Quantum subdivision capacities and continuous quantum coding

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joint work with David Reeb and Michael M. Wolf

arXiv:1310.2856

06.02.2014

TECHNISCHE UNIVERSITÄT MÜNCHEN

What is the optimal rate of information storage in a quantum memory?

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T_t

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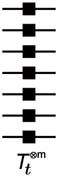

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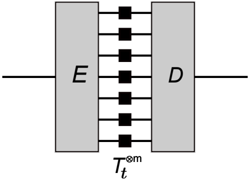

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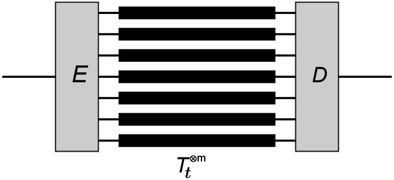

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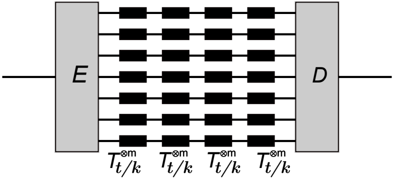

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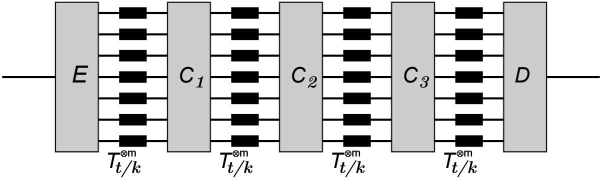

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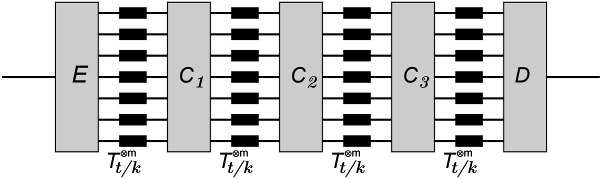


Quantum subdivision capacity: $\mathcal{Q}_{\mathfrak{C}}\,(t\mathcal{L})$

Different choices of $\mathfrak{C}$ lead to different capacities!

Definition (MH, Reeb, Wolf 2013)

The $\mathfrak{C}-$**quantum subdivision capacity** of $t\mathcal{L}$ is then defined as the supremum of asymptotical achievable rates

$$\mathcal{Q}_{\mathfrak{C}}(t\mathcal{L}) := \sup\{R \in \mathbb{R}^+ : \ \mathrm{R} = \limsup_{\nu\to\infty} \frac{n_\nu}{m_\nu}\}.$$

such that the asymptotic communication error vanishes

$$\inf_{k,\mathcal{E},\mathcal{D},\mathcal{C}_1,\ldots,\mathcal{C}_k} \left\| \mathrm{id}_2^{\otimes n_\nu} - \mathcal{D} \circ \prod_{l=1}^{k} \left(\mathcal{C}_l \circ \left(e^{\frac{t}{k}\mathcal{L}} \right)^{\otimes m_\nu} \right) \circ \mathcal{E} \right\|_\diamond \to 0 \quad \text{as } \nu \to \infty.$$

Infimum goes over:

- $k \in \mathbb{N}$ number of subdivisions
- $\mathcal{E} : \mathfrak{M}_2^{\otimes n_\nu} \to \mathfrak{M}_d^{\otimes m_\nu}$ and $\mathcal{D} : \mathfrak{M}_d^{\otimes m_\nu} \to \mathfrak{M}_2^{\otimes n_\nu}$ quantum channels
- $\mathcal{C}_l \in \mathfrak{C}$ channels from the subset $\mathfrak{C}$

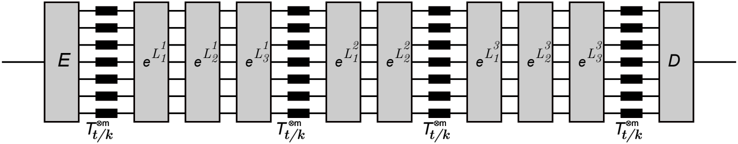
Outline

1. **Infinitesimal divisible coding maps**
2. **Unitary coding maps**
3. **Continuous quantum capacity**

Infinitesimal divisible coding maps

1. Example: Let $\mathfrak{C}$ be the set of **infinitesimal divisible quantum channels**

- $\mathcal{C}_l = \prod_{i=1}^{N} e^{\mathcal{L}_i^l}$ for some coding Liouvillians $\mathcal{L}_i^l$
- $\rightsquigarrow$ Denote this set by ID.



