Entanglement Rates and Area Laws

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Based on Arxiv: 1304.5931 with Karel Van Acoleyen, Frank Verstraete

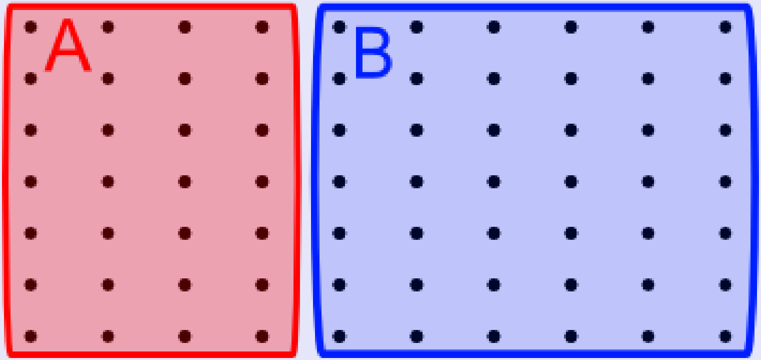
Arxiv: 1304.5935 by Koenraad Audenaert

Overview

Part I: Area Law

Quantum Spin System

- Total Hilbert space dimension is d^N
- Dimension of smallest subsystem (A) is D
- Interest is in both A, B big



A random state of a quantum system has entropy

$S\left(\mathrm{Tr}_A\left(|\psi\rangle\langle\psi|\right)\right) \sim \log D = N \log(d)$ Hayden, Leung, Winter (2004)

For many body systems: volume scaling of entropy

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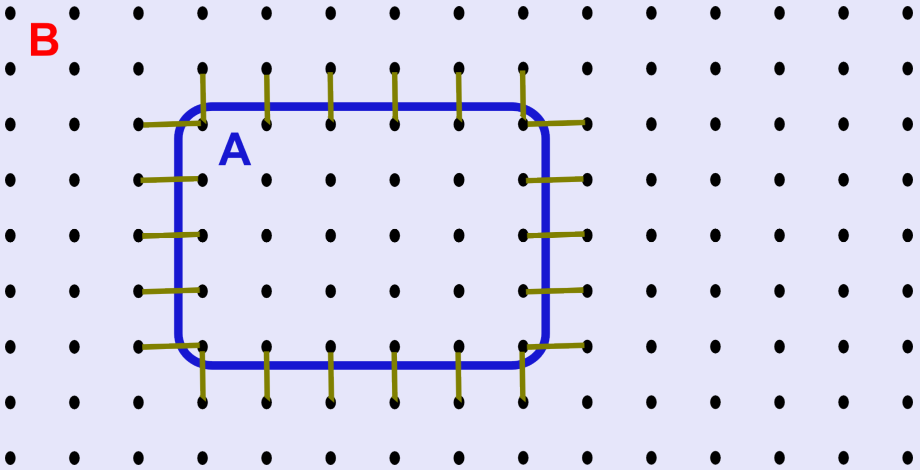
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The area law is the motivation behind variational classes: MPS and PEPS

- Hastings: in 1D, these states have an area law behaviour
- Arad, Kitaev, Landau, Vazirani: improved version

In more than 1 dimension, no rigorous results
Is entanglement a meaningful quantity for many body systems?

$$\begin{aligned} |S(\rho) - S(\sigma)| &\leqslant T \log(D-1) + H(\{T, 1-T\}) \quad \text{(Fannes-Audenaert)} \\ &\lesssim \text{volume scaling} \end{aligned}$$

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Take N qubits and ρ pure and

$$\sigma = (1-\varepsilon)\rho + \frac{\varepsilon}{2^N - 1}(\mathbb{1} - \rho) \Rightarrow |S(\rho) - S(\sigma)| \sim \varepsilon N$$

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In quantum many body theory, important concept of a phase: states in the same phase have similar properties (not expectation values)

Gapped Quantum Phase

When are two ground states of gapped Hamiltonians in the same phase?

Definition (X.G. Wen, Hastings et al.)

