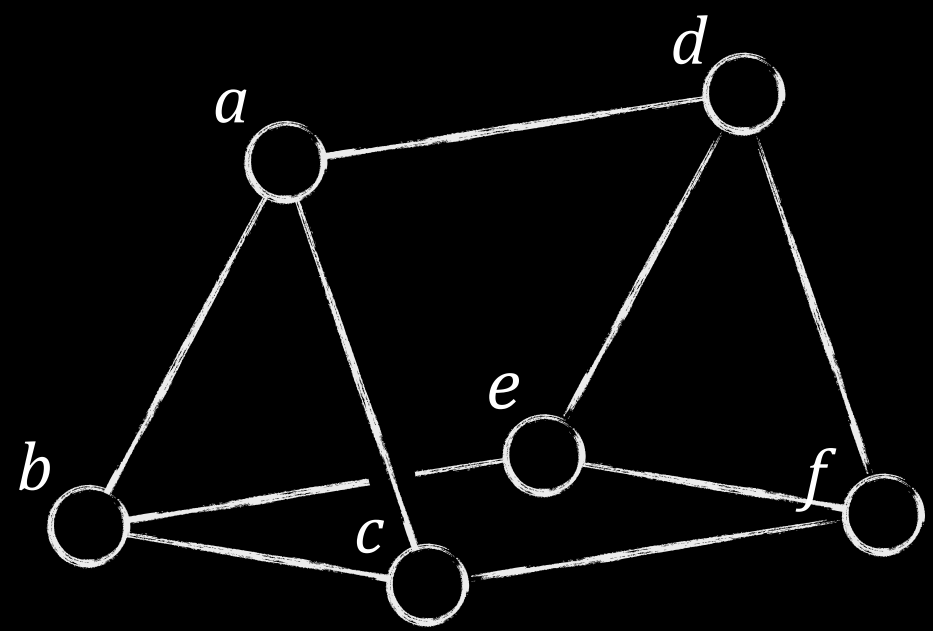

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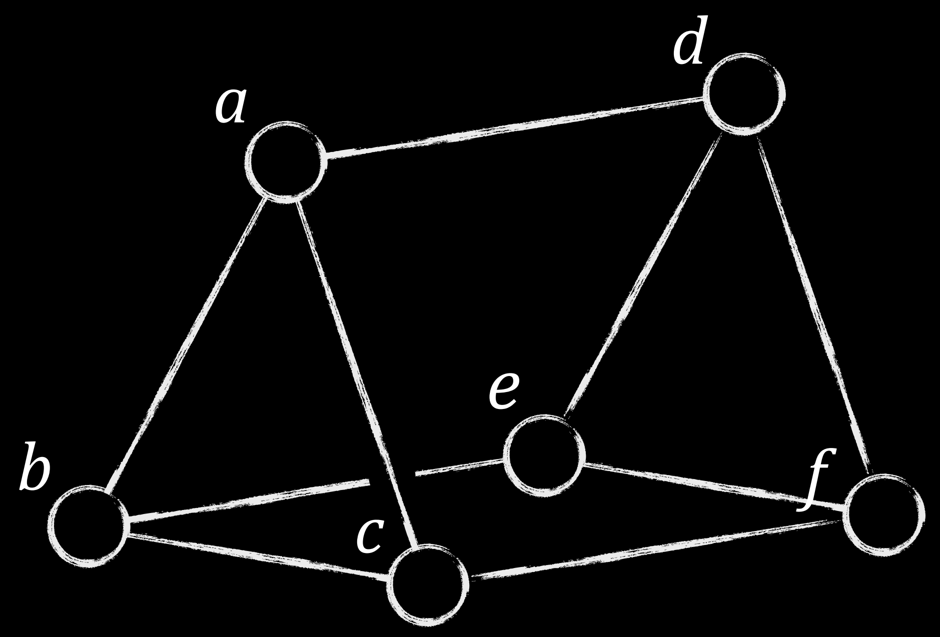

