Long-range entanglement is necessary for a topological storage of quantum information

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Isaac H. Kim

Perimeter Institute of Theoretical Physics
Waterloo, ON N2L 2Y5, Canada

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How do we protect quantum information?

- Virtually almost all the key proposals for protecting quantum information uses the idea of **quantum error correcting code**(QECC).
- Formally, one can encode $\log_2 N$ number of qubits into a set of states $\{|\psi_i\rangle\}_{i=1,\cdots,N}$ which forms a *subspace* of a larger Hilbert space.
- As it stands, the preceding construction is too general. Often times stronger statements can be obtained by enforcing additional structures.
  - ex) Stabilizer code(Gottesman 1997), Codeword stabilized code(Cross *et al.* 2007)

Fundamental bounds on general QECC

For stabilizer codes,

- Quantum Gilbert-Varshamov bound(Calderbank, Shor 1996)
- Quantum Hamming bound(Gottesman 1996)
- Quantum Singleton bound(Knill, Laflamme 1997)

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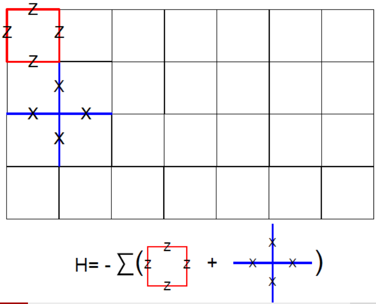
Still too general!

Local projector codes

One may assume that the code subspace is the degenerate ground state subspace of some Hamiltonian H, such that

$$H = \sum_i h_i,$$

where h_is are **geometrically local**. For many interesting systems, $[h_i, h_j] = 0$ and h_i annihilates ground states individually. ex) toric code and Levin-Wen model. Let's call these as **local projector codes.**



Local projector codes: Why are they studied?

- Stability : (Bravyi, Hastings, Michalakis 2010)
- Covers a large class of systems!
  - Toric code, quantum double model(Kitaev 1996)
  - Levin-Wen string-net model(Levin, Wen 2003)
  - Glass/Fractal codes(Chamon 2006, Haah 2011, Yoshida 2013)

Tradeoff bound for local projector codes

Bravyi et al. proved that for a **local projector code** on a D-dimensional lattice,

$$k \leq \frac{cn}{d^{2/(D-1)}}.$$

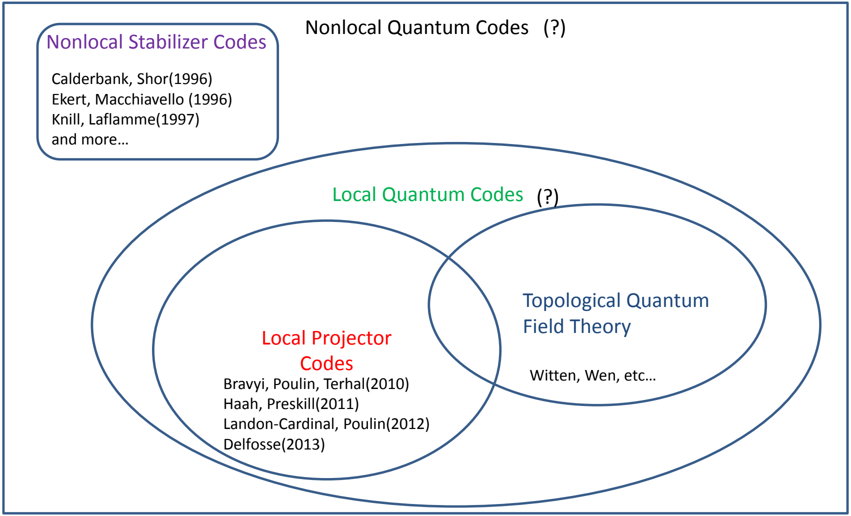
(2010)

- k : number of encoded qubits(ex. 4 for the toric code)
- n : number of particles(ex. $2L^2$ for a toric code on a $L \times L$ lattice)
- d : code distance(ex. L for a toric code on a $L \times L$ lattice)
- c : some constant that depends on the radius of interaction for each h_i(ex. say, 10)

Local projector codes $\neq$ everything

- Sometimes, you don't have a Hamiltonian.
  - Variational wavefunction for fractional quantum Hall effect(FQHE), Resonanting valence-bond state, etc...
- Sometimes, you have a parent Hamiltonian but the local terms don't commute with each other.
  - Rokhsar-Kivelson point, parent Hamiltonian of FQHE
- There are several resolutions.
  - Showing that the Hamiltonian is adiabatically connected to a known local projector code.(ex. Schuch et al. 2012)
  - Assuming the ground state can be described by a topological quantum field theory, in which case $k = O(1)$.

Landscape of QECC tradeoff bounds



Main result: tradeoff bound from entanglement entropy

We can produce a nontrivial inequality between n, k, and d, from entanglement entropy analysis of a **single codeword** alone. The bound has the following features.