- H_0 and H_1 local gapped Hamiltonians with ground states $|\psi_0\rangle, |\psi_1\rangle$
- The states $|\psi_0\rangle, |\psi_1\rangle$ are in the same phase if there exists a $\gamma > 0$ and a smooth path of gapped, local H_s interpolating between H_0, H_1

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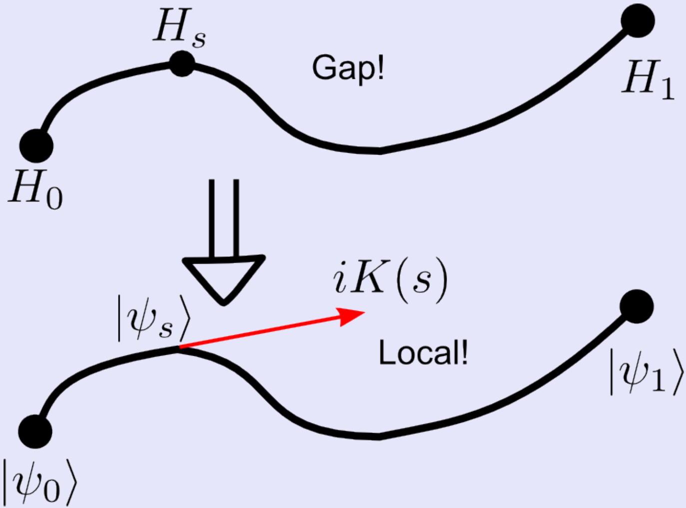

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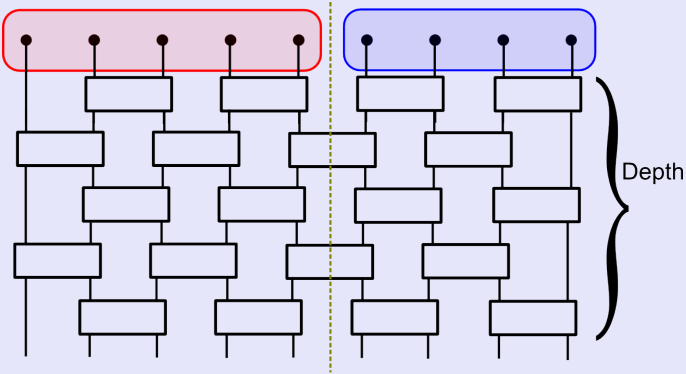
(Almost) equivalent intuitive definition:

- The states $|\psi_0\rangle, |\psi_1\rangle$ are in the same phase if there exists a constant depth local quantum circuit that connects them.

With this intuitive picture in mind:
$|\psi_0\rangle$ obeys an area law iff $|\psi_1\rangle$ does $\Rightarrow$ make this rigorous



$$\Delta S \leq O(\text{Depth.Area})$$

Given a gapped path, how can we go from $|\psi_0\rangle$ to $|\psi_1\rangle$?

Answer $\frac{\partial}{\partial s}|\psi(s)\rangle = iK(s)|\psi(s)\rangle$ with

$$K(s) = -i \int_{\mathbb{R}} F(\gamma t) e^{iH_s t} \left(\partial_s H_s\right) e^{-iH_s t} dt$$

The function F:

- is odd
- decays super polynomially in t
- $\hat{F}(\omega) = -\frac{1}{\omega}, \quad |\omega| \geqslant 1$
- exists, classic result in Fourier analysis

Quasi-Adiabatic Evolution

The existence of K is an exact version of the adiabatic theorem by Kato.

Hastings proved that K itself is a *quasi*-local Hamiltonian!