1. Commutativity
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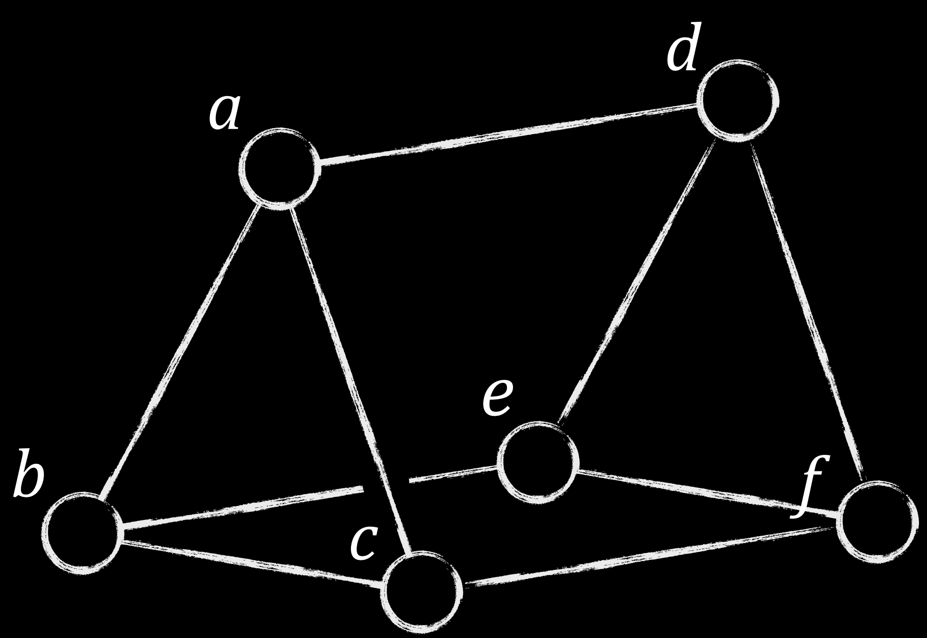

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2. Extendibility

Commutativity gadget

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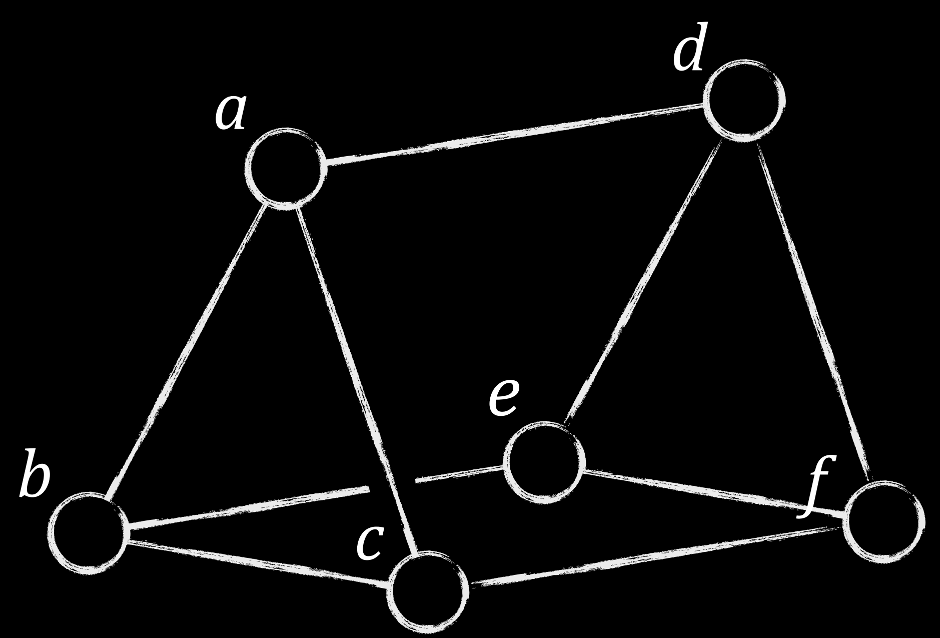

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Proof idea: commutator is in the ideal generated by the constraints. Non-commutative Grobner basis.

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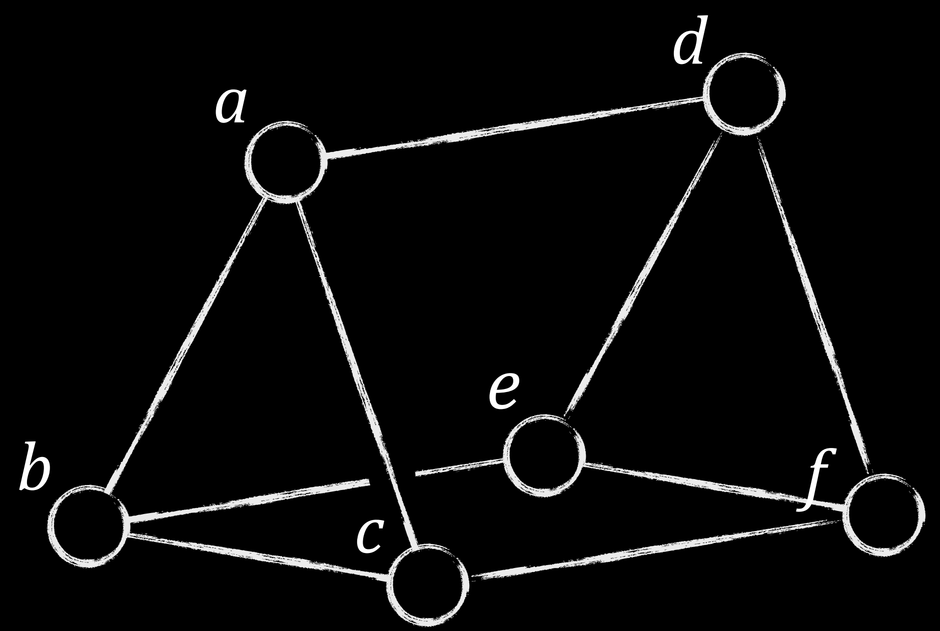

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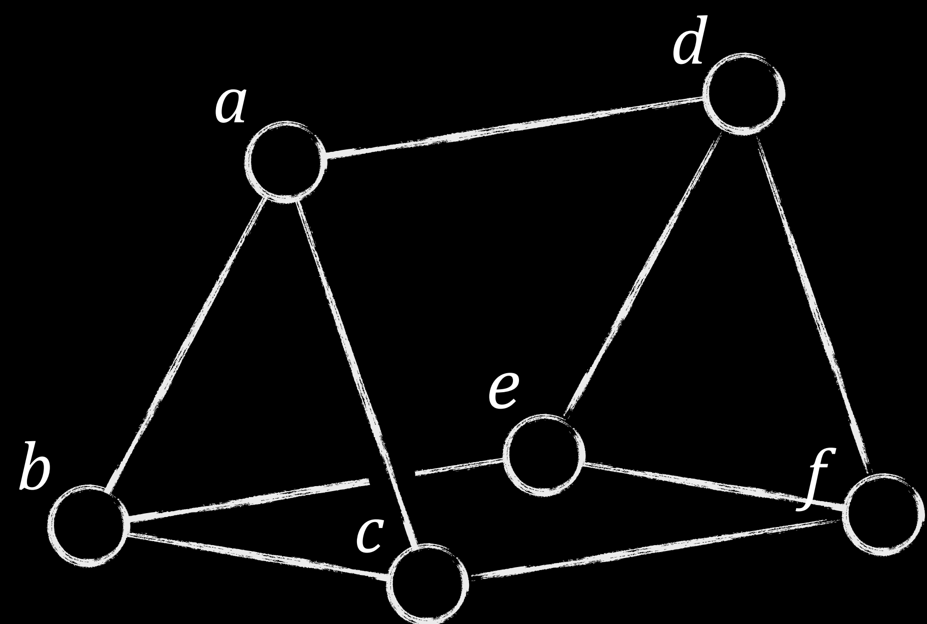