Theorem (MH, Reeb, Wolf 2013)

For any noise Liouvillian $\mathcal{L} : \mathfrak{M}_d \to \mathfrak{M}_d$ and any $t \in \mathbb{R}^+$ we have

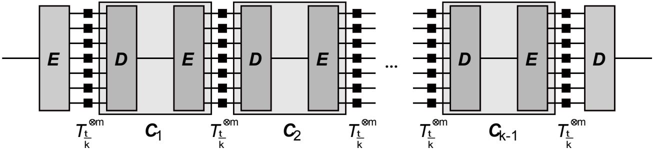
$$\mathcal{Q}_{ID}(t\mathcal{L}) = \log(d)$$

Proof of $\mathcal{Q}_{\mathbf{ID}}(t\mathcal{L}) = \log(d)$

Continuity of quantum capacity:

$$\mathcal{Q}\left(e^{\frac{t}{k}\mathcal{L}}\right) \to \log(d), \quad k \to \infty$$

Coding scheme achieving maximal subdivision capacity $\log(d)$:



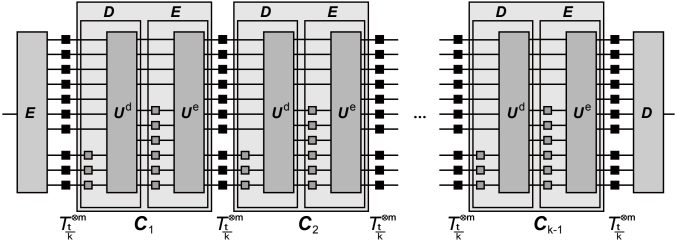
But the channel $\mathcal{D} \circ \mathcal{E}$ is not necessarily infinitesimal divisible!

Proof of $\mathcal{Q}_{\text{ID}}(t\mathcal{L}) = \log(d)$

Implement intermediate coding via unitaries and pure ancillas

Almost pure ancillas from the infinitely divisible channel:

$$\rho \mapsto (1 - e^{-rt})\mathrm{tr}(\rho)\,|0\rangle\langle 0| + e^{-rt}\rho\,, \quad \text{for large rate r}$$



How many pure ancillas are sufficient?

Proof of $\mathcal{Q}_{\mathbf{ID}}(t\mathcal{L}) = \log(d)$

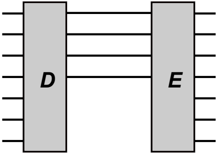
Decoupling approach to the quantum capacity:

$$\frac{1}{4}\inf_{\mathcal{D}}\left\|\mathrm{id} - \mathcal{D}\circ\mathcal{T}^{\otimes m}\circ\mathcal{E}\right\|_{\diamond}^{2} \leq \left\|\left(\mathcal{T}^{\otimes m}\circ\mathcal{E}\right)^{c} - \mathrm{tr}(\bullet)\,\sigma^{E}\right\|_{\diamond}$$

- Random isometry $\mathcal{E}$ leads to asymptotically vanishing r.h.s.
- Then $\mathcal{D}$ has the form

$$\mathcal{D}(\rho) = \mathrm{tr}_E\left(V\rho V^{\dagger}\right)$$

for isometry V, where $|E| = \mathrm{rank}\left(\sigma^E\right)$.



Schumacher Compression:

$$\mathrm{rank}\left(\sigma^E\right) \simeq 2^{mS\left(\sigma^E\right)}$$

$\rightsquigarrow$ **number of qubit ancillas sublinear in m**

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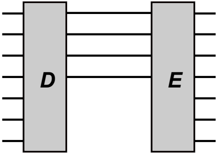

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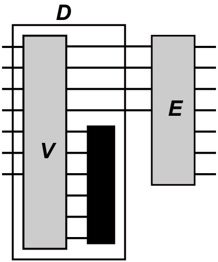

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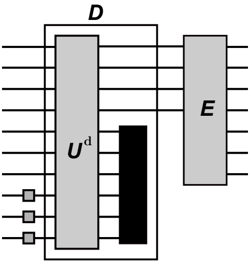