- Use Lieb-Robinson bounds
- K can be written as $\sum_i \sum_{r \geqslant 0} k_i(r)$ and $\|k(r)\| \leqslant cF(r)$

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Conclusion: $K(s)$ is generator we need

1. Brings $|\psi_0\rangle$ to $|\psi_1\rangle$ in short 'time' $s \in [0, 1]$
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Michalakis (2012):

Extra condition on spectrum of reduced density matrices (decay): use the quasi-adiabatic theorem and techniques from Hasting's proof to find that entanglement changes $\sim A \log A$

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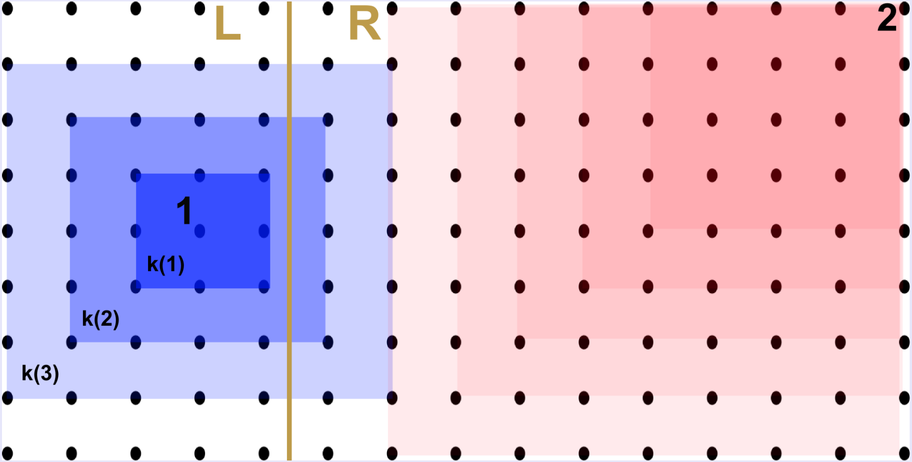
Extra assumption (proof in second part talk):

The maximal rate at which a Hamiltonian H acting on system of dimension D can generate entanglement is $\Gamma(H) \lesssim \|H\| \log D$ independently of ancillas.

Divide a regular 2D lattice in a left and right part with straight cut



Divide a regular 2D lattice in a left and right part with straight cut

$$\begin{aligned}\frac{dS_L(|\psi_s\rangle)}{ds} &= i\,\mathrm{Tr}\left(K(s)[|\psi_s\rangle\langle\psi_s|\,,\log\rho_L\otimes\mathbb{1}_R]\right)\\ &= i\sum_{r\geqslant 0}\sum_{x}\sum_{y}\mathrm{Tr}\left(k_{(x,y)}(r)[|\psi_s\rangle\langle\psi_s)|\,,\log\rho_L\otimes\mathbb{1}_R]\right)\\ &= i\sum_{r\geqslant 0}\sum_{y}\sum_{x\leqslant r}\mathrm{Tr}\left(k_{(x,y)}(r)[|\psi_s)\rangle\langle\psi_s)|\,,\log\rho_L\otimes\mathbb{1}_R]\right).\end{aligned}$$

Hence,

$$\left|\frac{dS_L(|\psi_s\rangle)}{ds}\right| \leqslant \sum_{r\geqslant 0}\sum_{y}\sum_{x\leqslant r}\left|\mathrm{Tr}\left(k_{(x,y)}(r)[|\psi_s\rangle\langle\psi_s|\,,\log\rho_L\otimes\mathbb{1}_R]\right)\right|$$

Divide a regular 2D lattice in a left and right part with straight cut

$$\begin{aligned}
\left|\frac{dS_L(|\psi_s\rangle)}{ds}\right| &\leqslant \sum_{r\geqslant 0}\sum_{y}\sum_{x\leqslant r}\left|\mathrm{Tr}\left(k_{(x,y)}(r)[|\psi_s\rangle\langle\psi_s|\,,\log\rho_L\otimes\mathbb{1}_R]\right)\right| \\
&\leqslant \sum_{r\geqslant 0}\sum_{y}\sum_{x\leqslant r}\Gamma\left(k_{(x,y)}(r)\right) \\
&\leqslant cA_L\sum_{r\geqslant 0} r\|k(r)\|\log\left(d^{P(r)}\right) && \text{log is crucial!} \\
&= cA_L\sum r^3\|k(r)\| && P(r)\sim r^2 \text{ in 2D}
\end{aligned}$$

Since $k(r)$ decays super polynomially, the sum converges in any dimensions for regular lattices and all partitions.