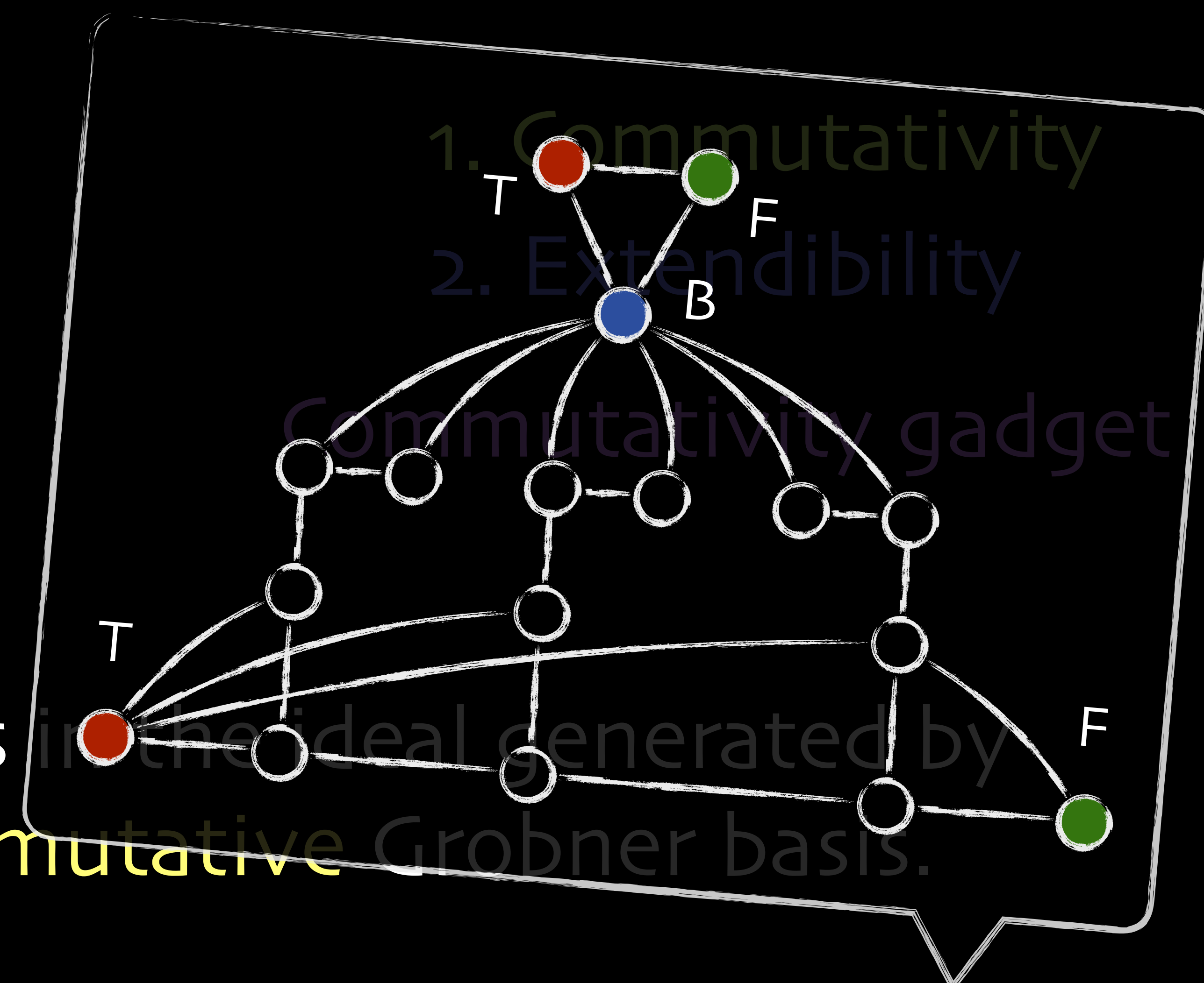

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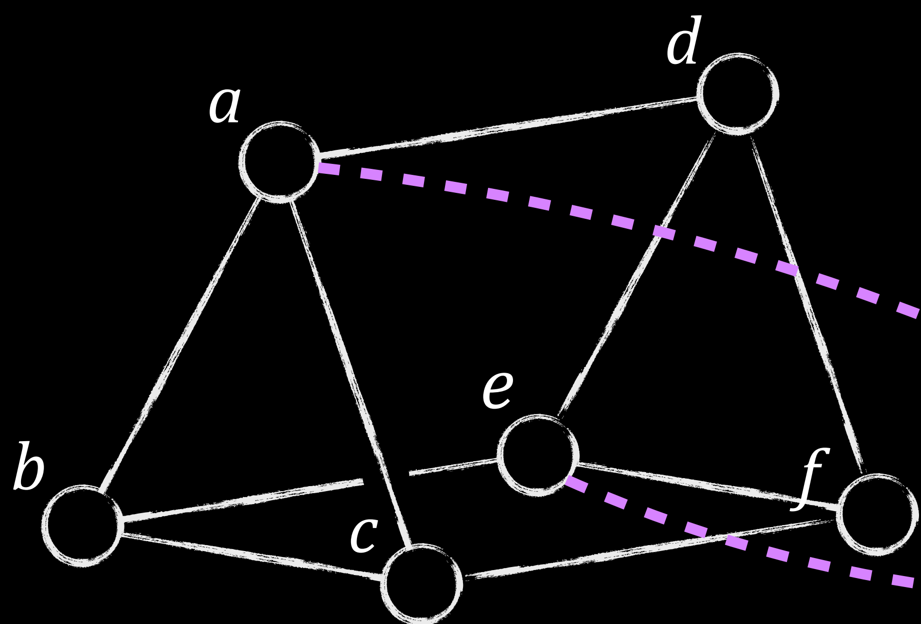