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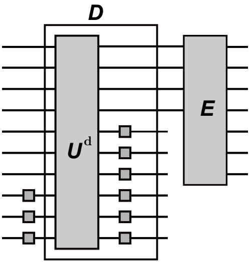

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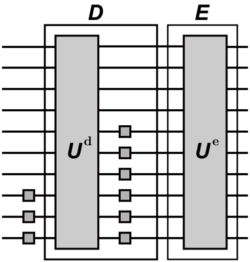

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Unitary coding maps

2. Example: $\mathfrak{C} = \mathfrak{U}$, i.e. unitary coding maps.

Theorem (MH, Reeb, Wolf 2013)

For any noise Liouvillian $\mathcal{L} : \mathfrak{M}_d \to \mathfrak{M}_d$ and any $t \in \mathbb{R}^+$ we have

$$\mathcal{Q}_{\mathfrak{U}}(t\mathcal{L}) > 0$$

$\rightsquigarrow$**Better bounds possible. Depending on the entropy of the fixed points.**

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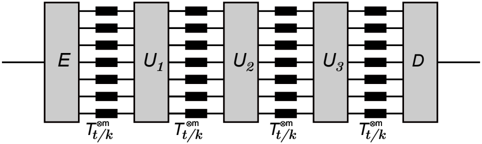

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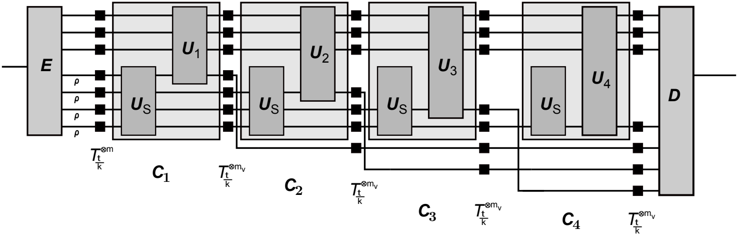

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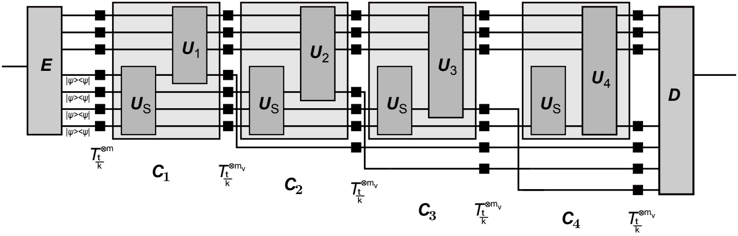

Maximal entropy of ancilla states never reached in finite time!

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$$\mathcal{Q}_{\mathfrak{U}}(t\mathcal{L}) > 0$$

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Is $\mathcal{Q}_{\mathfrak{U}}(t\mathcal{L})$ also $\log(d)$?

Answer: No!

Liouvillian depolarizing onto state $\rho_0 \in \mathfrak{M}_d$:

$$\mathcal{L}^{\text{dep}}(\rho) := \operatorname{tr}(\rho)\,\rho_0 - \rho$$

$\rightsquigarrow$ Generates depolarizing channel: $e^{t\mathcal{L}^{\text{dep}}}(\rho) = \left(1 - e^{-t}\right)\operatorname{tr}(\rho)\,\rho_0 + e^{-t}\rho$

Theorem (MH, Reeb, Wolf 2013)

For the noise Liouvillian $\mathcal{L}^{dep} : \mathfrak{M}_d \to \mathfrak{M}_d$ and $t \in \mathbb{R}^+$ we have

$$\mathcal{Q}_{\mathfrak{U}}\left(t\mathcal{L}^{dep}\right) \leq \log(d) - \left(1 - e^{-t}\right) S\left(\rho_0\right)$$

Proof of $\mathcal{Q}_{\mathfrak{U}}\left(t\mathcal{L}^{\mathbf{dep}}\right) \leq \log(d) - \left(1-e^{-t}\right) S\left(\rho_0\right)$

Consider subdivision coding scheme achieving rate R:

$$\inf_{k,\mathcal{E},\mathcal{D},\mathcal{C}_1,\ldots,\mathcal{C}_k} \left\| \mathrm{id}_2^{\otimes Rm} - \mathcal{D} \circ \prod_{l=1}^{k} \left(\mathcal{U}_l \circ \left(e^{\frac{t}{k}\mathcal{L}} \right)^{\otimes m} \right) \circ \mathcal{E} \right\|_{\diamond} \to 0$$