Part II: Entanglement Rate

Entanglement Rate

How fast can a Hamiltonian generate entanglement between two subsystems?

- Interaction H_{AB} between two subsystems: straightforward (Bravyi)

$$\Gamma(H) \leqslant c\|H\| \log D$$

- What if we allow for ancillas?

Do we really expect ancillas to have an influence on this rate for a local Hamiltonian?

The Swap Operator

Look at unitary gates instead of Hamiltonian evolution

- Can the total change of entanglement change by adding ancillas?

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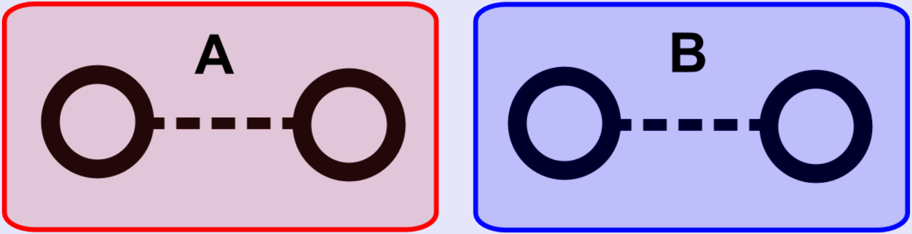

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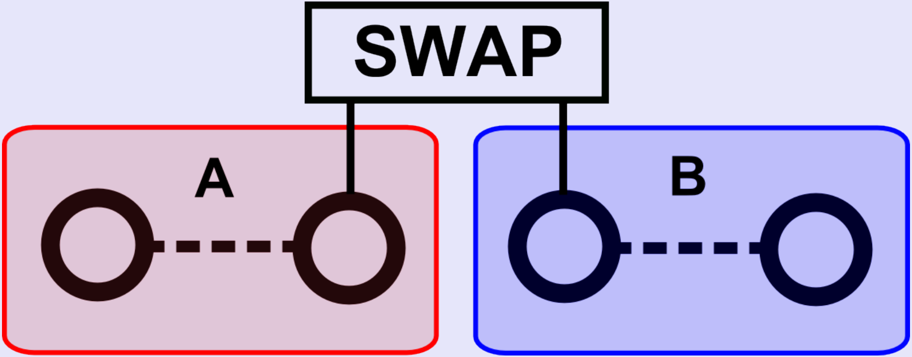

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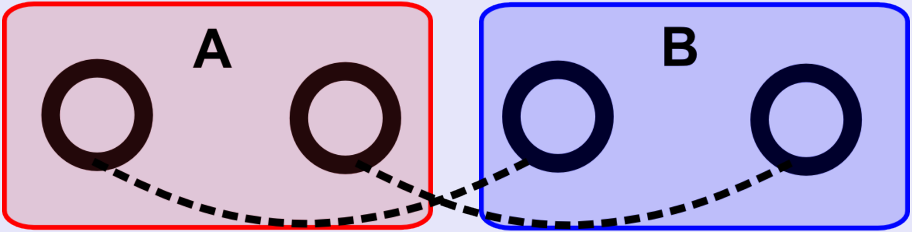

Influence of Ancillas

- The swap operator is the worst case scenario
- In general, the upper bound changes by factor (Bennett et al. 2003),

$$\log D \Rightarrow 2 \log D$$

Influence of Ancillas

- The swap operator is the worst case scenario
- In general, the upper bound changes by factor (Bennett et al. 2003),

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- How about the (infinitesimal) rate at which entanglement can be created?
- Kitaev conjectured the analogous bound

$$\Gamma := \left| \frac{dS(\rho_{Aa})}{dt} \right| \leqslant c \|H\| \log D$$

this conjecture is the *Small Incremental Entangling* (SIE)