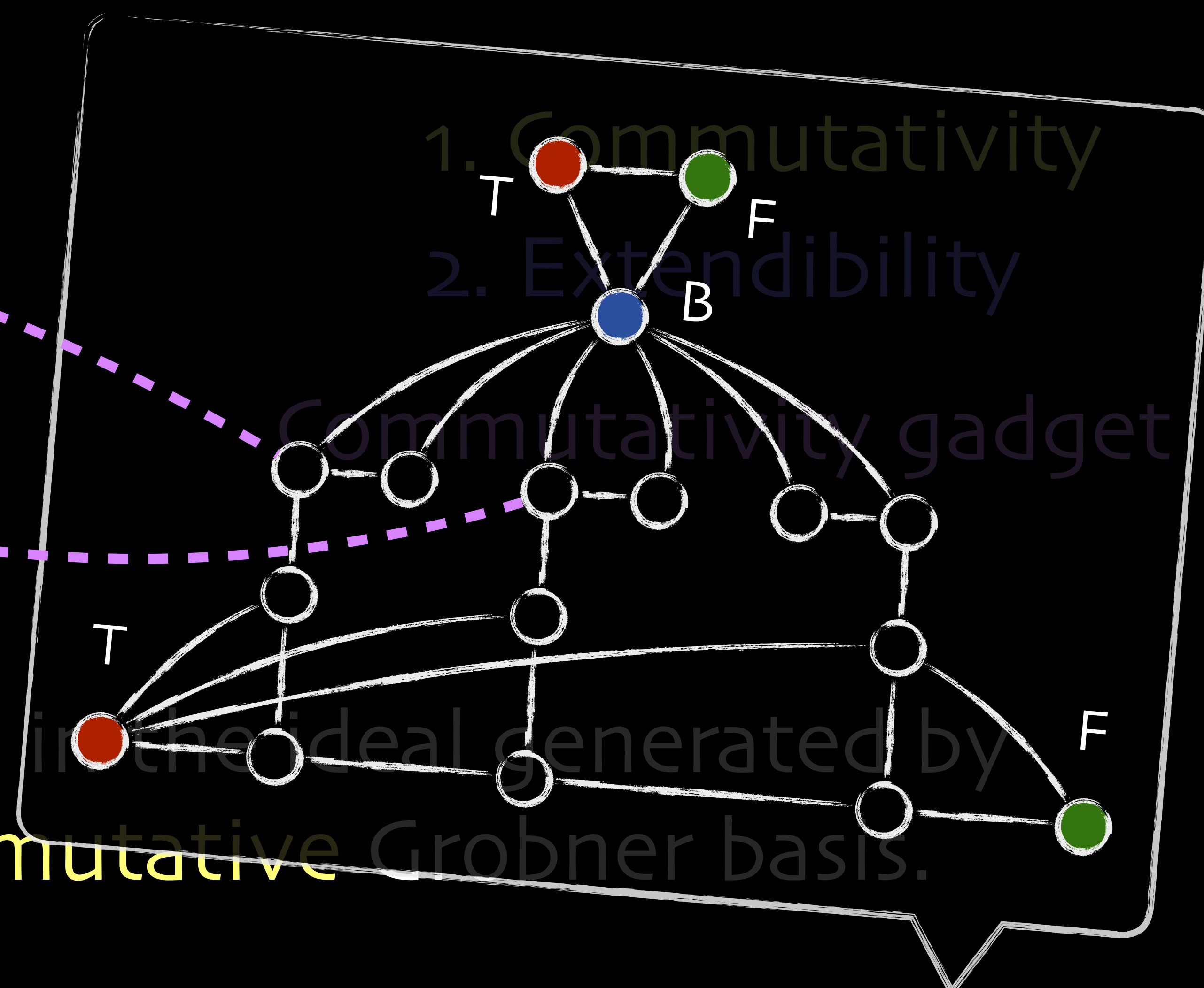

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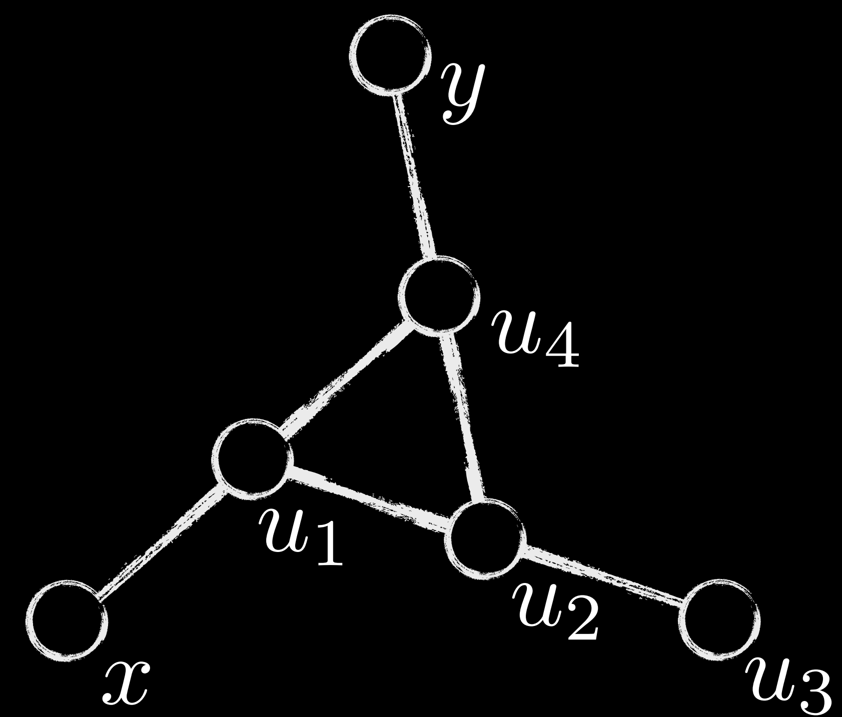