- Robustness : It even works for approximate quantum error correcting codes, with a logarithmic dimension-dependent overhead.
- Performance : For many interesting physical systems, the inequality is saturated with an equality.
- Generality : No assumptions on the degeneracy/ parent Hamiltonian are made.

Key idea 1 : local indistinguishability

Given two orthogonal states $|\psi_i\rangle, |\psi_j\rangle \in QECC$ with a code distance d,

$$\langle\psi_i| X |\psi_j\rangle = 0$$

$$\langle\psi_i| X |\psi_i\rangle = \langle\psi_j| X |\psi_j\rangle$$

for all X such that $wt(X) < d$.

- In particular, $Tr_{A^c} |\psi_i\rangle\langle\psi_i| = Tr_{A^c} |\psi_j\rangle\langle\psi_j|$ for any subsystem A whose size is smaller than d.

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- In particular, $Tr_{A^c} |\psi_i\rangle\langle\psi_i| = Tr_{A^c} |\psi_j\rangle\langle\psi_j|$ for any subsystem A whose size is smaller than d.
- **Therefore, for the reduced density matrix of subsystems smaller than d, we do not have to specify the codeword.**
  - For any QECC, reduced density matrices of such subsystems have an invariant meaning.

Key idea 2 : Petz's theorem

Strong subadditivity of entropy(Lieb, Ruskai 1972):

$$I(A:C|B) = S(AB) + S(BC) - S(B) - S(ABC) \geq 0.$$

If $I(A:C|B) = 0$, Petz's theorem states that there is a nontrivial relationship between reduced density matrices over different subsystems:

- $\rho_{ABC} = \rho_{AB}^{\frac{1}{2}} \rho_B^{-\frac{1}{2}} \rho_{BC} \rho_B^{-\frac{1}{2}} \rho_{AB}^{\frac{1}{2}}$.
- The precise form of the relation is unimportant for our purpose.
- What matters is the fact that the global state(ρ_{ABC}) can be reconstructed from local states($\rho_{AB}, \rho_{BC}, \rho_B$).

Key idea 2 : Petz's theorem

A_1	A_2	A_3	A_4	A_5	A_6

Recall that $I(A{:}C|B) = S(AB) + S(BC) - S(B) - S(ABC) \geq 0$

$I(A_3{:}A_1|A_2) + I(A_4{:}A_1A_2|A_3) + I(A_5{:}A_1A_2A_3|A_4) + I(A_6{:}A_1A_2A_3A_4|A_5) = 0$ implies

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$$\rho_{123456} = \rho_{56}^{\frac{1}{2}}\rho_5^{-\frac{1}{2}}\rho_{12345}\rho_5^{-\frac{1}{2}}\rho_{56}^{\frac{1}{2}} = \rho_{56}^{\frac{1}{2}}\rho_5^{-\frac{1}{2}}\rho_{45}^{\frac{1}{2}}\rho_4^{-\frac{1}{2}}\rho_{1234}\rho_4^{-\frac{1}{2}}\rho_{45}^{\frac{1}{2}}\rho_5^{-\frac{1}{2}}\rho_{56}^{\frac{1}{2}} = \cdots$$

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Key insight: The global state(ρ_{123456}) is uniquely determined by an overlapping set of reduced density matrices ($\rho_{12}, \rho_{23}, \cdots$), if all the conditional mutual information are equal to 0.

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?????? **Conflict!**

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  - Recall that there were other terms as well. If we expand all the conditional mutual information in terms of entanglement entropies, we will see that all but one term involves only local information about the density matrix. The only global term is the entropy of the system, which is something we can choose.

Let's expand all the terms.

We only need entanglement entropies of some local regions and a global entropy.

$$I(A_3 : A_1|A_2) = S_{12} + S_{23} - S_2 - S_{123}$$

$$I(A_4 : A_1A_2|A_3) = S_{123} + S_{34} - S_3 - S_{1234}$$

$$I(A_5 : A_1A_2A_3|A_4) = S_{1234} + S_{45} - S_4 - S_{12345}$$

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If LHS=0, RHS has to be equal to 0.(Why? Entropy is nonnegative.)

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- RHS becomes $\log N$.
- LHS $=$ local information. RHS=size of the code subspace

General protocol

1. Assume there are N orthogonal states that are indistinguishable in regions smaller than the code distance(d).
2. $\log N \leq S(A_1) + \sum_{i=2}^{m} S(A_i B_i) - S(B_i)$, where
   - $A_i B_i$ are local and
   - $\{A_i\}_{i=1,\cdots m}$ forms a partition of the system and
   - $B_i \subset \cup_{j=1}^{i-1} A_j$
3. Optimize over all $\{A_i B_i\}$.

Application 1 : Average entanglement entropy bounds the code rate

Using subadditivity of entropy,

$$\begin{aligned}\log N &\leq S(A_1) + \sum_{i=2}^{m} S(A_i B_i) - S(B_i) \\ &\leq \sum_{i=1}^{m} S(A_i)\end{aligned}$$

Application 1 : Average entanglement entropy bounds the code rate

For a $[[n, k, d]]$ code,

$$\frac{k}{n} \leq \frac{\sum_{i=1}^{m} S(A_i)}{n},$$

where $|A_i| < d$.