History of the Problem

- Example were ancillas increase the entanglement rate given by Dür et al. (2001)
- Several authors obtained partial results,
  1. Dür, et al. (2001): qubits without ancillas
  2. Childs, et al. (2002): Ising and anisotropic Heisenberg interaction
  3. Wang, et al. (2002): Self-inverse product Hamiltonians
  4. Childs, et al. (2004): Simulation of product Hamiltonians
- Bennett, Harrow, Leung, Smolin: first general bound independent of ancillas

The last authors found an upper bound of the form

$$\Gamma \leqslant O(\|H\| D^4)$$

The last bound is a polynomial in the system's dimension, further refinements:

- Bravyi (2007): obtained several results,
    1. $\Gamma \leqslant O(\|H\| D^2)$
    2. general case without ancillas: $\Gamma \lesssim c\|H\| \log D$ (tight, $c \approx 2$)
    3. rewrote the problem to make it tractable (see later)
- Lieb, Vershynina (2013): corollary $\Gamma \leqslant O(\|H\| D) \sim O(\|H\| d^N)$

Numerical evidence suggests that Kitaev was right,

$$\Gamma \leqslant 2\|H\| \log D \sim 2\|H\| N \log d \qquad \text{SIE-Conjecture}$$

Bravyi's Trick

Suppose $D_A \geqslant D_B$, we replace $A \Rightarrow A \otimes a$.

- The entanglement rate reads

$$\Gamma = -i \operatorname{Tr}\left(H_{AB}[\rho_{AB}, \log(\rho_A) \otimes \mathbb{1}_B]\right)$$

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- The entanglement rate reads

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- Find an ensemble $\{(1-p, \rho_0), (p, \rho_{AB})\}$ such that

$$p = \frac{1}{D_B^2} \quad \text{and} \quad (1-p)\rho_0 + p\rho_{AB} = \rho_A \otimes \frac{\mathbb{1}_B}{D_B}$$

Look at *Small Incremental Mixing* (SIM)

$$\Lambda(p) = \frac{dS}{dt} \underbrace{\left((1-p)\rho_0 + p e^{-iHt} \rho_{AB} e^{iHt}\right)}_{\tau(t)}\Bigg|_{t=0}$$

Bravyi's Trick

We see that for this ensemble

$$\Lambda(p) = p\Gamma$$

If we proof that

$$\Lambda(p) \leqslant c\|H\| p \log(1/p) \qquad \text{SIM-Conjecture}$$

we conclude that

$$\Gamma \leqslant c\|H\| \log(D_B^2) = 2c\|H\| \log D_B$$

Bravyi's Trick

We now bound $\Lambda(p)$ under the restrictions $\|H\| = 1$ and $p < e^{-2}$

It suffices to proof that

$$|\Lambda(p)| \leqslant \max_{X,Y} \|[X, \log(Y)]\|_1 \leqslant -cp \log p$$

with

$$\operatorname{Tr} X = p, \quad \operatorname{Tr} Y = 1, \quad 0 \leqslant X \leqslant Y$$

We use variational characterization of the trace norm

$$\|[X, \log(Y)]\|_1 \leqslant 2 \max_{0 \leqslant P \leqslant \mathbb{1}} |\operatorname{Tr}(P[X, \log(Y)])|$$

Proof

1. Use the eigenbasis of Y,

$$2\left|\sum_{i<j}\log\frac{y_i}{y_j}\left(X_{ij}P_{ji}-X_{ji}P_{ij}\right)\right|$$

2. Order its eigenvalues $y_{i_k}\in[p^k,p^{k-1})$ and the summation

$$\sum_{i<j}=\underline{\left(\sum_{i_1<j_1}+\sum_{i_2,j_2}+\sum_{i_2<j_2}\right)}+\underline{\left(\sum_{i_2<j_2}+\sum_{i_2,j_3}+\sum_{i_3<j_3}\right)}+\dots$$
$$-\underline{\left(\sum_{i_2<j_2}\right)}-\underline{\left(\sum_{i_3<j_3}\right)}-\dots\dots+\underline{\left(\sum_{i_1,i_{k>2}}+\sum_{i_2,i_{k>3}}+\dots\right)}$$