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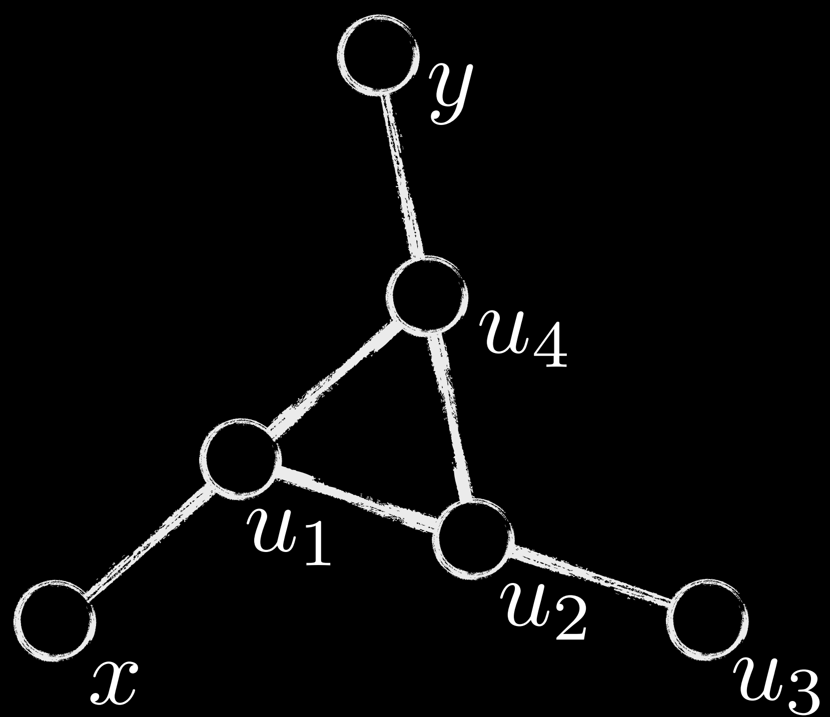

$$[x + u_1 + u_4 - 1, -x + u_1 + u_3] = [x, u_3] + [u_4, u_3],$$
$$[y + u_2 + u_4 - 1, -x] = [x, y] + [x, u_2],$$
$$[u_1 + u_2 + u_3 - 1, x + u_4] = [u_2, x] + [u_3, x] + [u_3, u_4].$$

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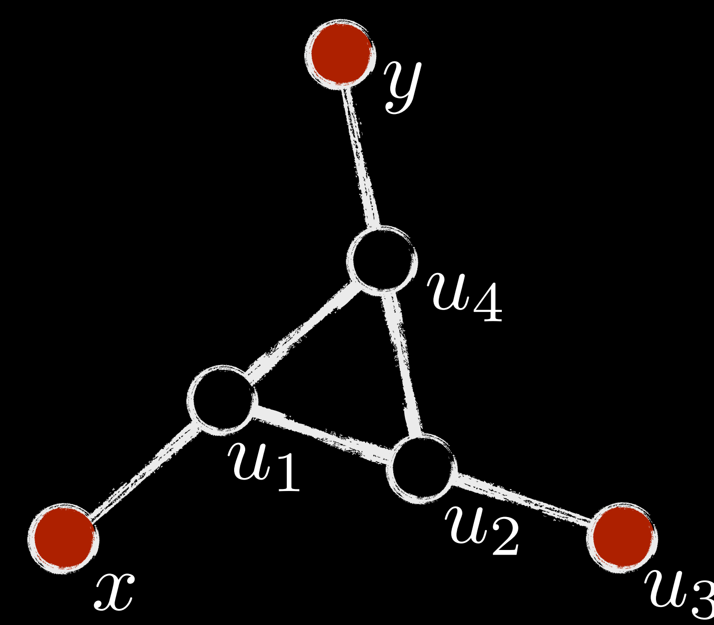