Entropy growth for depolarizing channels [Aharonov, et al.]:

$$S\left(\left(e^{t\mathcal{L}^{\mathrm{dep}}}\right)^{\otimes m}(\rho)\right) \geq \left(1-e^{-t}\right) mS\left(\rho_0\right) \quad \forall \rho$$

$$\begin{aligned} m\log(d) &\geq S\left(\frac{1}{2^{Rm}} \sum_{i=1}^{2^{Rm}} \tilde{\mathcal{T}}_m\left(|i\rangle\langle i|\right)\right) \\ &\simeq H\left(\left\{\frac{1}{2^{Rm}}\right\}\right) + \sum_{i=1}^{2^{Rm}} \frac{1}{2^{Rm}} S\left(\tilde{\mathcal{T}}_m\left(|i\rangle\langle i|\right)\right) \\ &\geq Rm + \left(1-e^{-t}\right) mS\left(\rho_0\right) \end{aligned}$$

Proof of $\mathcal{Q}_{\mathfrak{U}}\left(t\mathcal{L}^{\text{dep}}\right) \leq \log(d) - \left(1 - e^{-t}\right) S\left(\rho_0\right)$

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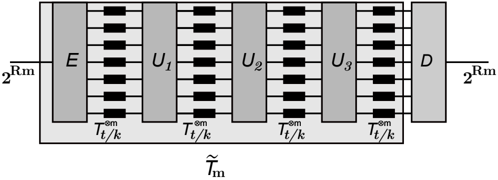

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Continuous quantum capacity

Definition (MH, Reeb, Wolf 2013)

The $\mathfrak{C}-$**continuous quantum capacity** of noise Liouvillian $t\mathcal{L}$ is then defined as the supremum of asymptotical achievable rates

$$\mathcal{Q}_{\mathfrak{C}}^{\text{cont}}\left(t\mathcal{L}\right) := \sup\{R \in \mathbb{R}^{+} : \ \mathrm{R} = \limsup_{\nu\to\infty} \frac{n_\nu}{m_\nu}\}.$$

such that the asymptotic communication error vanishes

$$\inf_{\mathcal{E},\mathcal{D},\mathcal{L}_c} \left\| \mathrm{id}_2^{\otimes n_\nu} - \mathcal{D} \circ T\exp\left(\int_0^t \mathcal{L}^{\oplus m_\nu} + \mathcal{L}_c\left(t'\right) dt'\right) \circ \mathcal{E} \right\|_{\diamond} \to 0 \quad .$$

as $\nu \to \infty$. Infimum goes over:

- $\mathcal{E} : \mathfrak{M}_2^{\otimes n_\nu} \to \mathfrak{M}_d^{\otimes m_\nu}$ and $\mathcal{D} : \mathfrak{M}_d^{\otimes m_\nu} \to \mathfrak{M}_2^{\otimes n_\nu}$ quantum channels
- $\mathcal{L}_c \in \mathfrak{C}$ time-dependent coding Liouvillians from the subset $\mathfrak{C}$

Can dissipation improve quantum capacity?

Answer: Yes

Even for usual quantum capacity we have:

Theorem (MH, Reeb, Wolf 2013)

There exist time-independent Liouvillians $\mathcal{L} : \mathfrak{M}_d \to \mathfrak{M}_d$ *and* $\mathcal{L}' : \mathfrak{M}_d \to \mathfrak{M}_d$ *where* $\mathcal{L}'$ *is purely dissipative such that*

$$\mathcal{Q}\left(e^{\mathcal{L}}\right) < \mathcal{Q}\left(e^{\mathcal{L}+\mathcal{L}'}\right).$$

Some riddles...

To what extend can $\mathcal{Q}\left(e^{\mathcal{L}+\mathcal{L}'}\right)$ **differ from** $\mathcal{Q}\left(e^{\mathcal{L}}\right)$**?**

What are good choices for $\mathfrak{C}$ **in the various capacities?**

Is there a closed formula for $\mathcal{Q}_{\mathfrak{U}}\left(t\mathcal{L}\right)$**?**

Thank you for your attention.

For more information see: arXiv:1310.2856