Reordering the Summations

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} \end{pmatrix}$$

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Reordering the Summations

$$a_{ij} = \log \frac{y_i}{y_j} \left(X_{ij} P_{ji} - X_{ji} P_{ij} \right)$$

$$\begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\
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a_{71} & \frac{y_j}{y_i} \le p & a_{73} & a_{74} & a_{75} & a_{76} & \frac{y_j}{y_i} \ge p^2 & a_{78} \\
a_{81} & & a_{83} & a_{84} & a_{85} & a_{86} & & a_{88}
\end{pmatrix}$$

Last braces has summations over pairs of eigenvalues far from each other:

$$y_j < py_i \Rightarrow \sqrt{y_j/y_i}\log\left(\frac{y_i}{y_j}\right) \leqslant -\sqrt{p}\log(p)$$

We use Cauchy-Schwarz and $X = Y^{1/2}ZY^{1/2}$ with $0 \leqslant Z \leqslant \mathbb{1}$,

$$\begin{aligned}
\text{Summations} &= 2\left|\tilde{\sum}_{i<j}\log\frac{y_i}{y_j}\left(X_{ij}P_{ji} - X_{ji}P_{ij}\right)\right| \\
&\leqslant \left(\tilde{\sum}\log\frac{y_i}{y_j}\sqrt{y_iy_j}Z_{ij}Z_{ji}\right)^{1/2}\left(\tilde{\sum}\log\frac{y_i}{y_j}\sqrt{y_iy_j}P_{ij}P_{ji}\right)^{1/2} \\
&\leqslant 4\sqrt{p}\log(1/p)\left(\sum y_iZ_{ij}Z_{ji}\right)^{1/2}\left(\sum y_iP_{ij}P_{ji}\right)^{1/2} \\
&\leqslant 4p\log(1/p)
\end{aligned}$$

Restricted Subspaces

First braces: matrices restricted to small subspaces spanned by eigenvectors with close eigenvalues

$$\begin{aligned}\text{First term} = 2\left|\sum_{i}^{n_2}\sum_{j>i}^{n_2}\log\frac{y_i}{y_j}\left(X_{ij}P_{ji}-X_{ji}P_{ij}\right)\right| \\ \leqslant \left\|[\tilde{X},\log\tilde{Y}]\right\|_1 \\ \leqslant \left\|[\tilde{X},\log\tilde{Y}/\tilde{y}_{\min}]\right\|_1 \\ \leqslant \left\|\log\left(\tilde{Y}/\tilde{y}_{\min}\right)\right\| \|X\|_1\end{aligned}$$

We continue:

$$\begin{aligned}\text{First term} &= \log\frac{\tilde{y}_{\max}}{\tilde{y}_{\min}}\operatorname{Tr}\tilde{X}\\ &\leqslant 2\log(1/p)\sum_{i}^{n_2} X_{ii}\end{aligned}$$

The first line in the decomposition is bounded by $4p\log(1/p)$, the last contribution is bounded by $p\log(1/p)$

We obtain the final bound

$$\Lambda(p)\leqslant 9p\log(1/p) \quad\Rightarrow\quad \Gamma\leqslant 18\|H\|\log D$$

The quantum relative entropy

$$S(\rho||\sigma) = \mathrm{Tr}\,\rho(\log\rho - \log\sigma)$$

has the well known problem of divergence if $\mathrm{supp}(\rho) \subsetneq \mathrm{supp}(\sigma)$

Quantum Skew Divergence

The quantum relative entropy

$$S(\rho||\sigma) = \mathrm{Tr}\,\rho(\log\rho - \log\sigma)$$

has the well known problem of divergence if $\mathrm{supp}(\rho) \subsetneq \mathrm{supp}(\sigma)$

One solution is:

$$SD_\alpha(\rho||\sigma) = \frac{1}{-\log\alpha} S(\rho||\alpha\rho + (1-\alpha)\sigma)$$

Quantum Skew Divergence

Is the Quantum Skew Divergence SD_α useful?