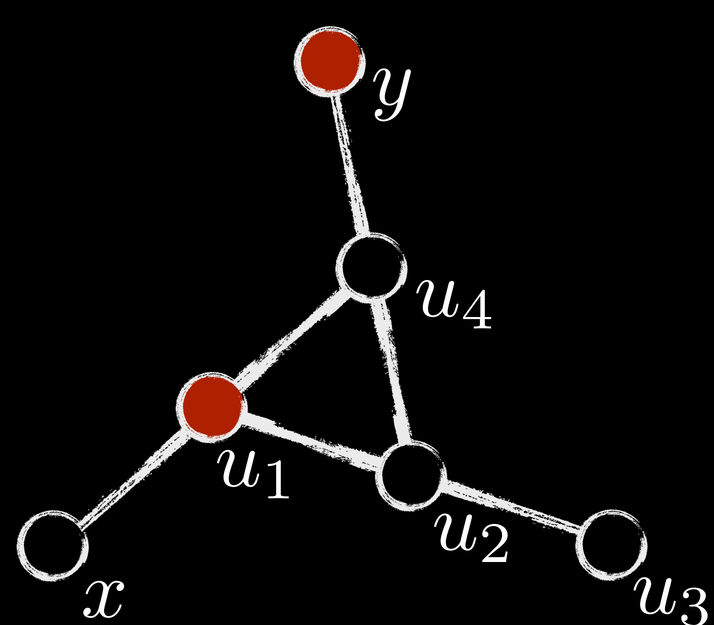

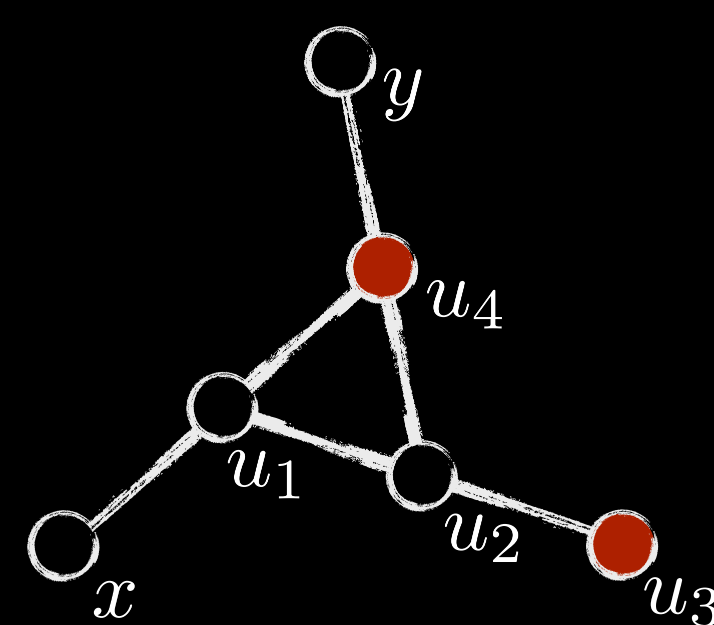

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  Anti-commutativity gadget + Clifford algebra

Conclusions

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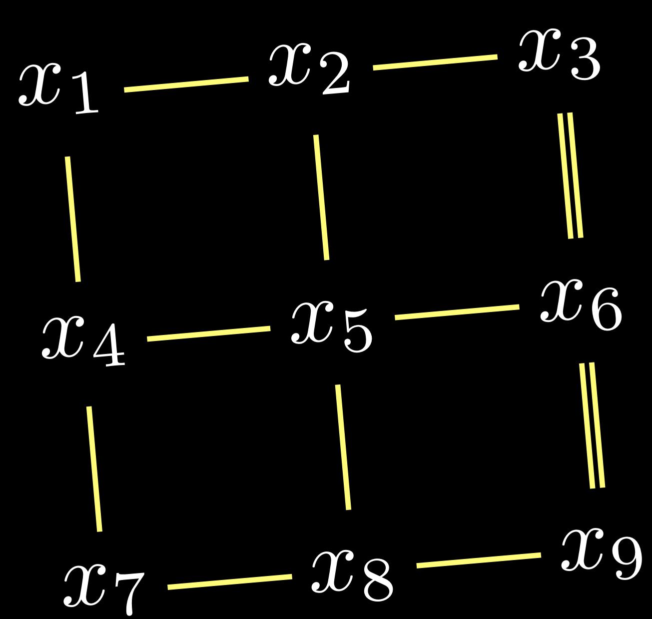
- Exact case vs. approximate case.

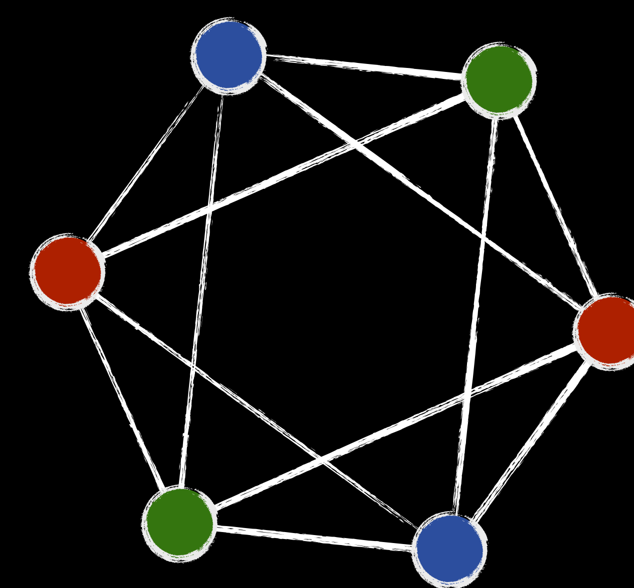
“Connecting the Dots”