- Brandão and Harrow (2013): If entanglement entropy satisfies a subvolume law, there exists a subexponential sized classical witness that approximates the ground state energy density to a fixed accuracy.
- QECC must have lots of entanglement in order to be good.

Application 2 : Topological entanglement entropy bounds ground state degeneracy

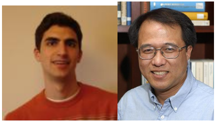
There is a general belief that if a quantum many-body system has a constant energy gap between its ground state sectors and its first excited state, entanglement entropy satisfies area law:

$$S(A) = a|\partial A|^{D-1} + b|\partial A|^{D-2} + \cdots .$$

In particular, in 2D,

$$S(A) = a|\partial A| - \gamma + o(1)$$

(Kitaev and Preskill, Levin and Wen 2006)



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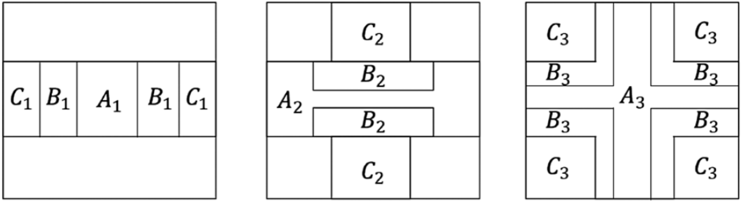
On a 2D torus, if

$$S(A) = al - \gamma + o(1),$$

$\sum_{i=1}^{3} I(A_i : C_i | B_i) \geq 0$ yields:

$$\log N \leq 2\gamma + o(1).$$



Application 2 : Topological entanglement entropy bounds # of locally indistinguishable states

Some remarks:

- The inequality is almost saturated with an equality for Abelian anyon models(up to $o(1)$ correction).
- No assumptions are made about quasiparticle statistics, braiding rule, S-matrix, etc...
- # of locally indistinguishable states $\geq$ # of locally indistinguishable *ground states*

$D = 3$?

For Cubic/Fractal codes k scales as $\Theta(L)$ for certain choices of L on a $L \times L \times L$ lattice.(Chamon 2006, Bravyi *et al.* 2010, **Haah 2011**, Yoshida 2013)

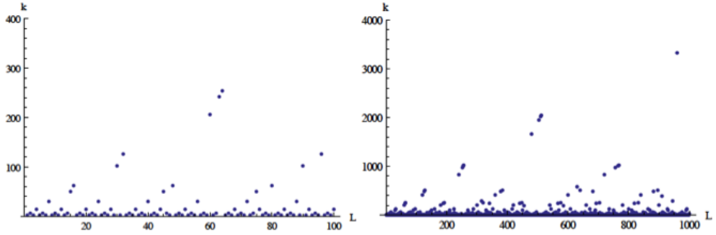


Figure 5.2: Number of encoded qubits k of the cubic code defined on $L \times L \times L$ periodic lattice

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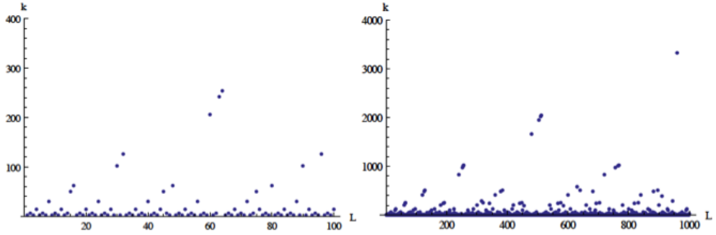

Figure 5.2: Number of encoded qubits k of the cubic code defined on $L \times L \times L$ periodic lattice

The $\Theta(L)$ contribution of the entanglement entropy that cannot be canceled out in any ways. What is the meaning of this?

Summary

- A family of states cannot be locally indistinguishable and conditionally independent at the same time.
- Entanglement entropy dependence of Brandão-Harrow product state ground energy approximation guarantee(STOC '13) becomes trivial if the ground state is a codeword of a good quantum error correcting code.
- Tight tradeoff bounds for local codes can be obtained from entropy analysis alone.
- In the absence of long range entanglement ($\gamma \approx 0$), consistent local reduced density matrices uniquely determine the global wavefunction.
- A perspective from quantum marginal problem : See T. Osborne's work(arXiv:0806.2962).

Open questions

- Tradeoff bound for stabilizer low-density parity check codes?
- If TEE is close to 0, there is only one state that is locally consistent with the local reduced density matrices. Does this mean we can **reconstruct** the global state from the local reduced density matrices with good accuracy?
- Generalization of Petz's theorem to the case when $I(A:C|B) \approx 0$.
- What is this mysterious $\Theta(L)$ term that appears in the entanglement entropy of Haah's model?
- Stability of the topological entanglement entropy?
  - Only perturbative analysis exists so far. (I. Kim, 2012)
- Approximate quantum error correcting code?