Closed formula, linear and operator monotonous, jointly convex, contractivity, ...

- $0 \leqslant SD_\alpha \leqslant 1$ and $SD_\alpha = 1$ iff $\rho \perp \sigma$, $SD_\alpha = 0$ iff $\rho = \sigma$
- $SD_\alpha(\rho||\sigma) \leqslant \|\rho - \sigma\|_1/2$
- Continuity in first and second argument

Special case $\sigma_2 = \sigma$ and $\sigma_1 = e^{itH}\sigma e^{-itH}$:

$$|SD_\alpha(\rho||\sigma_1) - SD_\alpha(\rho||\sigma_2)| \leqslant \frac{1-\alpha}{-\alpha\log\alpha} t\|H\|$$

Holevo Chi

Consider an ensemble of states $\mathcal{E} = \{(p, \rho), (1-p, \sigma)\}$.

The Holevo-Chi quantity is given by

$$\begin{aligned}\chi &= S(p\rho + (1-p)\sigma) - pS(\rho) - (1-p)S(\sigma) \\ &= -p \log pSD_p(\rho||\sigma) - (1-p)\log(1-p)SD_{1-p}(\sigma||\rho) \\ &\leqslant h(\{p, 1-p\})\|\rho - \sigma\|_1/2\end{aligned}$$

Improvement of both $\chi \leqslant h(\{p, 1-p\})$ and $\chi \leqslant \|\rho - \sigma\|_1/2$.

Small Incremental Mixing

Remember the small incremental mixing from Bravyi's trick:

$$\Lambda(p) = \frac{dS}{dt} \underbrace{\left((1-p)\rho_1 + pe^{-iHt}\rho_2 e^{iHt}\right)}_{\tau_t} = \frac{d\chi(\mathcal{E})}{dt}$$

We obtain

$$\begin{aligned} S(\tau(t)) - S(\tau(0)) &= \chi(\mathcal{E}(t)) - \chi(\mathcal{E}(0)) \\ &\leqslant t\|H\| \end{aligned}$$

by rewriting χ and using continuity of SD_α

The factor $h(\{p, 1-p\})$ is missing.

Differential Skew Divergence

Improve the continuity inequality to give us the correct bound

We need to look at *Differential Skew Divergence*

$$\begin{aligned} DSD_\alpha(\rho||\sigma) &= \frac{d}{d(-\log(\alpha))} S(\rho||\alpha\rho + (1-\alpha)\sigma) \\ &= -\alpha \frac{d}{d\alpha} S(\rho||\alpha\rho + (1-\alpha)\sigma) \end{aligned}$$

Improve the continuity inequality to give us the correct bound

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$$DSD_\alpha(\rho||\sigma) = \frac{d}{d(-\log(\alpha))} S(\rho||\alpha\rho + (1-\alpha)\sigma)$$
$$= -\alpha \frac{d}{d\alpha} S(\rho||\alpha\rho + (1-\alpha)\sigma)$$

Same nice properties as Skew Divergence itself, similar proofs, stronger bounds.

Relation is given by averaging procedure:

$$SD_\alpha(\rho||\sigma) = \frac{1}{-\log\alpha} \int_0^{-\log\alpha} DSD_\alpha(\rho||\sigma) d(-\log\alpha)$$

Conclusion

- We considered the rate at which entanglement can be generated by a Hamiltonian H_{AB} in the most general case with ancillas.
- We used Bravyi's trick to rewrite the problem
- Two different methods to proof the upper bound: direct calculation and quantum skew divergence

$$\left| \frac{dS(\rho_{Aa})}{dt} \right| \leqslant c\|H\| \log D$$

- The log and quasi-adiabatic evolution gives the stability of the area law
- Area law is property of phase: suffices to find one state in each phase (fixed point, string net models, ...)
- Details in Arxiv:1304.5931 and Arxiv:1304.5935

THANK YOU