Magic Square

INDEPENDENCE*

CLIQUE*



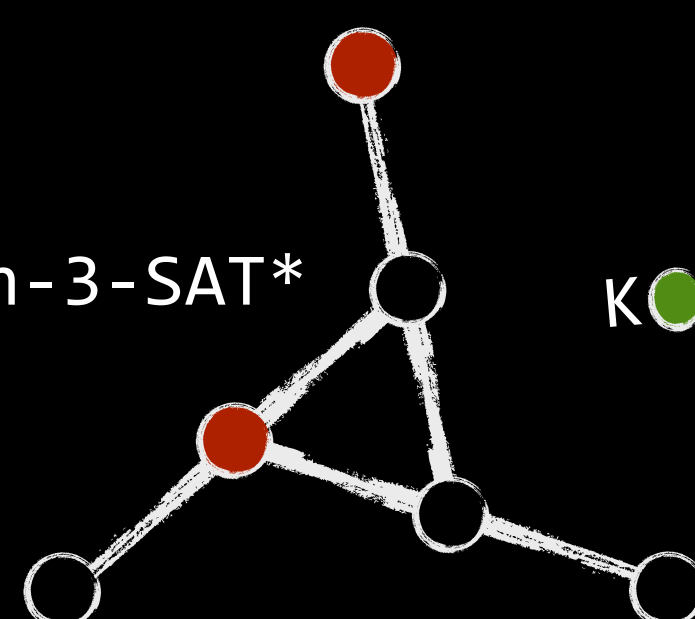
3-COLORING*

$x_1 \vee x_2 \vee \neg x_3$

3-SAT*

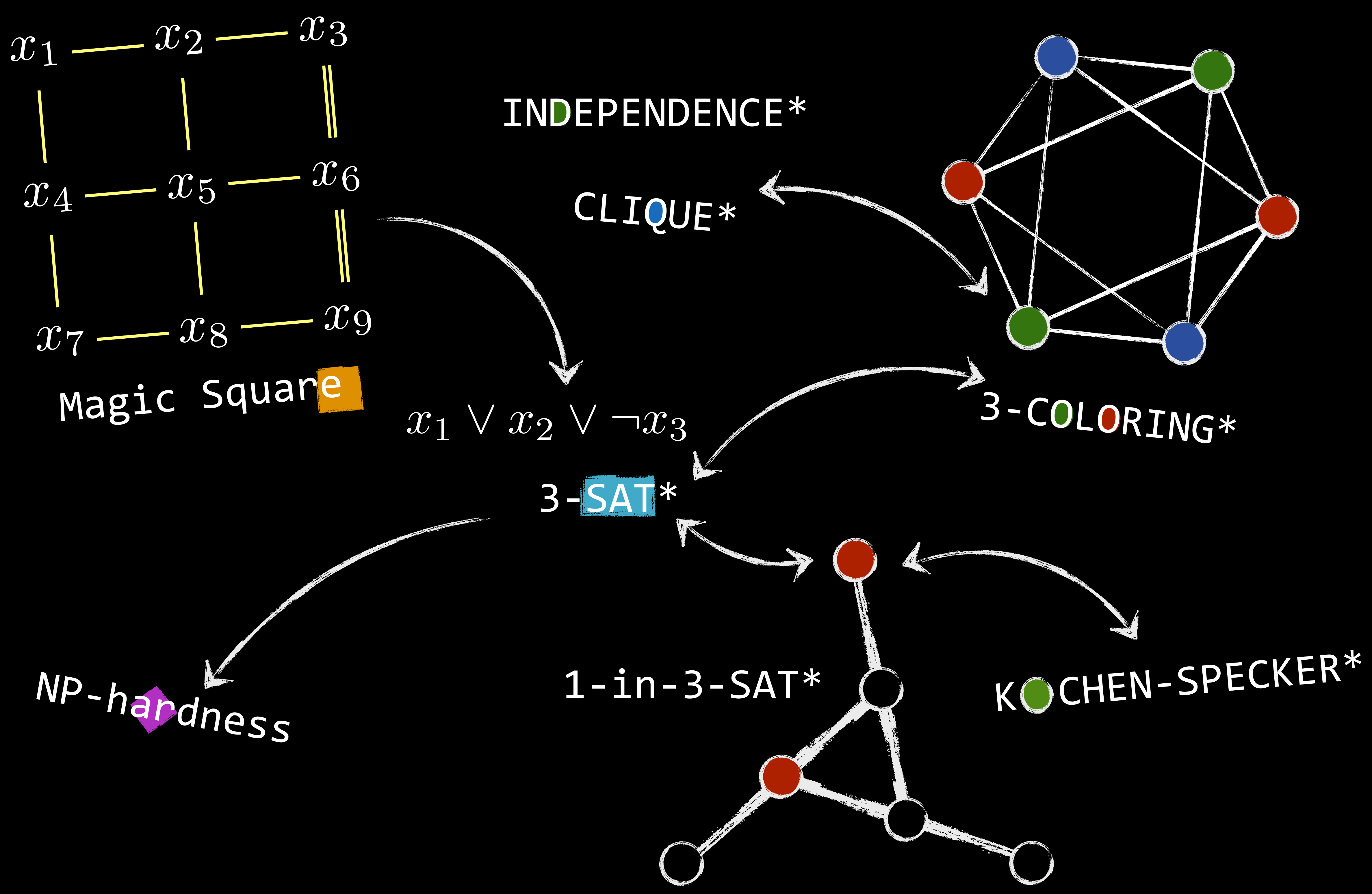
NP-hardness

1-in-3-SAT*



KOCHEN-SPECKER*

"Connecting the Dots"



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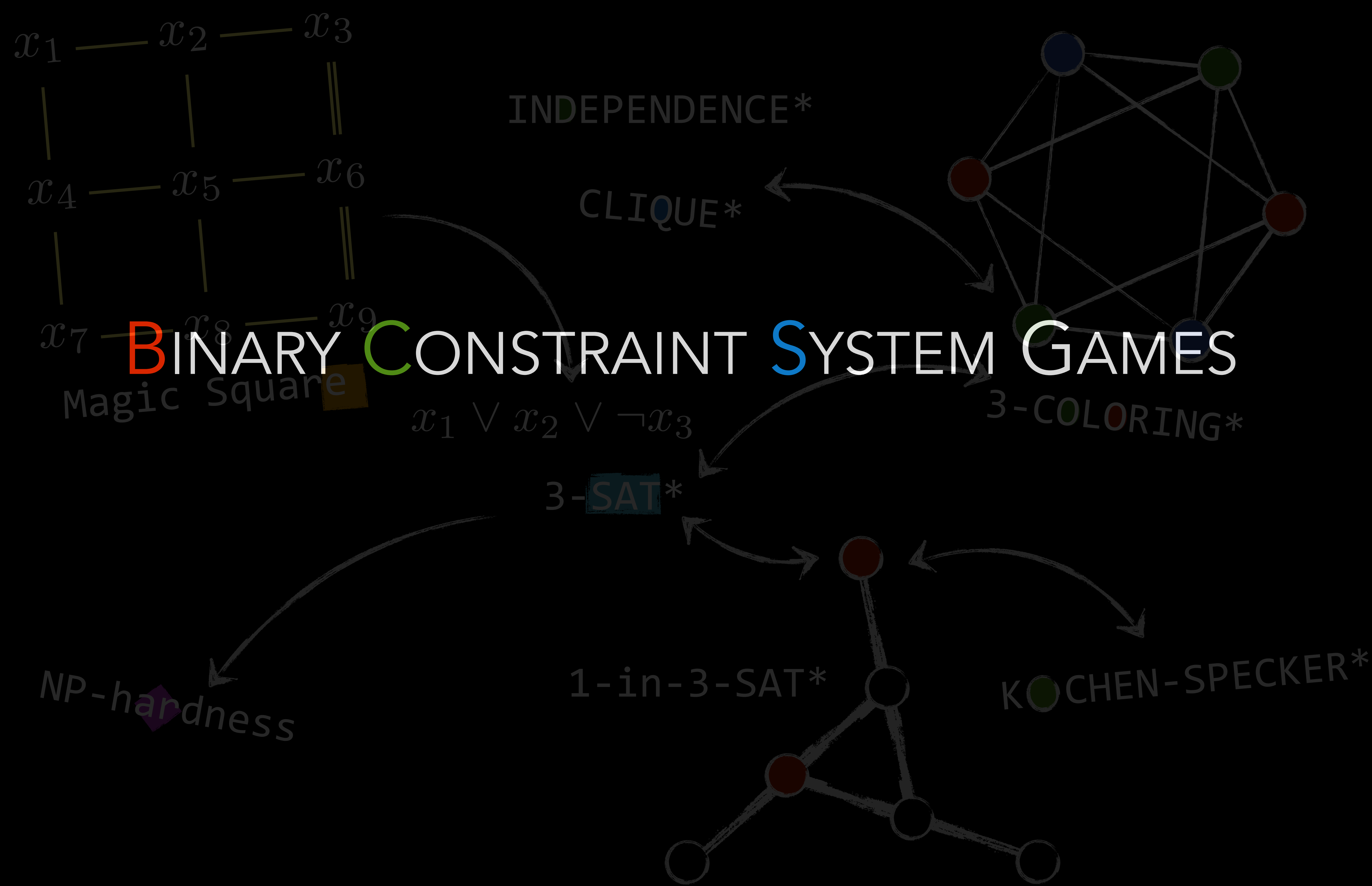
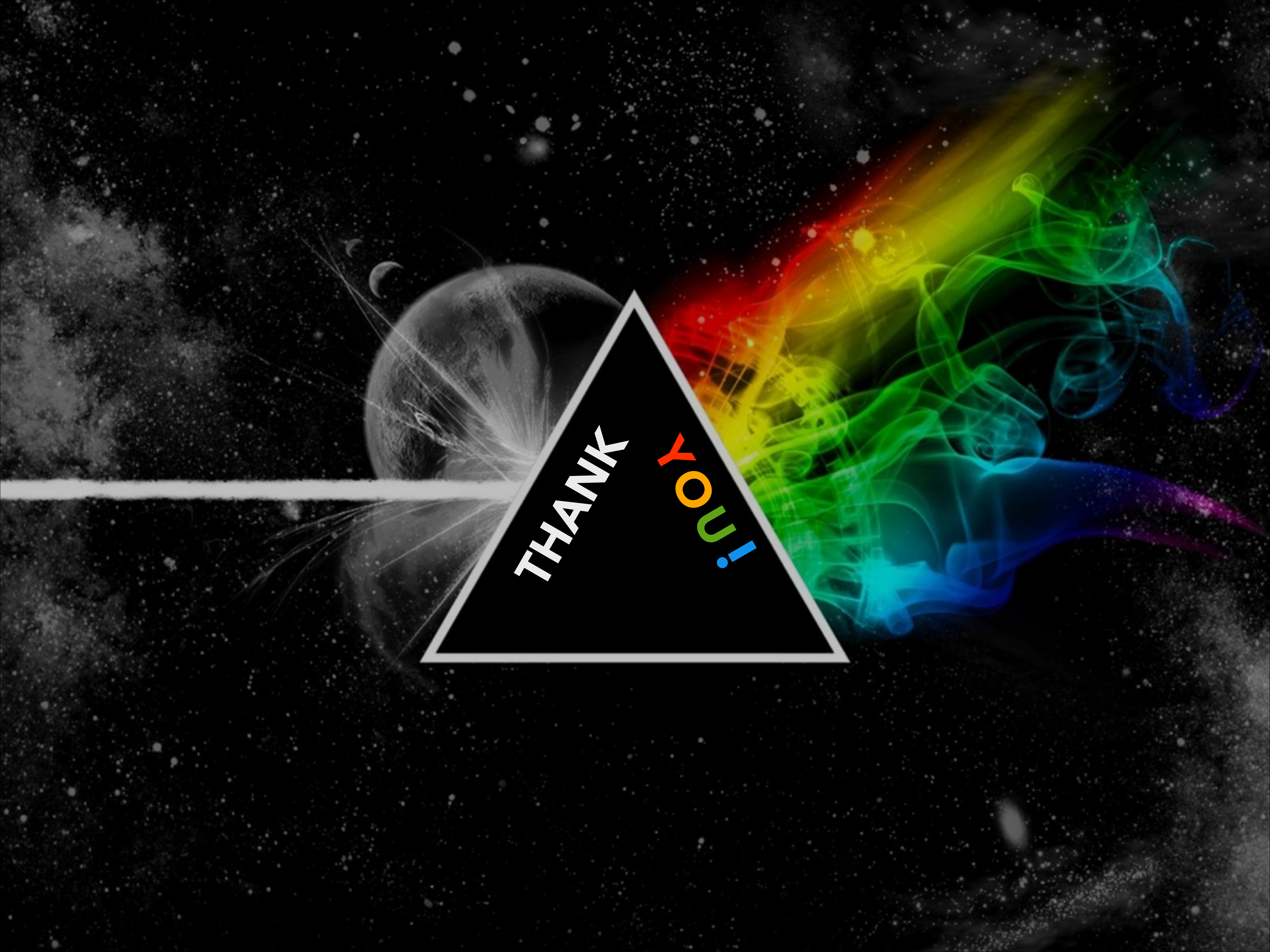
BINARY CONSTRAINT SYSTEM GAMES




THANK YOU!